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Note on markaracter tables of finite groups

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Abstract. The markaracter table of a finite group G is a matrix obtained from the mark table of G in which we select rows and columns corresponding to cyclic subgroups of G. This concept was introduced by a Japanese chemist Shinsaku Fujita in the context of stereochemistry and enumeration of molecules. In this note, the markaracter table of generalized quaternion groups and finite groups of order pqr, p, q and r are prime numbers and $p \geq q \geq r$, are computed.

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§1. Introduction

Let G be a finite group acting transitively on a finite set X. Then it is well-known that X is $G-$isomorphic to the set of left cosets $G/H = \{(e = g_1)H, \cdots, g_mH\}$, for some subgroup H of G. Moreover, two transitive $G-$sets G/H and G/K are $G-$isomorphic if and only if H and K are conjugate. If U is a subgroup of G, then the mark $\beta_X(U)$ is defined as $\beta_X(U) = |Fix_X(U)|$, where $Fix_X(U) = \{x \in X : ux = x, \forall u \in U\}$. Set $Sub(G) = \{U|U \leq G\}$. The group G is acting on $Sub(G)$ by conjugation. Assume that the set of orbits of this action is $\Gamma_G/G = \{G_i^G\}_{i=1}^r$, where $G_1(= 1)$, G_2, …, $G_r(= G)$ are representatives of the conjugacy classes of subgroups of G and $|G_1| \leq |G_2| \leq \cdots \leq |G_r|$. The **table of marks** of G, is the square matrix $M(G) = (M_{ij})_{i,j=1}^r$, where $M_{ij} = \beta_{G/G_i}(G_j)$ [3]. This table has substantial applications in isomer counting [1]. For the main properties of this matrix we refer to the interesting paper of Pfeiffer [14].

The matrix $MC(G)$ obtained from $M(G)$ in which we select rows and columns corresponding to cyclic subgroups of G is called the **markaracter table** of G. It is merit to mention here that the markaracter table of finite groups was firstly introduced by Shinsaku Fujita to discuss marks and characters of a finite group in a common basis. Fujita originally developed his theory to be the foundation for enumeration of molecules [4]. We encourage the interested readers to consult papers [5, 6, 7] for some applications in chemistry, the papers [2, 11] for applications in nanoscience and two recent books [8, 9] for more information on this topic. We also refer to [10], for a history of Fujita's theory.

The cyclic group of order n and the generalized quaternion group of order 2^n are denoted by Z_n and Q_{2^n}, respectively. The number of rows in the markaracter table of a finite group G is denoted by $NRM(G)$. Our other notations are standard and mainly taken from the standard books of group theory such as, e.g., [13, 15].

§2. Main Result

The aim of this section is to calculate generally the markaracter tables of groups of order p, pq and pqr, where p, q and r are distinct prime numbers and $p > q > r$.

Theorem 2.1. *Suppose G is a finite group, $MC(G) = (M_{i,j})$ and G_1, G_2, $\dots$, G_r are all non conjugated cyclic subgroups of G, where $|G_1| \leq |G_2| \leq \cdots \leq |G_r|$. Then*

a) *The matrix $MC(G)$ is a lower triangular matrix,*

b) *$M_{i,j}|M_{1,j}$, for all $1 \leq i, j \leq r$,*

c) *$M_{i,1} = \frac{|G|}{|G_i|}$, for all $1 \leq i \leq r$,*

d $M_{i,i} = [N_G(G_i) : G_i]$,

e *if G_i is a normal subgroup of G then M_{ij} is $|G|/|G_i|$ when $G_j \subseteq G_i$, and zero otherwise.*

Proof. The proof follows from definition and the fact that $M_{i,j} = \beta_{G/G_i}(G_j) = |Fix_{G/G_i}(G_j)| = |\{xG_i \mid G_j \subseteq xG_ix^{-1}\}|$. □

As an immediate consequence of Theorem 2.1, the markaracter table of a cyclic group G of prime order p can be computed as:

Table 1. The Markaracter Table of Cyclic Group of Order p, p is Prime.

$MC(G)$	G_1	G_2
G/G_1	p	0
G/G_2	1	1

where $G_1 = 1$ and $G_2 = G$.

