1 Functions, Limits and Differentiation

1.1 Introduction

Calculus is the mathematical tool used to analyze changes in physical quantities. It was developed in the 17th century to study four major classes of scientific and mathematical problems of the time:

- Find the tangent line to a curve at a point.
- Find the length of a curve, the area of a region, and the volume of a solid.
- Find minima, maxima of quantities , such as the distance of a planet from sun
- Given a formula for the distance traveled by a body in any specified amount of time, find the velocity and acceleration or velocity at any instant, and vice versa.

1.2 Functions

1.2.1 Definition, Range, Domain

The term *function* was first used by Leibniz in 1673 to denote the dependence of one quantity on another. In general, if a quantity y depends on a quantity x in such a way that each value of x determines exactly one value of y, then we say that y is a "function" of x.

A function is a rule that assigns to each element in a nonempty set A one and only one element in set B. (A is the domain of the function, while B is the range of the function).

- *Domain:* the set in which the independent variable is restricted to lie. Restrictions on the independent variable that affect the domain of the function generally are due to: physical or geometric considerations, natural restrictions that result from a formula used to define the function. and artificial restrictions imposed by a problem solver.
- *Range:* the set of all images of points in the domain (f(x), x∈A).
- *The vertical line test*: A curve in the xy-plane is the graph of $y = f(x)$ for some function f iff no vertical line intercepts the curve more than once.
- *The horizontal line test*: A curve in the xy-plane is the graph of $x = f(y)$ for some function f iff no horizontal line intercepts the curve more than once.
- *Explicit definition of a function: e.g.:* $y = \pm\sqrt[3]{1-x}$
- *Implicit definition of a function: e.g.:* $1 + xy^3 - \sin(x^2y) = 0$,one can not define, by means of simple algebra, whether the y is explicitly defined by x or vice versa.

1.2.2 Arithmetic Operations on functions

- *Sum:* $(f + g)(x) = f(x) + g(x)$,domain: the intersection of the domains of f and g.
- *Difference:* $(f - g)(x) = f(x) - g(x)$,domain: the intersection of the domains of f and g.
- *Product:* $(f * g)(x) = f(x) * g(x)$,domain: the intersection of the domains of f and g.
- *Quotient:* $(f/g)(x) = f(x)/g(x)$,domain: the intersection of the domains of f and g with the points where $g(x) = 0$ excluded.

1.2.3 Composition of functions

- *Composition* of f with g : $(f \circ g)(x) = f((g(x))$,the domain of $f \circ g$ consists

  of all x in the domain of g for which $g(x)$ is in the domain of f.

1.2.4 Classification of functions

- *Constant functions:* $f(x) = c$
- *Polynomial functions:*$f(x) = a_0 + a_1x_1 + \cdots + a_{n-1}x^{n-1} + a_nx_n$
- *Rational functions:* ratio of polynomials functions,
  $f(x) = \frac{a_0+a_1x_1+\cdots+a_{n-1}x^{n-1}+a_nx_n}{b_0+b_1x_1+\cdots+b_{n-1}x^{n-1}+b_nx_n}$
- *Irrational functions:* Root extractions,
  $f(x) = \sqrt[m]{\frac{a_0+a_1x_1+\cdots+a_{n-1}x^{n-1}+a_nx_n}{b_0+b_1x_1+\cdots+b_{n-1}x^{n-1}+b_nx_n}}$
- *Piece-wise functions.e.g.* $f(x) =\mid x - 1 \mid$
- *Transcendental:* trigonometric expressions, exponentials and logarithms[1].

1.2.5 One-to-one functions

- A function f is one-to-one if its graph is cut at most once by any horizontal line, or if it does not have the same value at two distinct points in its domain, or $\forall x_1, x_2 \in D(f), x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$
- Thus, a function has an inverse if it is one-to-one.

1.2.6 Monotone functions

- A function f defined on an interval, x_1,x_2 points in the interval is said to be:
  - *increasing* on the interval if $f(x_1) \prec f(x_2)$, whenever $x_1 \prec x_2$
  - *decreasing* on the interval if $f(x_1) \succ f(x_2)$, whenever $x_1 \prec x_2$
  - *constant* on the interval if $f(x_1) = f(x_2)$,for all points x_1, x_2

[1] see Appendix A

1.2.7 Inverse functions

- *Inverse:* If the functions f and g satisfy the two conditions $f(g(x)) = x \forall x \in D(g)$ and $g(f(x)) = x \forall x \in D(f)$,then f and g are inverse functions.
- *Notation:* $f(f^{-1}(x)) = x$, and $f^{-1}(f(x)) = x$
- Range of f^{-1} = domain of f and domain of f^{-1} = range of f
- If a function has an inverse then the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are symmetric about the line $y = x$.
- *The horizontal line test:* a function f has an inverse, if and only if no horizontal line intersects its graph more than once.
- If the domain of f is an interval if f is either an increasing or decreasing function on that interval, then f has an inverse.

1.3 Limits

The development of calculus was stimulated by two geometric problems: finding areas of plane regions and finding tangent lines to curves. Both these problems are related to the concept of "limit". The portion of calculus arising from the tangent problem is called *differential calculus* and that arising from the area problem is called *integral calculus.*

1.3.1 Notation

One-sided limits of $f(x)$ at x_0 : $\lim\limits_{x \to x_{0-}} f(x)$ and $\lim\limits_{x \to x_{0+}} f(x)$: the limit of $f(x)$ as x approaches x_0 from the left (right).

Two-sided limit of $f(x)$ at x_0 : $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_{0-}} f(x) = \lim\limits_{x \to x_{0+}} f(x)$: the limit of $f(x)$ as x approaches x_0 f.

Limits at infinity: $\lim\limits_{x \to +\infty} f(x)$, $\lim\limits_{x \to -\infty} f(x)$

1.3.2 Computational techniques

- $\lim\limits_{x \to x_0} k = k$, $\lim\limits_{x \to +\infty} k = \lim\limits_{x \to -\infty} k = k$
- $\lim\limits_{x \to x_0} x = x_0$, $\lim\limits_{x \to +\infty} x = +\infty$, $\lim\limits_{x \to -\infty} x = -\infty$,
- $\lim\limits_{x \to +0^+} \frac{1}{x} = +\infty$, $\lim\limits_{x \to 0^-} \frac{1}{x} = -\infty$, $\lim\limits_{x \to +\infty} \frac{1}{x} = 0$, $\lim\limits_{x \to -\infty} \frac{1}{x} = 0$.
- $\lim\,[f(x) + g(x)] = \lim f(x) + \lim g(x)$
- $\lim\,[f(x) - g(x)] = \lim f(x) - \lim g(x)$
- $\lim\,[f(x) * g(x)] = \lim f(x) * \lim g(x)$
- $\lim\,[f(x)/g(x)] = \lim f(x)/\lim g(x)$,if $\lim g(x) \neq 0$
- $\lim \sqrt[n]{f(x)} = \sqrt[n]{\lim f(x)}, \lim f(x) \succeq 0$ if n is even.
- $\lim\,[f(x)]^n = [\lim f(x)]^n$
- $\lim\limits_{x \to +\infty} x^n = +\infty$, $\lim\limits_{x \to -\infty} x^n = +\infty$, if $n = 2, 4, 6, \cdots$, $\lim\limits_{x \to -\infty} x^n = -\infty$, if $n = 1, 3, 5, \cdots$
- $\lim\limits_{x \to \pm\infty} \left(a_0 + a_1 x_1 + \cdots + a_{n-1} x^{n-1} +\right) = \lim\limits_{x \to \pm\infty} (a_n x_n)$
- $\lim\limits_{x \to \pm\infty} \left(\frac{a_0 + a_1 x_1 + \cdots + a_{n-1} x^{n-1} + a_n x_n}{b_0 + b_1 x_1 + \cdots + b_{n-1} x^{n-1} + b_n x_n}\right) = \lim\limits_{x \to \pm\infty} \left(\frac{a_n x_n}{b_n x_n}\right)$

1.3.3 Limits (a formal approach)

- *Definition:* $\lim_{x \to a} f(x) = L$, if $\forall \varepsilon \succ 0, \exists \delta(\varepsilon) \succ 0 : |f(x) - L| \prec \varepsilon$,with $0 \prec |x - a| \prec \delta$
- We assume that an arbitrary positive number ε is given to us and then we try to find a positive number δ dependent on ε such that the above formula is satisfied. Once we find it, any $\delta_1 \prec \delta$ satisfies it, too.
- *Definition:* $\lim_{x \to +\infty} f(x) = L$, if $\forall \varepsilon \succ 0, \exists N \succ 0 : |f(x) - L| \prec \varepsilon$,with $x \succ N$
- *Definition:* $\lim_{x \to -\infty} f(x) = L$, if $\forall \varepsilon \succ 0, \exists N \succ 0 : |f(x) - L| \prec \varepsilon$,with $x \prec N$
- *Definition:* $\lim_{x \to a} f(x) = +\infty$, if $\forall N \succ 0\ \exists \delta \succ 0 : f(x) \succ N$,with $0 \prec |x - a| \prec \delta$
- *Definition:* $\lim_{x \to a} f(x) = -\infty$, if $\forall N \succ 0\ \exists \delta \succ 0 : f(x) \prec N$,with $0 \prec |x - a| \prec \delta$

1.3.4 The Squeezing Theorem

Let f, g, h be functions satisfying $g(x) \preceq f(x) \preceq h(x)$ for all x in some open interval containing the point a. If $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$,then $\lim_{x \to a} f(x) = L$.

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  - $\lim_{x \to 0} \frac{\sin x}{x} = 1$
  - $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$
  - $\lim_{x \to 0} \frac{\tan x}{x} = 1$

1.4 Continuity

A moving physical object cannot vanish at some point and reappear someplace else to continue its motion. The path of a moving object is a single, unbroken curve without gaps, jumps or holes. Such curves are described as continuous.

1.4.1 Definitions

- A function f is said to be *continuous at a point* c if the following conditions are satisfied:
  - $f(c)$ is defined
  - $\lim_{x\to c} f(x)$ exists
  - $\lim_{x\to c} f(x) = f(c)$.

  Examples: $f(x) = x^2 - x - 1$ is a continuous function,

  $f(x) = \frac{x^2-4}{x-2}$ is not a continuous function at $x = 2$, because it is not defined at this point.

