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On Matricial Ranges of Some Matrices

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Abstract. The matricial ranges of 2×2 complex matrices are revisited. Moreover, using the standard block-matrix techniques, we describe matricial ranges of some special non-quadratic higher order matrices. Finally, we obtain the matricial ranges of some specific 3×3 matrices. Various examples are given as well.

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1. Introduction

One of the most well-known concept in study of operator algebras is the notion of numerical range of an operator. Let $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $\mathbb{B}(\mathscr{H})$ be the C^*-algebra of all bounded linear operators on $\mathscr{H}$ with the identity operator I. When $\mathscr{H}$ has finite dimension n, we identify $\mathbb{B}(\mathscr{H})$ with the algebra $\mathbb{M}_n := \mathbb{M}_n(\mathbb{C})$ of all $n \times n$ complex matrices and I_n denotes the $n \times n$ identity matrix. The numerical range of an operator $T \in \mathbb{B}(\mathscr{H})$ is defined by

$$W(T) = \{\langle Tx, x\rangle;\ x \in \mathscr{H},\ \|x\| = 1\}$$

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and the numerical radius of T is defined by $\omega(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle|$. This set is a powerful tool which gives many information about T, particularly about its eigenvalues and eigenspaces. The numerical range has a unique nature in numerical analysis and differential equations. It has many desirable properties, which probably the most famous of them is the Toeplitz-Hausdorff result. It asserts that $W(T)$ is convex for every $T \in \mathbb{B}(\mathscr{H})$; see, e.g., [8, Theorem 1.1-2].

As a noncommutative extension of the numerical range, the matricial ranges of an operator $T \in \mathbb{B}(\mathscr{H})$ is defined by Arveson in [1] as the set of all $n \times n$ matrices of the form $\Phi(T)$ where Φ ranges over all unital completely positive linear mapppings of $C^*(T)$ into $\mathbb{M}_n$ in which $C^*(T)$ is the unital C^*-algebra generated by T and n is a positive integer. This definition is extensively connected with a noncommutative type of convexity, called C^*-convexity. A set $\mathcal{K} \subset \mathbb{B}(\mathscr{H})$ is called C^*-convex, if $X_1, \ldots, X_m \in \mathcal{K}$ and $A_1, \ldots, A_m \in \mathbb{B}(\mathscr{H})$ with $\sum_{j=1}^m A_j^* A_j = I$ imply that $\sum_{j=1}^m A_j^* X_j A_j \in \mathcal{K}$ [13]. Indeed, this is a noncommutative generalization of linear convexity. It is evident that the C^*-convexity of a set implies its convexity in the usual sense. But the converse is not true in general.

Matricial ranges are closely connected with C^*-convex sets. In fact, the matrix ranges turns out to be the compact C^*-convex sets. However, except in some special cases, it is not routine to obtain the matricial ranges of an operator. The reader is referred to [1, 5, 6, 13, 14, 15, 17] and the references therein for more information about C^*-convexity and matricial ranges.

The main purpose of this paper is to describe the matricial ranges of some matrices. In Section 2, we provide preliminaries concerning the matricial ranges. In Section 3, the matricial ranges of all 2×2 matrices are revisited and then the matricial ranges of some specific non-quadratic higher order matrices have been described, accoding to the matricial ranges of 2×2 matrices. Section 4 is devoted to describe the matricial ranges of some special 3×3 matrices. Various examples are given as well.

2. Matricial Ranges

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras with units $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$, respectively. A mapping $\Phi : \mathcal{A} \to \mathcal{B}$ is said to be positive if $\Phi(A) \geqslant 0$ in $\mathcal{B}$ whenever $A \geqslant 0$ in $\mathcal{A}$. It is called unital if $\Phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$. Let $\mathcal{M}_n(\mathcal{A})$ be the C^*-algebra of all $n \times n$ matrices with entries in $\mathcal{A}$. A linear mapping $\Phi : \mathcal{A} \to \mathcal{B}$ is called completely positive if the associated mappings

$$\Phi_{(n)} : \mathcal{M}_n(\mathcal{A}) \to \mathcal{M}_n(\mathcal{B}), \quad \Phi_{(n)}([A_{ij}]) = [\Phi(A_{ij})], \quad (n \geqslant 1),$$

are all positive [2].

We denote by $\mathcal{CP}(C^*(T), \mathbb{M}_n)$ the space of all unital completely positive linear mappings from $C^*(T)$ into $\mathbb{M}_n$. For a positive integer n, the nth matricial range of $T \in \mathbb{B}(\mathscr{H})$ [1] is defined by

$$W^n(T) = \{\Phi(T); \quad \Phi \in \mathcal{CP}(C^*(T), \mathbb{M}_n)\},$$

and the sequence $\{W^1(T), W^2(T), \ldots\}$ is called the matrix range of T.

Let us collect some basic properties of this set. The set $W^n(T)$ is compact and clearly it is contained in the ball of radius $\|T\|$. Moreover, as a noncommutative version of Toeplitz-Hausdorff theorem, it is known that $W^n(T)$ is C^*-convex [13, Example 4].

Also, a subset $\mathcal{K}$ of $\mathbb{M}_n$ is compact and C^*-convex if and only if there exists a bounded operator T with $W^n(T) = \mathcal{K}$. More precisely, the C^*-convex set generated by $T \in \mathbb{M}_n$ is the matricial range $W^n(T)$ of T [13, Corollary 20], that is, if $T \in \mathbb{M}_n$, then

$$W^n(T) = \{\sum_{i=1}^{\infty} V_i^* T V_i \ : \ \sum_{i=1}^{\infty} V_i^* V_i = I_n \text{ (in norm)}\}.$$

Evidently, $W^1(T)$ is the closure of the numerical range of T. Indeed, since $\mathcal{CP}(C^*(T), \mathbb{M}_1 = \mathbb{C}) = \mathcal{S}(C^*(T))$ is the state space of $C^*(T)$, it is known that

$$\{\varphi(T) \ ; \ \varphi \in \mathcal{S}(C^*(T))\} = \overline{W(T)}.$$

Therefore $W^1(T) = \overline{W(T)}$; see, e.g., [2, 18].