Suppose A and B are $m \times n$ and $p \times q$ matrices, respectively. The tensor product $A \otimes B$ of matrices A and B is the $mp \times nq$ block matrix:

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

Lemma 2.2. *Suppose that G_1 and G_2 are two finite groups with co-prime orders. Then the markaracter table of $G_1 \times G_2$ is obtained from the tensor product of $MC(G_1)$ and $MC(G_2)$ by permuting rows and columns suitably.*

Proof. Let A, A_1 and A_2 be the set of all non-conjugate cyclic subgroups of $G_1 \times G_2$, G_1 and G_2, respectively. Suppose that $U = \langle u \rangle \in A_1$ and $V = \langle v \rangle \in A_2$, then $U \times V$ is a cyclic group generated by (u, v). So, $U \times V$ is conjugate with a cyclic subgroup in A. On the other hand, if $H = \langle h \rangle \in A$, then $h = (u, v)$ such that $u \in G_1$, $v \in G_2$ and $gcd(o(u), o(v)) = 1$. Then there are $U \in A_1$ and $V \in A_2$ conjugate with $\langle u \rangle$ and $\langle v \rangle$, respectively, such that $H = U \times V$. Therefore, $NRM(G_1 \times G_2) = NRM(G_1)NRM(G_2)$ and the result follows from Theorem 2.1. □

Let G be a cyclic group of order $n = p_1^{\alpha_1} \ldots a_r^{\alpha_r}$. Then Lemma 2.2 shows that $MC(Z_n) = MC(Z_{p_1^{\alpha_1}}) \otimes \ldots \otimes MC(Z_{p_r^{\alpha_r}})$. Let p be a prime number and q be a positive integer such that $q|p-1$. Define the group $F_{p,q}$ to be presented by $F_{p,q} = \langle a, b : \ a^p = b^q = 1, b^{-1}ab = a^u \rangle$, where u is an element of order q in multiplicative group $\mathbb{Z}_p^*$ [13, Page 290]. It is easy to see that $F_{p,q}$ is a Frobenius group of order pq.

Theorem 2.3. *Let p be a prime number and q be a positive integer such that $q|p-1$ and $q = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_s^{\alpha_s}$ be its decomposition into distinct primes $q_1 < q_2 < \cdots < q_s$. Suppose $\tau(n)$ denotes the number of divisors of n and $d_1 < \cdots < d_{\tau(q)}$ are positive divisors of q. Then the markaracter table of the Frobenius group $F_{p,q}$ can be computed as Table 2.*

Proof. The group $F_{p,q}$ has order pq and its non-conjugate cyclic subgroups are $G_i = \langle\ b^{k_i} \rangle$ where $k_i = \frac{q}{d_i}$ for $1 \leq i \leq \tau(q)$ and $G_{\tau(q)+1} = \langle a \rangle$. Set $MC(F_{p,q}) = (M_{i,j})$. The first column of this table can be computed from Theorem 2.1 (c). The normalizer of G_i, $1 < i \leq \tau(q)$, is equal to $\langle b \rangle$ and so for each $1 < i \leq \tau(q)$, we have $M_{i,i} = \frac{q}{d_i} = d_{\tau(q)-i+1}$. But by Sylow theorem, $G_{\tau(q)+1}$ is normal subgroup of $F_{p,q}$ and by using Theorem 2.1, $M_{\tau(q)+1,1} = M_{\tau(q)+1,\tau(q)+1} = q$ and $M_{\tau(q)+1,j} = 0$, where $2 \leq j \leq \tau(q) - 1$.

Since $M_{i,j} = |\{xG_i \mid G_j \subseteq xG_ix^{-1}\}|$, $1 < j < i \leq \tau(q)$, $G_j \subseteq xG_ix^{-1}$ if and only if $x \in G_{\tau(q)}$ and therefore it is sufficient to compute the number of cosets of G_i in $G_{\tau(q)}$. Finally, this equals to $\frac{q}{d_i}$ if and only if $d_j|d_i$. This completes the proof. □

Table 2. The Markaracter Table of the Frobenius Group $F_{p,q}$.