  $f(x) = \frac{x^2-4}{x-2}, x \neq 2$, and $f(x) = 3, x = 2$ is not a continuous function because $\lim_{x\to 2} f(x) \neq f(2)$.

A function f is said to be *continuous from the left at a point* c if the following conditions are satisfied:

- - $f(c)$ is defined
  - $\lim_{x\to c^-} f(x)$ exists
  - $\lim_{x\to c^-} f(x) = f(c)$.

A function f is said to be *continuous from the right at a point* c if the following conditions are satisfied:

- - $f(c)$ is defined
  - $\lim_{x\to c^+} f(x)$ exists
  - $\lim_{x\to c^+} f(x) = f(c)$.

A function f is said to be *continuous on a closed interval* $[a, b]$ if the following conditions are satisfied:

- - f is continuous on (a, b)
  - f is *continuous from the right at* a.
  - f is *continuous from the left at* b.

1.4.2 Properties

- Polynomials are continuous functions.
- Rational functions are continuous everywhere except at the points, where the denominator is zero.
- $\lim f(g(x)) = f(\lim(g(x))$, if $\exists \lim g(x)$ and if $f(x)$ is continuous at $\lim g(x)$.
- If the function g is continuous at the point c and the function f is continuous at the point $g(c)$, then the composition $f \circ g$ is continuous at c.
- If a function f is continuous and has an inverse, then $f^{-1}$is also continuous.
- The functions $\sin x$ and $\cos x$ are continuous.
- The functions $\tan x, \cot x, \sec x$ and $\csc x$ are continuous except at the points that they are not defined, the denominator is zero.

1.4.3 The Intermediate value theorem

If f is continuous on a closed interval $[a, b]$ and $C \in [f(a), f(b)]$, then $\exists$at least one $x \in (a, b) : f(x) = C$.

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  - If f is continuous on a closed interval $[a, b]$, and if $f(a), f(b)$ have opposite signs, then there is at least one solution of the equation $f(x) = 0$ in the interval (a, b) .

1.5 Differentiation

Many physical phenomena involve changing quantities- the speed of a rocket, the inflation of a currency, the number of bacteria in a culture, the shock intensity of an earthquake , the voltage of an electric signals. A relationship exists between tangent lines and rates of change.

1.5.1 Tangent lines and rates of change

Tangent versus secant line Slope of the secant line: $m_{\text{sec}} = \frac{f(x_1)-f(x_0)}{x_1-x_0}$

Slope of the tangent line: $m_{\tan} = \lim\limits_{x_1 \to x_0} \frac{f(x_1)-f(x_0)}{x_1-x_0} = \lim\limits_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$

Tangent line: $y - y_0 = m_{\tan}(x_1 - x_0)$

Average rate of change of $y = f(x)$ with respect to x over the interval $[x_0, x_1]$ is the slope m_{sec} of the secant line joining the points $(x_o, f(x_0))$ and $(x_1, f(x_1))$ on the graph of f.

Instantaneous rate of change of $y = f(x)$ with respect to x at the point x_0 is the slope $m_{\tan}$ of the tangent line to the graph of f at the point x_o.

The Derivative The function $f' = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$ is called the derivative with respect to x of the function f. The domain of f'consists of all the points for which the limit exists.

Geometric interpretation of the derivative: Slope of the tangent

Rate of change interpretation: function whose value at x is the instantaneous rate of change of y with respect to x at the point x.

Existence of derivatives The most commonly encountered points of non-differentiability can be classified as corners, vertical tangents, and points of discontinuity.

Differentiability and continuity If a function is differentiable, then it is continuous.

The opposite does not hold.

1.5.2 Techniques of differentiation

- If f is a constant function, $f(x) = c$, for all x, then $f'(x) = 0$ or $\frac{d}{dx}[c] = 0$.
- If n positive integer, then for every real value of x, $\frac{d}{dx}[x^n] = nx^{n-1}$.
- Let c be a constant. If f is differentiable at x,then so is cf, and $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$

- If f and g are differentiable at x,then so is $f \pm g$, and $\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]$
- If f and g are differentiable at x,then so is $f * g$, and $\frac{d}{dx}\left[f(x) * g(x)\right] = f(x)\frac{d}{dx}\left[g(x)\right] + g(x)\frac{d}{dx}\left[f(x)\right]$
- If f and g are differentiable at x,and $g(x) \neq 0$, then so is $\frac{f}{g}$, and $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$
- If g is differentiable at x,and $g(x) \neq 0$, then so is $\frac{1}{g}$, and $\frac{d}{dx}\left[\frac{1}{g(x)}\right] = -\frac{\frac{d}{dx}[g(x)]}{[g(x)]^2}$
- *Higher derivatives:* $f^{(n)}(x) = \frac{d^n}{dx^n}\left[f\left(x\right)\right]$
- *Trigonometric functions*
  - $\frac{d}{dx}\left[\sin x\right] = \cos x$
  - $\frac{d}{dx}\left[\cos x\right] = -\sin x$
  - $\frac{d}{dx}\left[\tan x\right] = \sec^2 x$
  - $\frac{d}{dx}\left[\cot x\right] = -\csc^2 x$
  - $\frac{d}{dx}\left[\sec x\right] = \sec x \tan x$
  - $\frac{d}{dx}\left[\csc x\right] = -\csc x \cot x$
- $\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}, \frac{d}{du}\left[\ln(u)\right] = \frac{1}{u} * \frac{du}{dx}$
- $\frac{d}{dx}\left[\log_b(x)\right] = \frac{1}{x}\log_b(e)$
- $\frac{d}{dx}\left[\exp(x)\right] = \exp x$
- $\frac{d}{dx}\left[b^x\right] = b^x \ln b$
- *Inverse function:* If f has an inverse and the value of $f^{-1}(x)$ varies over an interval on which f has a nonzero derivative, then $f^{-1}$is differentiable and the derivative is given by the formula: $f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$
- *The Chain Rule*

  If g is differentiable at the point x and f is differentiable at the point $g(x)$, then the composition $f \circ g$ is differentiable at the point x.If $y = f(g(x))$ and $u = g(x)$, then $y = f(u)$ and $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
- I*implicit differentiation*

  Example: Find $\frac{dy}{dx}$ if $5y^2 + \sin y = x^2$

  $$\frac{d}{dx}\left[5y^2 + \sin y\right] = \frac{d}{dx}\left[x^2\right] \Rightarrow$$

$5\left(2y\frac{dy}{dx}\right) + (\cos y)\frac{dy}{dx} = 2x \Longrightarrow$

$(10y + \cos y)\frac{dy}{dx} = 2x \Longrightarrow$

$\frac{dy}{dx} = \frac{2x}{10y+\cos y}$

- $\Delta-$*notation; differentials*
  - Increments: $\Delta x =$ change in the value of x, $\Delta y =$ change in the value of y, so $\frac{dy}{dx} = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. The increments ($\Delta y$) represent changes of the curve ($f(x)$).
  - The symbols dx, dy are called differentials and represent changes of the tangent line.
  - If $dx = \Delta x$, Δy represents the change in y that occurs when we start at x and travel along the curve $y = f(x)$ until we have moved $\Delta x (= dx)$ units in the $x-$direction, while dy represents the change in y that occurs when we start at x and travel along the tangent line until we have moved $\Delta x (= dx)$ units in the $x-$direction.

- *Tangent line approximations:*

  $f(x_0 + \Delta x) \approx f(x_0) + f^{'}(x_0)\Delta x$. When $\Delta x \to 0$,this result is a good (linear) approximation of f near x_0.
  - Example: Approximate $\sqrt[2]{1.1}, \cos 62°$

- *Error propagation:* A measurement error Δx propagates to produce an error Δy in the calculated value of y.

  $\Delta y \approx f^{'}(x_0)\Delta x$.
  - Example: estimate the possible error in the computed volume of a sphere with radius measured ti be 50 cm with a possible measurement error $\pm$ 0.02 cm ($V = \frac{4}{3}\pi r^3$).

1.5.3 Applications of differentiation

Related rates problems In this kind of problem, one tries to find the rate at which some quantity changes by relating it to other quantities whose rates of change are known.

- Examples
  - If a rocket is rising vertically at $880 ft/\sec$, when it is at 4000ft up, how fast is the camera-to-rocket distance changing at that instant?(horizontal distance between camera and rocket is 3000 ft).
  - How fast should the camera elevation angle change at that instant to keep the rocket in sight?

– A 5-feet ladder, leaning against a wall slips so that its base moves away from the wall at a rate of 2 ft/sec. How fast will the top of the ladder be moving down the wall when the base is 4 ft from the wall?

Intervals of increase and decrease; concavity

- Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b).
  - if $f'(x) \succ 0$ for every value of x in (a, b) , then f is increasing on $[a, b]$.
  - if $f'(x) \prec 0$ for every value of x in (a, b) , then f is decreasing on $[a, b]$
  - if $f'(x) = 0$ for every value of x in (a, b) , then f is constant on $[a, b]$
- Let f be a function that is differentiable on an interval.
  - if f' is increasing on the interval, then f is concave up on the interval.. $\Leftrightarrow f'' \succ 0$
  - if f' is decreasing on the interval, then f is concave down on the interval.$\Leftrightarrow f'' \prec 0$
- *Inflection point* of f is the point that f changes direction of its concavity.($f'' = 0$)

Relative extrema

- A critical point for a function f is any value of x in the domain of f at which $f'(x) = 0$ or at which f is not differentiable; the critical values where $f'(x) = 0$ are called stationary points of f .
- First derivative test: The relative extrema of a continuous nonconstant function f if any occur at those critical points where f ' changes sign.
- Second derivative test: Suppose f is twice differentiable at a stationary point x_0.
  - If $f''(x_0) \succ 0$, then f has a relative minimum at x_0.
  - If $f''(x_0) \prec 0$, then f has a relative maximum at x_0.

Optimization problems These are problems concerned with finding the best way to perform a task. a large class of these problems can be reduced to finding the largest and smallest value of a function and determining whether this value occurs.

- Extreme value or absolute extremum is a maximum or minimum in the whole domain of the function.

- *Extreme value theorem:* if a function f is continuous on a closed interval $[a,b]$, then f has both a maximum and a minimum value on $[a,b]$.
- If a function f has an extreme value (either a maximum and a minimum value on (a,b)), then the extreme value occurs at a critical point of f .