$MC(F_{p,q})$	G_1	G_2	G_3	$\ldots$	G_i	$\ldots$	$G_{\tau(q)}$	$G_{\tau(q)+1}$
G/G_1	pq	0	0	$\ldots$	0	$\ldots$	0	0
G/G_2	$\frac{pq}{d_2}$	$d_{\tau(q)-1}$	0	$\ldots$	0	$\ldots$	0	0
G/G_3	$\frac{pq}{d_3}$	0	$d_{\tau(q)-2}$	$\ldots$	0	$\ldots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
G/G_i	$\frac{pq}{d_i}$	$m_{i,3}$	$m_{i,4}$	$\ldots$	$d_{\tau(q)-i+1}$	$\ldots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$G/G_{\tau(q)}$	p	1	1	$\ldots$	1	$\ldots$	1	0
$G/G_{\tau(q)+1}$	q	0	0	$\ldots$	0	$\ldots$	0	q

where $m_{i,j} = \begin{cases} \frac{q}{d_i}, & d_j|d_i \\ 0, & o.w. \end{cases}$.

Corollary 2.4. *Let p and q be two prime numbers such that $p > q$ and G is isomorphic to $F_{p,q}$. Then the group $F_{p,q}$ has three non-conjugate subgroups $G_1 = \langle id \rangle$, $G_2 = \langle a \rangle$ and $G_3 = \langle b \rangle$ and the markaracter table of $F_{p,q}$ is as follows:*

Table 3. The Markaracter Table of Non-abelian Group of Order pq.

$MC(F_{p,q})$	G_1	G_2	G_3
G/G_1	pq	0	0
G/G_2	p	1	0
G/G_3	q	0	q

where $|G_1| = 1$, $|G_2| = q$ *and* $|G_3| = p$.

Suppose $\mathfrak{G}(p, q, r)$ be the set of all groups of order pqr where p, q and r are distinct prime numbers with $p > q > r$. Hölder [12] classified groups in $\mathfrak{G}(p, q, r)$. By his result, it can be proved that all groups of order pqr, $p > q > r$, are isomorphic to one of the following groups:

- $G_1 = \mathbb{Z}_{pqr}$,
- $G_2 = \mathbb{Z}_r \times F_{p,q}(q|p - 1)$,
- $G_3 = \mathbb{Z}_q \times F_{p,r}(r|p - 1)$,
- $G_4 = \mathbb{Z}_p \times F_{q,r}(r|q - 1)$,
- $G_5 = F_{p,qr}(qr|p - 1)$,

- $G_{i+5} = \langle a,b,c : a^p = b^q = c^r = 1, ab = ba, c^{-1}bc = b^u, c^{-1}ac = a^{v^i}\rangle$, where $r|p-1, q-1, o(u) = r$ in $\mathbb{Z}_q^*$ and $o(v) = r$ in $\mathbb{Z}_p^*$ $(1 \le i \le r-1)$.

Theorem 2.5. *Let p, q and r be prime numbers such that $p > q > r$ and $G \in \mathfrak{G}(p,q,r)$. Then the markaracter table of G has one of the following shapes:*

1. $MC(G) = MC(\mathbb{Z}_p) \otimes MC(\mathbb{Z}_q) \otimes MC(\mathbb{Z}_r)$,
2. $MC(G) = MC(F_{p,q}) \otimes MC(\mathbb{Z}_r)(q|p-1)$,
3. $MC(G) = MC(F_{p,r}) \otimes MC(\mathbb{Z}_q)(r|p-1)$,
4. $MC(G) = MC(F_{q,r}) \otimes MC(\mathbb{Z}_p)(r|q-1)$,
5. $MC(G) = MC(F_{p,qr})(qr|p-1)$,
6. $MC(G) = MC(G_{i+5})$ $(r|p-1, q-1)$ *and the markaracter table* $MC(G_{i+5})$ *is as follows:*

Table 4. The Markaracter Table of Group $G \cong G_{i+5}$ of Order pqr.

$MC(G)$	H_1	H_2	H_3	H_4	H_5
G/H_1	pqr	0	0	0	0
G/H_2	pq	1	0	0	0
G/H_3	pr	0	pr	0	0
G/H_4	qr	0	0	qr	0
G/H_5	r	0	r	r	r

Proof. If $G \cong G_1$, then the markaracter table of G can be computed by Theorem 2.1. If G is isomorphic to G_2, G_3 or G_4 then by applying Lemma 2.2 and Corollary 2.4, the result is obtained. If G is isomorphic to G_5 then the markaracter of G can be computed directly from Theorem 2.3. It is remained to compute the markaracter table of groups $G \cong G_{i+5}$.