Applied maximum and minimum problems

- Examples (continuous function over a closed interval)
  - Find the dimensions of a rectangle with perimeter 100 ft whose area is as large as possible.
  - Find the radius and the height of the right circular cylinder of largest volume that can be inscribed in a right-circular cone with radius 6cm and height 10cm.(V=$\pi r^2 h$)
- Examples (continuous function over open or infinite intervals)
  - A closed cylindrical can is to hold 1liter(1000cm^3) of liquid. how should we choose the height and radius to minimize the amount of material needed to manufacture the can? ($S = 2\pi r^2 h + 2\pi r h, V = \pi r^2 h$)
  - Find a point on the curve $y = x^2$ that is closest to the point(18,0)

Newton's method This technique is an efficient method of approximating the solution of an equation.
$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Rolle's Theorem, Mean-Value Theorem

- Rolle's Theorem: Let f be differentiable on (a,b) and continuous on $[a,b]$. If $f(a) = f(b) = 0$,then there is at least one point c in (a,b) where $f'(c) = 0$.(the tangent line to the curve is horizontal)
- Mean Value Theorem: Let f be differentiable on (a,b) and continuous on $[a,b]$. Then there is at least one point c in $(a,b)$where $f'(c) = \frac{f(b)-f(a)}{b-a}$. (the tangent parallel to the secant line).

Motion along a line

- Instantaneous velocity: $v(t) = s'(t) = \frac{ds}{dt}$, where $s(t)$ is the position function of a particle moving on a coordinate line.
- Instantaneous acceleration: $a(t) = v'(t) = \frac{dv}{dt} = s''(t) = \frac{d^2s}{dt^2}$
- instantaneous speed: absolute velocity.

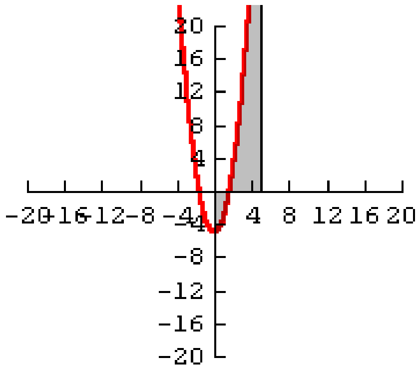

Figure 1:

2 Integration

In this section we are concerned with finding areas: *integral calculus,* so we will discuss techniques of finding areas.

The first real progress in finding areas was made by Archimedes who used the *method of exhaustion:* He inscribed a succession of regular polygons in a circle of radius r and allowed the number of sides n to increase indefinitely. As n increases, the polygons exhaust the the region inside the circle and the areas of polygons become better and better approximations to the exact area of the circle.

We will also discuss the "Fundamental theorem of Calculus" that relates the problem of finding tangent lines and areas. In fact, the distinction between differential and integral calculus is often hard to discern.

2.1 The area problem

- Given a function f that is continuous and non-negative on an interval $[a, b]$, find the area between the graph of f and the interval $[a, b]$ on the x-axis.

- There are two basic methods of finding the area of a region as defined above:
  - The *rectangle method*, and
  - The *antiderivative method*

2.1.1 The rectangle method

This method stems from the method of exhaustion that explicitly incorporates the notion of a limit. It makes use of rectangles that tend to exhaust a region.

We divide the interval [a,b] into n equal subintervals and construct a rectangle that extends from x-axis to any point on the curve $y = f(x)$ that is above the subinterval. For each n the total area of the rectangles can be viewed as an approximation to the exact area under the curve. As n increases these approximations tend to get better and will approach the area as a limit.

Drawback: The limits involved can be evaluated directly only in special cases.

2.1.2 The antiderivative method

This method came from Isaac barrow and Isaac Newton in Great Britain and Leibniz in Germany. To find the area under a curve, one should first find the area $A(x)$ between the graph of f and the interval $[a, x], x \in [a, b]$, then substituting for $x = b$, we take the area. The derivative of the area function $A(x)$ is the function whose graph forms the upper boundary of the region.

The connection between the two methods is given by the Fundamental Theorem of Calculus.

2.2 The indefinite integral; integral curves and direction fields

- *Definition:* A function F is called an *antiderivative* of a function f on a given interval I if $F'(x) = f(x)$ for all x in the interval.

- *Notation:* $\int f(x)dx = F(x) + C$.

  $\int f(x)dx$: indefinite integral
  $f(x)$: integrand
  dx: differential symbol that is used to identify the independent variable.
  C: constant of integration.

- *Formulas:*
$$\begin{array}{ll} \int dx = & x + C \\ \int x^r dx = & \frac{x^{r+1}}{r+1} + C \ (r \neq -1) \\ \int \cos x dx = & sinx + C \\ \int \sin x dx = & -cosx + C \\ \int \sec^2 x dx = & tanx + C \\ \int \csc^2 x dx = & -cotx + C \\ \int \sec x \tan x dx = & secx + C \\ \int \csc x \cot x dx = & -cscx + C \end{array}$$

- *Theorem*:
$$\begin{array}{ll} \int cf(x)dx = & c \int f(x)dx \\ \int [f(x) + g(x)]dx = & \int f(x)dx + \int g(x)dx \\ \int [f(x) - g(x)]dx = & \int f(x)dx - \int g(x)dx \end{array}$$

- *Integral curves:* Graphs of antiderivatives of a function f. (Example: $\frac{dy}{dx} = x^2$ or $y = \frac{x^3}{3} + C$)

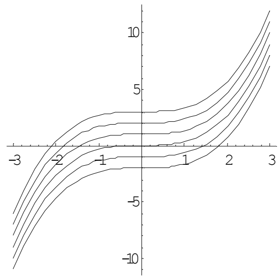

Figure 2:

2.3 Area as a limit

- *Definition (Area under a curve):* if the function f is continuous on $[a, b]$ and if $f(x) \succeq 0$ for all $x \in [a, b]$, then the area under the curve $y = f(x)$ over the interval $[a, b]$ is defined by $A = \lim\limits_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*) \Delta x_k$. We can choose left endpoint approximation, right endpoint approximation or the midpoint approximation.
- If $f(x)$ takes both positive and negative values over the interval, we find the integral by subtracting the negative "areas" from the positive ones.

2.4 Riemann sums and the definite integral

- A partition of the interval $[a, b]$ is a collection of numbers $a = x_o \prec x_1 \prec ... \prec x_{n-1} \prec x_n = b$ that divides $[a, b]$ into n subintervals of lengths $\Delta x_1 = x_1 - x_o, ...\Delta x_n = x_n - x_{n-1.}$ The partition is said to be regular if the subintervals have the same length $\Delta x_k = \frac{b-a}{n}$
- *Definition(The Riemann Sum)*: A function f is said to be integrable on a finite closed interval $[a, b]$ if the limit $\lim\limits_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k$ exists and does not depend on the choice of partitions or on the choice of the numbers x_k^* in the subintervals. When this is the case we denote the limit by the symbol $\int_a^b f(x)dx = \lim\limits_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k$ which is called the definite integral of f from a to b.
- If a function f is continuous on an interval $[a, b]$, then f is integrable on $[a, b]$.
- *Properties of the definite integral:*
  - If a is in the domain of f, we define $\int_a^a f(x)dx = 0$
  - If f is integrable on $[a, b]$, then we define $\int_a^b f(x)dx = -\int_b^a f(x)dx$

- If f is integrable on a closed interval containing a, b, c, then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$no matter how the numbers are ordered.
- If f is integrable on $[a,b]$ and $f(x) \succeq 0$ for all $x \in [a,b]$, then $\int_a^b f(x)dx \succeq 0$.
- If f and g are integrable on $[a,b]$ and $f(x) \succeq g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)dx \succeq \int_a^b g(x)dx$.

- *Discontinuities and Integrability*
  - A function f that is defined on an interval I is said to be bounded on I if there is a positive number M such that -$M \preceq f\ (x) \preceq M$ for all x in the interval I. Geometrically this means that the graph of f over the interval I lies between the lines $y = -M$ and $y = M$.
  - Let f be a function that is defined on the finite closed interval $[a,b]$.
    * If f has finitely many discontinuities in $[a,b]$ but is bounded on $[a,b]$, then f is integrable on $[a,b]$.
    * If f is not bounded on $[a,b]$, then f is not integrable on $[a,b]$.

2.5 The Fundamental theorem of Calculus

Its formulation by Newton and Liebniz is generally regarded to be the discovery of calculus.

The first part of this theorem relates the rectangle and antiderivative methods for calculating areas and the second part provides a powerful method for evaluating definite integrals using antiderivatives.

- Part 1: If f is continuous on $[a,b]$ and F is any antiderivative of f on $[a,b]$, then $\int_a^b f(x)dx = F(b) - F(a)$ Note: It stems from the Mean Value Theorem, which holds for each term in the Riemann sum. We omit the constant of integration.

- What kinds of functions have antiderivatives? All continuous functions.

- Part 2: If f is continuous on an interval I, then f has an antiderivative on I .In particular, if a is any number in I, then the function F defined by $F(x) = \int_a^x f(t)dt$ is an antiderivative of f on I; that is, $F'(x) = f(x) \forall x \in I$, or in an alternative notation $\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$.

2.5.1 Mean Value Theorem for Integrals

If f is continuous on $[a,b]$, then there is at least one number x^*in $[a,b]$ such that $\int_a^b f(x)dx = f(x^*)(b-a)$

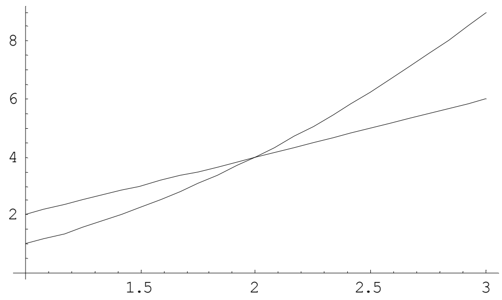

Figure 3:

2.6 Applications

2.6.1 Rectilinear motion revisited

- If the velocity of a partcle is known, then its position can be obtained by $s(t) = \int v(t)dt$ provided that we know the position s_o of the particle at time t_o in order to evaluate the constant of integration.
- Similarly, its velocity can be obtained by $v(t) = \int a(t)dt$.
- Example: find the position of the particle with $v(t) = \int \cos \pi t dt$ ($s_o = 4, t_o = 0$).

2.6.2 Average value of a function

If f is continuous on $[a, b]$,then the average value (or mean value)of f on $[a, b]$ is defined to be $f_{ave} = \frac{1}{b-a} \int_a^b f(x)dx$

2.6.3 Area between two curves

Other Applications:Volumes, Length of a plane curve,Area of a surface of revolution, Work, Fluid pressure and Force

2.6.4 Functions defined by Integrals

- The natural logarithm of x is formally defines by $\ln(x) = \int_1^x \frac{1}{t} dt, t \succ 0$.
- the Error function: $\text{erf}(x) = \frac{2}{\sqrt[2]{\pi}} \int_0^x e^{-t^2} dt$
- Fresnel sine and cosine functions
  - $S(x) = \int_0^x \sin(\frac{\pi t^2}{2})dt$
  - $C(x) = \int_0^x \cos(\frac{\pi t^2}{2})dt$

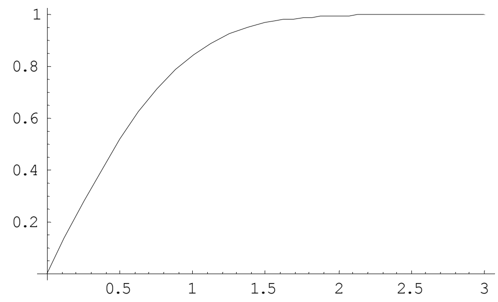

Figure 4:

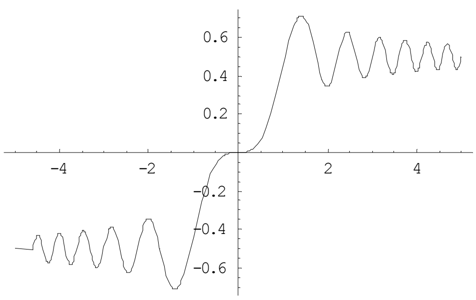

Figure 5:

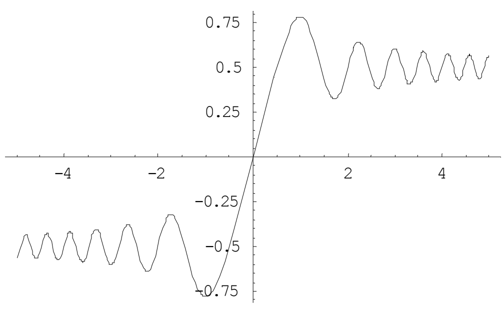

Figure 6:

2.7 Formulas of antiderivatives

2.7.1 Constants, Powers and Exponentials

$\int dx = \quad x + C$
$\int x^r dx = \quad \frac{x^{r+1}}{r+1} + C \ (r \neq -1)$
$\int a dx = \quad ax + C$
$\int \frac{1}{x} dx = \quad \ln|x| + C$
$\int b^x dx = \quad \frac{b^x}{\ln b} + C$
$\int e^x dx = \quad e^x + C$

2.7.2 Trigonometric functions

$\int \cos x dx = \quad sinx + C$
$\int \sin x dx = \quad -cosx + C$
$\int \sec^2 x dx = \quad tanx + C$
$\int \csc^2 x dx = \quad -cotx + C$
$\int \sec x \tan x dx = \quad secx + C$
$\int \csc x \cot x dx = \quad -cscx + C$

$\int \tan x dx = \quad -\ln|\cos x| + C$
$\int \cot x dx = \quad \ln|\sin x| + C$

2.7.3 Hyperbolic functions

$\int \cosh x dx = \quad \sinh x + C$
$\int \sinh x dx = \quad \cosh x + C$
$\int \sec h^2 x dx = \quad \tanh x dx + C$
$\int \csc h^2 x dx = \quad -\coth x + C$
$\int \sec hx \tanh x dx = \quad -\sec hx + c$
$\int s \csc hx \coth x dx = \quad -\csc hx + C$

2.7.4 Algebraic functions

$\int \frac{1}{\sqrt[2]{1-x^2}} dx = \quad \sin^{-1} x + C$
$\int \frac{1}{1+x^2} dx = \quad \tan^{-1} x + C$
$\int \frac{1}{x\sqrt[2]{x^2-1}} dx = \quad \sec^{-1}|x| + C$
$\int \frac{1}{\sqrt[2]{a^2-x^2}} dx = \quad \sin^{-1} \frac{x}{a} + C$
$\int \frac{1}{a^2+x^2} dx = \quad \frac{1}{a} \tan^{-1} \frac{x}{a} + C$
$\int \frac{1}{x\sqrt[2]{x^2-a^2}} dx = \quad \frac{1}{a} \sec^{-1} \left|\frac{x}{a}\right| + C$
. . . .
. . . .

2.8 Techniques of integration

2.8.1 Integration by substitution

- It stems from the chain rule of the derivatives as follows:

$\frac{d}{dx}\left[F(g(x))\right] = F'(g(x))g'(x) \Longrightarrow$
$\int F'(g(x))g'(x)dx = F(g(x)) + C \Longrightarrow$

$\int f(g(x))g'(x)dx = F(g(x)) + C \Longrightarrow$
$\int f(u)du = F(u) + C \qquad (u = g(x) \Longrightarrow \frac{du}{dx} = g'(x) \Longrightarrow du = g'(x)dx)$

- If g' is continuous on $[a, b]$ and f is continuous on an interval containing the values of $g(x)$ for $a \leq x \leq b$, then $\int_a^b f\left(g(x)\right) g'(x)dx = \int_{g(a)}^{g(b} f(u)du$

- Example: $\int \frac{dx}{(\frac{1}{3}x-8)^5} = -\frac{3}{4}\left(\frac{1}{3}x - 8\right)^{-4} + C$

- Verify the following:
  - $\int \sin^2 x \cos x dx = \frac{\sin^3 x}{3} + C \ (u = \sin x)$
  - $\int \frac{\cos \sqrt[2]{x}}{\sqrt[2]{x}} dx = 2\sin\sqrt[2]{x} + C \ (u = \sqrt[2]{x})$
  - $\int t^4 \sqrt[3]{3 - 5t^5} dt = -\frac{3}{100}\left(3 - 5t^5\right)^{\frac{4}{3}} + C$

2.8.2 Integration by Parts

- $\int f(x)g(x)dx = f(x)G(x) - \int f'(x)G(x)dx$
- Examples:$\int xe^x dx, \int \ln x dx, \int e^x \cos x dx$

2.8.3 Trigonometric Substitutions

- $x = a\sin u \ (\frac{-\pi}{2} \leq u \leq \frac{\pi}{2})$ to evaluate expressions such as $\sqrt{a^2 - x^2}$
- $x = a\tan u \ (\frac{-\pi}{2} \leq u \leq \frac{\pi}{2})$ to evaluate expressions such as $\sqrt{a^2 + x^2}$
- $x = a\sec u \ (0 \leq u \leq \frac{\pi}{2}$ if $x \geq a, \frac{\pi}{2} \leq u \leq \pi \ if \ x \leq -a))$ to evaluate expressions such as $\sqrt{x^2 - a^2}$
- Examples: $\int \frac{dx}{x^2\sqrt{4-x^2}} (x = 2\sin u)$

2.8.4 Rational functions by partial fractions

- We decompose proper rational functions (the degree of the numerator is less than the degree of the denominator) into a sum of partial frections.

  $\frac{P(x)}{Q(x)} = F_1(x) + F_2(x) + ... + F_n(x)$, where $F_1(x), F_2(x), ..., F_n(x)$ are rational functions of the form $\frac{A}{(ax+b)^k}, \frac{ax+B}{(ax^2+bx+c)^k}$

- Examples: $\int \frac{dx}{x^2+x-2}, \int \frac{2x+4}{x^3-2x^2}dx, \int \frac{dx}{x^2+x-2}, \int \frac{x^2+x-2}{3x^3-x^2+3x-1}dx$

2.9 Numerical integration

If an antiderivative of an integral can not be found we must find it using numerical approximation for the integral.

2.9.1 Trapezoidal approximation

If we take the average of the left-hand and right-hand approximation endpoint approximations, we ovtain trapezoidal approximation:
$\int_a^b f(x)dx \approx \left(\frac{b-a}{2n}\right)[y_o + 2y_1 + ... + 2y_{n-1} + y_n] = T_n$

2.9.2 Midpoint approximation (tangent approximation)

$\int_a^b f(x)dx \approx \left(\frac{b-a}{n}\right)[y_{m1} + y_{m2} + ... + y_{mn}] = M_n$

2.9.3 Simpson's Rule

$\int_a^b f(x)dx = \frac{1}{3}(2M_n + T_n)$ (It is like fitting a quadrating curve)

2.9.4 Evaluation of the three methods

Simpson's Rule generally produces more accurate results.

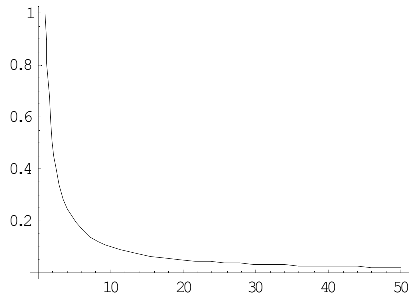

Figure 7:

3 Infinite Series

Definition: Infinite series are sums that involve infinitely many terms. They are used to approximate trigonometric functions and logarithms, to solve differential equations, to evaluate difficult integrals, to create new functions and to construct mathematical models of physical laws. Not all infinite series have a sum, so our aim is to develop tools for determining which infinite series have sums and which do not.

3.1 Sequences

- *Definition:* A sequence is a function whose domain is a set of integers. Specifically, we will regard the expression $\{a_n\}_{n=1}^{+\infty}$to be an alternative expression for the function $f(n) = a_n, n = 1, 2, 3, ...$

Informally, the term "sequence" is used to denote a succession of numbers whose order is determined by a rule or a function.