Let $G = G_{i+5}$ for $1 \le i \le r-1$. It is easy to see that $\langle a^\alpha\rangle = \langle a^\beta\rangle$, $\langle b^\delta\rangle = \langle b^\eta\rangle$, $\langle c^\theta\rangle = \langle c^\lambda\rangle$ and $\langle b^\mu a^\nu\rangle = \langle b^\rho a^\varphi\rangle$, where $1 \le \alpha,\beta,\nu,\varphi \le p-1$, $1 \le \delta,\eta,\mu,\rho \le q-1$ and $1 \le \theta,\lambda \le r-1$. Therefore, all of non-conjugate cyclic subgroups of G are $\langle id\rangle$, $\langle a\rangle$, $\langle b\rangle$, $\langle ab\rangle$, $\langle c\rangle$. Let $H_1 = \langle id\rangle$, $H_2 = \langle c\rangle$, $H_3 = \langle b\rangle$, $H_4 = \langle a\rangle$ and $H_5 = \langle ab\rangle$. One can easily check that $N_G(H_2) = H_2$ and $N_G(H_3) = N_G(H_4) = N_G(H_5) = G$ and so by applying Theorem 2.1, the entries of diagonal and the first column of markaracter table can be calculated. Since p,q,r are distinct prime numbers, $M_{3,2} = M_{4,2} = M_{4,3} = M_{5,2} = 0$ and the proof is completed. □

We notice that by our results, the markaracter table of cyclic groups Z_{pqr}, $p < q < r$ are primes, can be computed by Table 5.

Table 5. The Markaracter Table of Cyclic Groups $G \cong Z_{pqr}$, $p < q < r$ are Primes.

$MC(G)$	H_1	H_2	H_3	H_4	H_5
G/H_1	pqr	0	0	0	0
G/H_2	pq	pq	0	0	0
G/H_3	pr	0	pr	0	0
G/H_4	qr	0	0	qr	0
G/H_5	r	0	r	r	r

In the end of this paper, we compute the markaracter table of the generalized quaternion groups. For $n \geq 3$, the generalized quaternion groups can be defined as:

$$Q_{2^n} = \frac{(Z_{2^{n-1}} \rtimes Z_4)}{\langle (2^{n-2}, 2) \rangle},$$

where the semi-direct product has group law $(a, b)(c, d) = (a + (-1)^b c, b + d)$. The order of Q_{2^n} is equal to 2^n.

Theorem 2.6. *The markaracter table of* $G \cong Q_{2^n}$ *is as follows:*

$MC(Q_{2^n})$	G_1	G_2	G_3	G_4	G_5	G_6	$\cdots$	G_r
G/G_1	2^n	0	0	0	0	0	$\cdots$	0
G/G_2	2^{n-1}	2^{n-1}	0	0	0	0	$\cdots$	0
G/G_3	2^{n-2}	2^{n-2}	2^{n-2}	0	0	0	$\cdots$	0
G/G_4	2^{n-2}	2^{n-2}	0	2	0	0	$\cdots$	0
G/G_5	2^{n-2}	2^{n-2}	0	0	2	0	$\cdots$	0
G/G_6	2^{n-3}	2^{n-3}	2^{n-3}	0	0	2^{n-3}	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
G/G_r	2	2	2	0	0	2	$\cdots$	2

where r is the number of non-conjugate subgroups of G.

Proof. Suppose $a = \overline{(1, 0)}$ and $b = \overline{(0, 1)}$. It is well-known that,

- $|\langle a \rangle| = 2^{n-1}$ and $|\langle b \rangle| = 4$,
- $a^{2^{n-2}} = b^2$, $bab^{-1} = a^{-1}$ and for all $g \in Q_{2^n} \setminus \langle a \rangle$, g has order 4 and $gag^{-1} = a^{-1}$,
- the elements of this group have the forms a^x or $a^y b$ where $x, y \in \mathbb{Z}$,
- the $2^{n-2} + 3$ conjugacy classes of Q_{2^n} with representatives $1, a, a^2, \ldots, a^{2^{n-2}-1}, a^{2^{n-2}}, b, ab$.

Therefore, all non-conjugate cyclic subgroups of Q_{2^n} are $\langle b \rangle$, $\langle ab \rangle$ and all non-conjugate subgroups of $\langle a \rangle$. Note that the table obtained from removing the rows and columns 3 and 4, is equal to the markaracter table of $Z_{2^{n-1}}$. □

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