- Graphs of Sequences: Some examples
  - $a_n = \frac{1}{n}, n = 1, 2, 3, ...$
  - $a_n = \frac{n}{n+1}, n = 1, 2, 3, ...$
  - $a_n = 1 + (-\frac{1}{2})^n, n = 1, 2, 3, ...$
  - $a_n = (2^n + 3^n)^{\frac{1}{n}}, n = 1, 2, 3, ..$
- *Definition:* A sequence $\{a_n\}$ is said to converge to the limit L if given any $\epsilon \succ 0$, there is a positive integer N such that $|a_n - L| \prec \epsilon$ for $n \succeq N$ ($\lim\limits_{n\to+\infty} a_n = L$).
- *Theorem: Suppose that the sequences* $\{a_n\}, \{b_n\}$ converge to limits L_1 and L_2, respectively and c is a constant. Then,
  - $\lim\limits_{n\to+\infty} c = c$

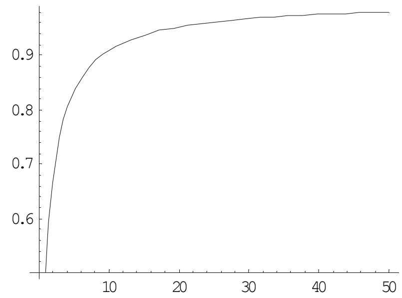

Figure 8:

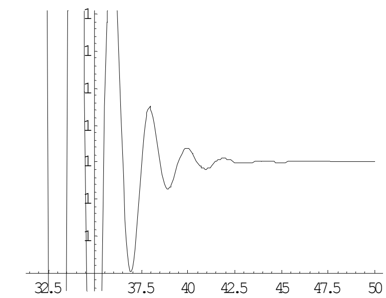

Figure 9:

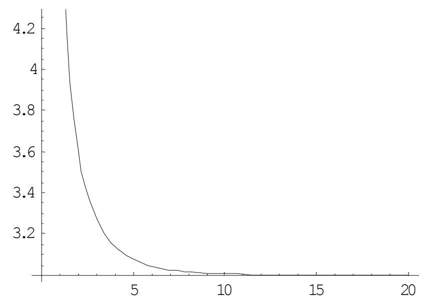

Figure 10:

- $\lim\limits_{n\to+\infty} ca_n = c\lim\limits_{n\to+\infty} a_n = cL_1$
- $\lim\limits_{n\to+\infty}(a_n + b_n) = \lim\limits_{n\to+\infty} a_n + \lim\limits_{n\to+\infty} b_n = L_1 + L_2,$
- $\lim\limits_{n\to+\infty}(a_n - b_n) = \lim\limits_{n\to+\infty} a_n - \lim\limits_{n\to+\infty} b_n = L_1 - L_2,$
- $\lim\limits_{n\to+\infty}(a_n * b_n) = \lim\limits_{n\to+\infty} a_n * \lim\limits_{n\to+\infty} b_n = L_1 * L_2,$
- $\lim\limits_{n\to+\infty}(a_n/b_n) = \lim\limits_{n\to+\infty} a_n/\lim\limits_{n\to+\infty} b_n = L_1/L_2$ (if $L_2 \neq 0$)

- Examples: $\lim\limits_{n\to+\infty} \frac{1}{2^n} = 0$, $\lim\limits_{n\to+\infty} \frac{n}{e^n} = 0$, $\lim\limits_{n\to+\infty} \sqrt[n]{n} = 1$
- *Theorem:* A sequence converges to a limit L if and only if the sequences of even-numbered terms and odd-numbered terms both converge to L.
- *Theorem:* If $\lim\limits_{n\to+\infty} |a_n| = 0$, then $\lim\limits_{n\to+\infty} a_n = 0$
- *Definition: Recursion formulas:* $a_1, a_{n+1} = f(a_n)$

3.2 Monotone sequences

- *Definition:* A sequence $\{a_n\}_{n=1}^{+\infty}$ is called
  - *strictly increasing* if $a_1 \prec a_2 \prec ... \prec a_n \prec ... \Longleftrightarrow a_{n+1} - a_n \succ 0 \Longleftrightarrow \frac{a_{n+1}}{a_n} \succ 1$
  - *increasing* if $a_1 \preceq a_2 \preceq ... \preceq a_n \preceq ... \Longleftrightarrow a_{n+1} - a_n \succeq 0 \Longleftrightarrow \frac{a_{n+1}}{a_n} \succeq 1$
  - *strictly decreasing* if $a_1 \succ a_2 \succ ... \succ a_n \succ ... \Longleftrightarrow a_{n+1} - a_n \prec 0 \Longleftrightarrow \frac{a_{n+1}}{a_n} \prec 1$
  - *decreasing* if $a_1 \succeq a_2 \succeq ... \succeq a_n \succeq ... \Longleftrightarrow a_{n+1} - a_n \preceq 0 \Longleftrightarrow \frac{a_{n+1}}{a_n} \preceq 1$
- *Definition:* If discarding finitely many terms from the beginning of a sequence produces a sequence with a certain property , then the original sequence is said to have the property **eventually.**
- Example: $\left\{\frac{10^n}{n!}\right\}_{n=1}^{\infty}$ is eventually strictly decreasing.
- *Theorem:* If a sequence $\{a_n\}$is eventually increasing, then there are two possibilities:
  - there is a constant M,called an upper bound for the sequence, such that $a_n \preceq M$ for all n, in which case the sequence converges to a limit L satisfying $L \preceq M$.
  - No upper bound exists, in which case $\lim\limits_{n\to+\infty} a_n = +\infty$

- *Theorem:* If a sequence $\{a_n\}$is eventually decreasing, then there are two possibilities:
  - there is a constant M,called a lower bound for the sequence, such that $a_n \succeq M$ for all n, in which case the sequence converges to a limit L satisfying $L \succeq M$.
  - No lower bound exists, in which case $\lim\limits_{n\to+\infty} a_n = -\infty$
- Example:$\{\frac{10^n}{n!}\}_{n=1}^{\infty}$ converges and its limit is 0.

3.3 Infinite series

- *Definition:* An infinite series is an expression that can be written in the form $\sum_{k=1}^{\infty} u_k = u_1 + ... + u_k + ...$The numbers $u_1, ..., u_k$ are called the terms of the series.
- *Definition:* The number $s_n = \sum_{k=1}^{n} u_k$ is called the nth partial sum of the series and the sequence $\{s_n\}_{n=1}^{+\infty}$ is called the sequence of partial sums.
- **Note: a sequence is a *succession*, while a series is a sum.**
- *Definition:* If the sequence $\{s_n\}$ converges to a limit S then the series is said to converge to S, and S is called the sum of the series : $S = \sum_{k=1}^{\infty} u_k$. If the sequence of partial sums diverges, then the series is said to diverge. a divergent series has no sum.
- *Definition:* A series of the form $\sum_{k=0}^{\infty} ar^k = a+ar+ar^2+...+ar^k+...(a \neq 0)$ is called a geometric series and the number r is called the ratio for the series.
- Theorem: A geometric series converges if $|r| \prec 1$ and diverges if $|r| \succeq 1$. if the series converges then the sum is $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$

3.4 Convergence tests

- **The Divergence test:**
  - If $\lim\limits_{k\to+\infty} u_k \neq 0$, then the series $\sum u_k$ diverges.
  - If $\lim\limits_{k\to+\infty} u_k = 0$, then the series $\sum u_k$ may either converge or diverge.
- Example: The following series both have the property $\lim\limits_{k\to+\infty} u_k = 0$.

  $\frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^k} + ...$ and $1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{k} + ...$
  The first is a convergent geometric series, while the second is a divergent harmonic series.
- If the series $\sum u_k$ converges, then $\lim\limits_{k\to+\infty} u_k = 0$.

- **The Integral test:**

  Let $\sum u_k$ be a series with positive terms, and let $f(x)$ be the function that results when k is replaced by x in the general term of the series. If f is decreasing and continuous on the interval $[a, +\infty]$, then $\sum_{k=1}^{\infty} u_k$ and $\int_a^{+\infty} f(x)dx$ both converge or both diverge.

- Examples: $\sum_{k=1}^{\infty} \frac{1}{k} = +\infty$ *and* $\int_1^{+\infty} \frac{1}{x}dx = +\infty$, while

  $\sum_{k=1}^{\infty} \frac{1}{k^2} = 1$ *and* $\int_1^{+\infty} \frac{1}{x^2}dx = 1$

- **Convergence of p-series**

  A p-series or hyperharmonic series is a series of the form:

  $\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + ... + \frac{1}{k^p} + ...$

  A p-series converges if $p \succ 1$ and diverges if $0 \prec p \preceq 1$.

- A series with nonnegative terms converges if and only if its sequence of partial sums is bounded above.

- **The Comparison test**

  Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be series with nonegative terms and suppose that $a_1 \preceq b_1, ...a_k \preceq b_k, ...$

  - if the bigger series $\sum_{k=1}^{\infty} b_k$ converges, then the smaller series $\sum_{k=1}^{\infty} a_k$ also converges.
  - if the smaller series $\sum_{k=1}^{\infty} a_k$ diverges, then the bigger series $\sum_{k=1}^{\infty} b_k$ also diverges.

- Techniques:

  - Constant summands in the denominator of u_k can usually be deleted without affecting the convergence or divergence of the series.

    Example:$\sum_{k=1}^{\infty} \frac{1}{\sqrt[2]{k} - \frac{1}{2}}$ diverges as does the $\sum_{k=1}^{\infty} \frac{1}{\sqrt[2]{k}}$
  - If a polynomial in k appears as a factor in the numerator or denominator of u_k , all but the leading term in the polynomial can usually be discarded without affecting the convergence or divergence of the series.

    Example:$\sum_{k=1}^{\infty} \frac{1}{2k^2+k}$ converges as does the $\sum_{k=1}^{\infty} \frac{1}{2k^2}$

- **The limit comparison test**

  Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be series with positive terms and suppose that $p = \lim\limits_{k \to +\infty} \frac{a_k}{b_k}$. If p is finite and $p \succ 0$, then the series both converge or both diverge.

  - Example:$\sum_{k=1}^{\infty} \frac{3k^3-2k^2+4}{k^7-k^3+2}$(compare with $\sum_{k=1}^{\infty} \frac{3}{k^4}$)

- **The Ratio Test**

  Let $\sum_{k=1}^{\infty} u_k$ be a series with positive terms and suppose that $p = \lim\limits_{k \to +\infty} \frac{u_{k+1}}{u_k}$

  - if $p \prec 1$, the series converges
  - if $p \succ 1$ *or* $p = +\infty$, the series diverges
  - if $p = 1$, another test must be tried.

    Example:$\sum_{k=1}^{\infty} \frac{1}{k!}$, $\sum_{k=1}^{\infty} \frac{k}{2^k}$

- **The Root Test**

  Let $\sum_{k=1}^{\infty} u_k$ be a series with positive terms and suppose that $p = \lim\limits_{k \to +\infty} \sqrt[k]{u_k}$

  - if $p \prec 1$, the series converges
  - if $p \succ 1$ *or* $p = +\infty$, the series diverges
  - if $p = 1$, another test must be tried.

    Example:$\sum_{k=1}^{\infty} \left(\frac{4k-5}{2k+1}\right)^k$, $\sum_{k=1}^{\infty} \frac{1}{(\ln(k+1))^k}$

3.5 Alternating series; Conditional convergence

- *Definition:* Series whose terms alternate between positive and negative are called alternating series. In general, an alternating series has one of the following forms:

  - $\sum_{k=1}^{\infty}(-1)^{k+1}a_k = a_1 - a_2 + a_3 - ...$
  - $\sum_{k=1}^{\infty}(-1)^{k}a_k = -a_1 + a_2 - a_3 - ...$

- **Alternating series test**

  An alternating series converges if the following two conditions are satisfied:

  - $a_1 \succeq a_2 \succeq ... \succeq a_k \succeq ...$
  - $\lim\limits_{k \to +\infty} a_k = 0$

    Example:$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$, $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+3}{k(k+1)}$(the first series is called alternating harmonic series)

- **Absolute convergence**

  If the series$\sum_{k=1}^{\infty} |u_k|$ converges (absolutely), then the series $\sum_{k=1}^{\infty} u_k$ converges.

- **Conditional convergence**

  A series that converges, but diverges absolutely is said to converge conditionally.

  Example: Alternating harmonic series.

- **The Ratio Test for Absolute Convergence**

  It holds as the ratio test for convergence.

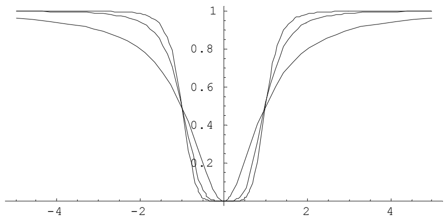

Figure 11:

3.6 Sequences of functions

These sequences are sequences $\{f_n\}$ whose terms are real-valued or complex-valued functions having a common domain on the real line R or in the complex plane C. For each x in the domain set, we can form another sequence $\{f_n(x)\}$ whose terms are the corresponding function values. Let S denote the set of x for which this second sequence converges. the function f defined by the equation $f(x) = \lim_{n\to\infty} f_n(x)$, if $x \in S$, is called the **limit function** of the sequence $\{f_n\}$, and we say that $\{f_n\}$ **converges pointwise** to f on the set S.

Pointwise convergence is usually not strong enough to transfer properties such as continuity, differentiability, or integrability to the limit function. Therefore we are led to study stronger methods of convergence that do preserve these properties. the most important of these is **uniform convergence**.

- Example 1: A sequence of continuous functions with a discontinuous limit function.

  $f_n(x) = \frac{x^{2n}}{1+x^{2n}}, x \in R, n = 1, 2, 3..., \lim_{n\to\infty} f_n(x)$ exists for every real x, and the limit function f is given by $f(x) = 0$, if $|x| \prec 1, f(x) = 1/2$, if $|x| = 1$,and $f(x) = 1$, if $|x| \succ 1$. Each f_n is continuous on R, but f is discontinuous at $x = \pm 1$.

- Example 2: A sequence of differentiable functions $\{f_n\}$ with limit 0 for which $\{f'_n\}$diverges.

  $f_n(x) = (\sin nx)/\sqrt{n}, x \in R, n = 1, 2, 3..., \lim_{n\to\infty} f_n(x) = 0\ \forall x$. But $f'_n(x) = \sqrt{n}\cos nx$ diverges. (see figures)

3.6.1 Uniform convergence of sequences

- *Example:* Consider the following sequence of functions: each function $f_n(x)$ is given for $0 \leq x \leq 2$, and the graph of $y = f_n(x)$ consists of three line segments joining the four points $(0, 0), (1/2n, 1), (1/n, 0), (2, 0)$. For fixed n, the curve $y = f_n(x)$ has a triangular hump with its apex at $(1/2n, 1)$ but except for this hump, $y = 0$. As n increases, the hump

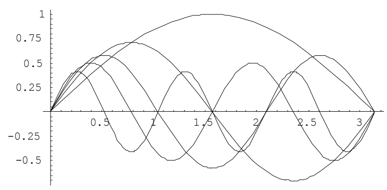

Figure 12:

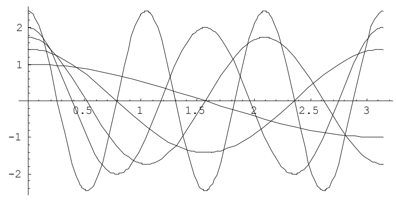

Figure 13:

moves farther to the left. If x is fixed and $0 \leq x \leq 2$,then $\lim_{n \to +\infty} s_n(x) = 0$,because eventually the hump is wholly to the left of x. the same condition holds for $x = 0$, since in this case $f_n(x) = 0$ for all n. Therefore, the sequence converges to 0, although the maximum value of each function is 1, but it does not *converge uniformly*, that is the difference between $f_n(x)$ and its limit can be made small for fixed x, by suitable choice of n,but it can not be made uniformly small for all x simultaneously.

- *Definition:* A sequence $\{f_n(x)\}$ converges uniformly to $f(x)$ in a given interval $[a, b]$ if $\forall\, \epsilon \succ 0\; \exists N$,**independent of** x, such that $|f_n(x) - f(x)| \prec \epsilon, \forall n \succ N, a \leq x \leq b$.
- *Geometric intepretation:* The graph of $y = f_n(x)$ lies in a strip of width 2ϵ centered on the graph of $y = f(x)$. No matter how narrow the strip may be, this condition must hold for all sufficiently large n; otherwise the convergence is not uniform.

3.6.2 Uniform Convergence of Series

Since the value of an infinite series is defined to be the limit of the sequence of partial sums, we can extend the concept of uniform convergence to series.

Let $\sum u_n(x)$ be a series of functions defined in a given interval $[a, b]$, with partial sums $s_n(x) = u_1(x) + u_2(x) + ... + u_n(x)$. If the sequence of partial sums converges uniformly to a function $s\,(x)$, then the series is said to be uniformly convergent.

Consider the remainder $r_n(x)$ after n terms $r_n(x) = s\,(x) - s_n(x) = u_{n+1}(x) + u_{n+2}(x) + ...$

Since the series converges to $s\,(x)$, then $\lim_{n\to+\infty} r_n(x) = 0$, which means that for any preassigned positive number ϵ, however small, one can find a number N such that $|r_n(x)| \prec \epsilon, \forall\forall n \succ N$. The magnitude of N depends not only on the chice of ϵ but also on the value of x. If it is possible to find a single, fixed N, for any preassigned positive ϵ, which will serve for all values of x in the interval, then the series is said to *converge uniformly.*

- *Definition:* The series $\sum u_n(x)$ converges uniformly to $s(x)$ in a given interval $[a, b]$ if $\forall\ \epsilon \succ 0\ \exists N$, **independent of** x, such that the remainder $|r_n(x)| \prec \epsilon, \forall n \succ N, a \leq x \leq b.$
- *Example 1: (* convergent but not uniformly convergent): consider the series $x + (x-1) * x + (x-1) * x^2 + ...$on the interval $[0, 1)$
- *Example 2: (* uniformly convergent): consider the series $\sum x^n$ on the interval $[-1/2, 1/2]$

Uniform Convergence tests- Properties

- Any test of convergence becomes a test of uniform convergence provided its conditions are satisfied uniformly, that is independently of x.
- Let $\sum u_k(x)$ be a series such that each $u_k(x)$ is a continuous function of x in the interval $[a, b]$. If the series is uniformly convergent in $[a, b]$, then the sum of the series is also a continuous function of x in $[a, b]$.
- If a series of continuous functions $\sum u_n(x)$ converges uniformly to $s(x)$, then $\int_\alpha^\beta s(x)dx = \int_\alpha^\beta u_1(x)dx + \int_\alpha^\beta u_2(x)dx + ... \int_\alpha^\beta u_n(x)dx + ...$,where $a \leq \alpha \leq b$ and $a \leq \beta \leq b$
- Let $\sum u_k(x)$ be a series of differentiable functions that converges to $s(x)$ in the interval $[a, b]$. If the series $\sum u'_k(x)$is uniformly convergent in $[a, b]$, then it converges to $s'(x)$.

3.6.3 Mean Convergence

Let $\{f_n\}$be a sequence of integrable functions defined on $[a, b]$. the sequence $\{f_n\}$ is said to converge in the mean to f on $[a, b]$ and we write l.i.m.$_{n\to\infty} f_n = f$ on $[a, b]$ if $\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)|^2\, dx = 0$.

- If the inequality $|f_n(x) - f(x)| \prec \epsilon$ holds for every x in $[a, b]$, then we have $\int_a^b |f_n(x) - f(x)|^2\, dx \leq \epsilon^2(b-a)$. Therefore uniform convergence implies mean convergence, provided that f is integrable on $[a, b]$.
- Convergence in the mean does not imply pointwise convergence at any point on the interval.

3.6.4 The Big Oh and Little oh notation

Given two sequences $\{a_n\}, \{b_n\}$ such that $b_n \geq 0$ for all n.

We write $a_n = O(b_n)$,if there exists a constant $M \succ 0 : |a_n| \leq Mb_n \forall n.$, and $a_n = o(b_n)$ as $n \to \infty$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

3.7 Power series

- One of the most useful types of infinite series is the power series: $a_0 + a_1x + a_2x^2 + ... + a_nx^n + ... = \sum_{n=0}^{\infty} a_nx^n$.
- The region of convergence is easily determined by the ratio test.
- Examples:
  - The series $\sum x^n n!$ converges only for $x = 0$. The ratio of two successive terms leads to $\left|\frac{x^n n!}{x^{n-1}(n-1)!}\right| = |xn| = |x| n \longrightarrow \infty$, for $x \neq 0$.
  - The series $\sum x^n (n!)^{-1}$ converges only for all x. The ratio of two successive terms leads to $\left|\frac{x^n (n-1)!}{x^{n-1}(n)!}\right| = |x| / n \longrightarrow 0$
  - The series $\sum x^n$ converges only for $|x| \prec 1$.
- Every power series, without exception, behaves like one of the previous examples.
- If a series converges for $|x| \prec r$. The number r is called the radius of convergence and the interval $|x| \prec r$ is the interval of convergence.
- *Theorem 1:* A power series may be differentiated or integrated term by term in any interval interior to its interval of convergence. The resulting series has the same interval of convergence as the original series and represents the derivative or integral of the function to which the original series converges.
  - Example: The geometric series $(1-x)^{-1} = 1 + x + x^2 + ... + x^n + ..., |x| \prec 1$. Differentiating this series term by term we obtain $(1-x)^{-2} = 1+2x+3x^2+...+nx^{n-1}+...$. This series converges for $|x| \prec 1$ Integrating this series term by term from zero to x we have the following expansion: $-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + ... + \frac{x^n}{n}$,for $|x| \prec 1$
- *Theorem 2:* If two power series converge to the same sum throughout an interval, the corresponding coefficients are equal.
- *Theorem 3:* Two power series can be multiplied like polynomials for values x which are interior to both intervals of convergence; that is, $\left(\sum a_nx^n\right)\left(\sum b_nx^n\right) = \left(\sum c_nx^n\right)$, where $c_n = a_0b_n + a_1b_{n-1} + a_2b_{n-2} + ... + a_nb_0$.

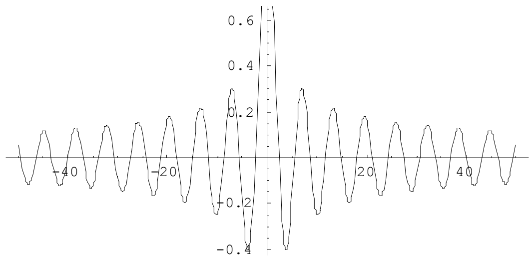

Figure 14:

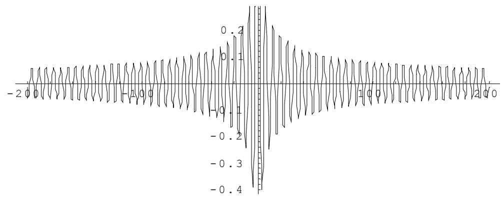

Figure 15:

- *Theorem 4 (Abel's theorem on the continuity of power series).* Suppose the power series $\sum a_n x^n$ converges for $x = x_0$, where x_0 may be an end point of the interval of convergence. then $\lim_{x \to x_0} \sum a_n x^n = \sum a_n x_0^n$ provided that $x \to x_0$ through values interior to the interval of convergence.
  - Example: $-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + ... + \frac{x^n}{n}$,for $|x| \prec 1$. If $x \to -1$,then $-\log 2 = \sum (-1)^n/n$, since the logarithm is continuous.
- Functions defined by power series (Bessel functions)
  - $J_0(x) = \sum \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}$, which is a solution to the differential equation $xy' + y' + xy = 0$ (Bessel equation of order zero) (see figures11,12)
  - $J_1(x) = \sum \frac{(-1)^k x^{2k+1}}{2^{2k+1}(k!)(k+1)!}$which is a solution to the differential equation $x^2y'' + xy' + (x^2-1)y = 0$ (Bessel equation of order one) (see figures 13,14)

3.8 Maclaurin and Taylor Polynomial approximations

- Local linear approximation near the point of tangency is given by the tangent line of a function: $f(x) \simeq f(x_0) + f'(x_0)(x - x_0)$. In this formula, the approximating function $p(x) = f(x_0) + f'(x_0)(x - x_0)$ is a first degree polynomial satisfying the following conditions: $p(x_0) = f(x_0)$ and $p'(x_0) = f'(x_0)$. The local linear approximation of f at x_0 has the property that its value and the values of its first derivatives match those of f at x_0.

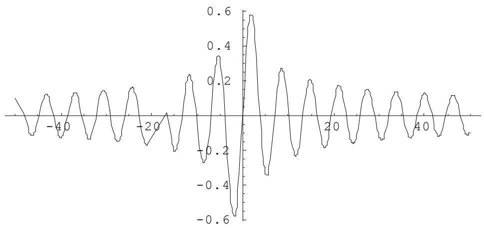

Figure 16:

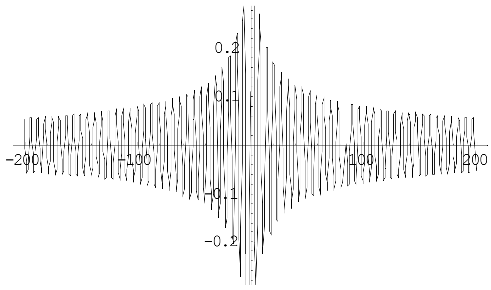

Figure 17:

- **Maclaurin polynomial:** If f can be differentiated n times at 0, then we define the nth Maclaurin polynomial for f to be $p_n(x) = f(0) + f'(x_0)x + \frac{f''(x_0)}{2!}x^2 + ... + \frac{f^{(n)}(x_0)}{n!}x^n$. This polynomial has the property that its value and the values of its first n derivatives match the values of f and its first n derivatives at $x = 0$.

- **Taylor polynomial:** If f can be differentiated n times at x_0, then we define the nth Maclaurin polynomial for f to be $p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$. This polynomial has the property that its value and the values of its first n derivatives match the values of f and its first n derivatives at $x = x_0$.

- The nth remainder: $R_n(x) = f(x) - p_n(x)$. Finding a bound for $R_n(x)$ gives an indication of the accuracy of the approximation $f(x) \approx p_n(x)$.

- The remainder estimation theorem: If the function f can be differentiated $n + 1$ times on an interval I containing the number x_0, and if M is an upper bound for $\left|f^{(n+1)}(x)\right|$ on I, that is $\left|f^{(n+1)}(x)\right| \leq M$ for all x in I, then $|R_n(x)| \leq \frac{M}{(n+1)!}|x - x_0|^{n+1}$ for all x in I.

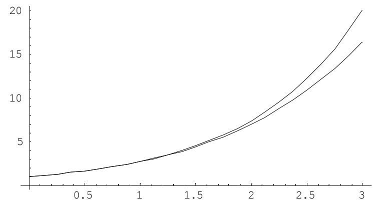

Figure 18:

3.8.1 Maclaurin and Taylor series

- Definition: If f has derivatives of all orders at x_0, then we call the series $\sum\limits_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + ...$,the Taylor series for f about $x = x_0$.

In the special case that $x_0 = 0$, the series becomes $\sum\limits_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x)^k = f(x_0) + f'(x_0)x + \frac{f''(x_0)}{2!}x^2 + ...$ and is called the Mclaurin series for f.

- Examples: The Mclaurin series for $e^x, \sin x, \cos x, \frac{1}{1-x}$.
  - $e^x = \sum\limits_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + ...$

  The figure shows the plot of $\exp x$ together with the 4th partial sum of the Maclaurin series.

3.8.2 Convergence of Taylor series; Computational methods

If $R_n(x) \to 0$,as $n \to \infty$,then $f(x) = \sum\limits_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$.

- **Example 1:** Show that the Maclaurin series for e^x converges to $e^x$for all x .
  $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^k}{k} + ...(-\infty \prec x \prec \infty)$
  $f^{(n+1)}(x) = e^x$. If $x \leq 0$, that is $c \in [x, 0]$, we have $\left|f^{(n+1)}(c)\right| \leq \left|f^{(n+1)}(0)\right| = e^0 = 1$. $(M = 1)$.If $x \succ 0$, that is $c \in [0, x]$, we have $\left|f^{(n+1)}(c)\right| \leq \left|f^{(n+1)}(x)\right| = e^x$ $(M = e^x)$.
  $R_n(x) \to 0$ in both cases.
- **Example 2:** Approximating π: $\tanh(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$ For $x = 1$, $\frac{\pi}{4} = \tanh(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... \Longrightarrow \pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...)$

3.8.3 Differentiating power series

- Suppose that a function f is represented by a power series in $x-x_0$ that has a non-zero radius of convergence R. Then the function f is differentiable on the interval $(x_0 - R, x_0 + R)$. If the power series representation for f is differentiated term by term , then the resulting series has the same radius of convergence R and converges to f' on the interval $(x_0 - R, x_0 + R)$, that is $f'(x) = \sum\limits_{k=0}^{\infty} \frac{d}{dx}[c_k(x - x_0)^k]$.

- If a function f can be represented by a power series in $x - x_0$ with a nonzero radius of convergence R, then f has derivatives of all orders on the interval $(x_0 - R, x_0 + R)$.

3.8.4 Integrating power series

- Suppose that a function f is represented by a power series in $x - x_0$ that has a non-zero radius of convergence R.

  - If the power series representation for f is integrated term by term , then the resulting series has the same radius of convergence R and converges to an antiderivative for $f(x)$ on the interval $(x_0 - R, x_0 + R)$, that is $\int f(x)dx = \sum\limits_{k=0}^{\infty} [\frac{c_{k+1}}{k+1}(x - x_0)^{k+1}] + C$, $x_0 - R \prec x \prec x_0 + R$
  - If a and b are points on the interval $(x_0 - R, x_0 + R)$, and if the power series representation of f is integrated term by term from a to b , then the resulting series converges absolutely on the interval $(x_0 - R, x_0 + R)$ and $\int_a^b f(x)dx = \sum\limits_{k=0}^{\infty} [\int_a^b c_k(x - x_0)^k dx]$.

- If a function f can be represented by a power series in $x - x_0$ on some open interval containing x_0, then the power series is the taylor series for f about $x = x_0$.

- Example: Approximate the integral $\int_0^1 \exp(-x^2)dx$.

  Replace x with $-x^2$ in the Maclaurin series: $\int_0^1 \exp(-x^2)dx = \sum\limits_{k=0}^{\infty} [\frac{(-1)^k}{(2k+1)k!}]$

3.8.5 Maclaurin series for the most important functions

	Maclaurin series	interval of convergence
$\frac{1}{1-x} =$	$\sum\limits_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + ...$	$-1 \prec x \prec 1$
$\frac{1}{1+x^2} =$	$\sum\limits_{k=0}^{\infty} (-1)^k x^{2k} = 1 - x^2 + x^4 - ...$	$-1 \prec x \prec 1$
$e^x =$	$\sum\limits_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$	$-\infty \prec x \prec \infty$
$\sin x =$	$\sum\limits_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$	$-\infty \prec x \prec \infty$
$\cos x =$	$\sum\limits_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...$	$-\infty \prec x \prec \infty$
$\ln(1+x) =$	$\sum\limits_{k=0}^{\infty} \frac{(-1)^{k+1} x^k}{k} = x - \frac{x^2}{2!} + \frac{x^3}{3!} - ...$	$-1 \prec x \leq 1$
$\tan^{-1}(x) =$	$\sum\limits_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - ...$	$-1 \leq x \leq 1$
$\sinh x =$	$\sum\limits_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$	$-\infty \prec x \prec \infty$
$\cosh x =$	$\sum\limits_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$	$-\infty \prec x \prec \infty$
$(1+x)^m =$	$1 + \sum\limits_{k=1}^{\infty} \frac{m(m-1)...(m-k+1)}{k!} x^k$	$1 \prec x \prec 1, (m \neq 0, 1, 2, ..)$

3.9 Fourier series

3.9.1 Introduction-Definitions

Fourier series are trigonometric series of the form $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$. These series are required in the study of many physical phenomena such as heat conduction, theory of sound, electric circuits, and mechanical vibrations. An important advantage of these series is that they can represent discontinuous functions, whereas Taylor series can only represent functions taht have derivatives of all orders.

The coefficients a_n,b_n are given by the following (Euler-Fourier) formulas for the interval $(-\pi, \pi)$ $\begin{array}{l} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx dx \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx \end{array}$. $f(x)$ should be absolutely integrable.

The distinction between a convergent trigonometric series and a Fourier series is important. The trigonometric series $\sum_{n=1}^{\infty} \frac{\sin nx}{\log(1+n)}$ is convergent for every value of x, and yet this is not a Fourier series, because there is no absolutely integrable $f(x)$ such that $\int_{-\pi}^{\pi} f(x) \cos nx dx = 0$ and $\int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{\pi}{\log(1+n)}$.

On the other hand, a series may be a Fourier series for some function and yet diverge. Such functions often arise in the theory of the Brownian motion, the problems of filtering and noise etc. Even when divergent, the Fourier series represents the main features of $f(x)$.

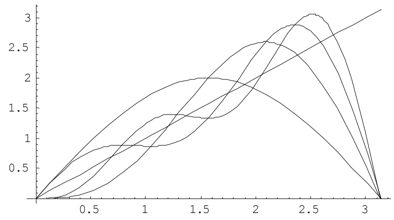

Figure 19:

- Example: Calculate the Fourier series for $f(x) = x$.
  $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx dx = 0$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx dx = -\frac{2}{n}\cos n\pi = \frac{2}{n}(-1)^{n+1} \Rightarrow$
  $x = 2(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - ...)$
  Figure 15 shows $f(x)$ together with the four first partial sums of the Fourier series for $f(x)$. As the number of terms increases, the approximating curves approach $y = x$ for each fixed x on $-\pi \prec x \prec \pi$, but not for $x = \pm\pi$.

- If $f(x)$ defined in the interval $-\pi \prec x \prec \pi$ is even, the Fourier series has cosine terms only and the coefficients are given by $\begin{array}{l} a_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx dx \\ b_n = 0 \end{array}$

- If $f(x)$ defined in the interval $-\pi \prec x \prec \pi$ is odd, the Fourier series has sine terms only and the coefficients are given by $\begin{array}{l} a_n = 0 \\ b_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx \end{array}$

3.9.2 Convergence

- Dirichlet's theorem: For $-\pi \leq x \prec \pi$, suppose that $f(x)$ is defined, bounded, has a finite number of minima nad maxima and has only a finite number of discontinuities. Let $f(x)$ be defined for other values of x by the periodicity condition $f(x + 2\pi) = f(x)$. Then the Fourier series for $f(x)$ converges to $\frac{1}{2}[f(x^+) + f(x^-)]$ at every value of x and hence it converges to $f(x)$ at points where $f(x)$ is continuous.

- Example: $f(x) = -\pi, -\pi \prec x \prec 0$ and $f(x) = x, 0 \prec x \prec \pi$

3.9.3 Extension of the interval

To obtain an expansion valid on the interval $(-l, l)$, change the variable from x to lz/π.If $f(x)$ satisfies the dirichlet conditions on $(-l, l)$, the function $f(lz/\pi)$ can be developed in a Fourier series in z.

3.9.4 Orthogonal and orthonormal functions

A sequence of functions $\theta_n(x)$is said to be orthogonal on the interval (a, b) if $\int_a^b \theta_m(x)\theta_n(x)dx = 0$, for $m \neq n$ and $\neq 0$ for $m = n$.($\theta_n(x) = \sin nx$ is orthogonal on $(0, \pi)$. If for $m = n$, $\int_a^b \varphi_m(x)\varphi_n(x)dx = 1$, then the functions form an orthonormal set. Series analogous to Fourier series are formed by means of any orthogonal set and are called generalised Fourier series. If $\int_a^b \theta_n^2(x)dx = A_n$ then $\varphi_n(x) = \sqrt{A_n}\theta_n(x)$. For example, $\int_0^{2\pi} \sin^2 nxdx = \pi, \varphi_n(x) = \pi^{\frac{-1}{2}} \sin x$.

Let $\{\varphi_n(x)\}$ be an orthonormal set of functions on (a, b) and $f(x)$ is to be expanded in the form $f(x) = c_1\varphi_1(x)+...+c_n\varphi_n(x)+...$ (multiply by $\varphi_n(x)$ and integrate)$\Longrightarrow \int_a^b f(x)\varphi_n(x)dx = \int_a^b c_n\varphi_n^2(x)dx = c_n$.. The coefficients obtained are the Fourier coefficients with respect to $\{\varphi_n(x)\}$. Orthogonal sets of functions are obtained in practice by solving differential equations.

3.9.5 Mean convergence of the Fourier series

When we try to approximate a function $f(x)$ by means of another function $p_n(x)$, the quantity $|f(x) - p_n(x)|$ or $[f(x) - p_n(x)]^2$ gives a measure of the error in the approximation. These maesures are appropriate in the case of convergence at any fixed point.

When we want a measure of error which applies to an interval we use $\int_a^b |f(x) - p_n(x)|\, dx$ or $\int_a^b [f(x) - p_n(x)]^2 dx$. These expressions are called the mean error and mean-square error. (converge in mean-mean convergence)

The partial sums of the Fourier series $c_1\varphi_1 + ... + c_n\varphi_n$, $c_{k=} \int_a^b f\varphi_k(x)dx$ give the smaller mean square error $\int_a^b (f - p_n)^2 dx$ than is given by any other linear combination $p_n = a_1\varphi_1(x) + ... + a_n\varphi_n(x)$.

The Fourier coefficient $c_n = \int_a^b f\varphi_n dx$ tend to zero as $n \to \infty$.

3.9.6 The pointwise convergence of the Fourier series

If $f(x)$ is periodic of period 2π , is piecewise smooth, and is defined at points of discontinuity by the Dirichlet's theorem, then the Fourier series for $f(x)$ converges to $f(x)$ at every value of x.

3.9.7 Integration and differentiation of the Fourier series

Any fourier series (whether convergent or not) can be integrated term by term between any limits. the integrated series converges to the integral of the periodic function corresponding the original series.

There is not much hope of being able to differentiate a fourier series, unless the periodic function generating the series is continuous at every value of x.

3.9.8 Integral transforms

Many functions in analysis can be expressed as improper Riemann integrals of the form $g(y) = \int_{-\infty}^{+\infty} K(x, y)f(x)dx$. The function g defined by an equation of this sort is called an **integral transform** of f. The function K which appears in the integrand is referred to as the **kernel** of the transform. They are especially useful in solving boundary value problems and certain types of integral equations. the more commonly used transforms are the following:

Exponential Fourier transform	$\int_{-\infty}^{+\infty} e^{-ixy} f(x)dx$
Fourier cosine transform	$\int_{-\infty}^{+\infty} \cos xy f(x)dx$
Fourier sine transform	$\int_{-\infty}^{+\infty} \sin xy f(x)dx$
Laplace transform	$\int_{-\infty}^{+\infty} e^{-xy} f(x)dx$
Mellin transform	$\int_{-\infty}^{+\infty} x^{y-1} f(x)dx$

4 Ordinary Differential Equations

4.1 Introduction

The power and effectiveness of mathematical methods in the study of natural sciences stem, to a large extent, from the unambiguous language of mathematics with the aid of which the laws governing natural phenomena can be formulated. Many natural laws especially those concerned with rates of change, can be phrased as equations involving derivatives or differentials. Whenever a mathematical model involves the rate of change of one variable with respect to another, a differential equation is apt to appear.

Differential equations arise in a variety of subject areas, including not only the physical sciences, but also such diverse fields as economics, medicine, psychology and operations research. The following examples provide evidence for it:

- The study of an electrical circuit consisting of a resistor, an inductor, and a capacitor driven by an electromotive force leads to the equation: $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$ (application of Kirchhoff's laws).
- The study of the gravitational equilibrium of a star, which is an application of Newton's law of gravity and of the Stefsn-Boltzmann law for gases leads to the equilibrium equation: $\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi\rho G$,where P is the sum of the gas kinetic and radiation pressure, $r$is the distance from the center of the star, ρ is the density and G is the gravitational constant.
- In psychology, a model of the learning of a task involves the equation $\frac{dy/dt}{y^{\frac{3}{2}}(1-y)^{\frac{3}{2}}} = \frac{2\rho}{\sqrt{n}}$, where the variable y represents the learner's skill level as afunction of time t. The constants ρ and n depend on the individual learner and the nature of the task.

4.2 Definitions

- A **differential equation** is an equation involving some of the derivatives of a function.
- Differential equations are divided into two classes: ordinary and partial. **Ordinary** differential equations contain only one independent variable and derivatives with respect to it, while **partial** differential equations contain more than one independent variable.
- The **order** of the highest derivative contained in a differential equation is the order of the equation.
- A function $y = y(x)$ is a **solution** of a differential equation on an open interval I if the equation is satisfied identically on I when y and its derivatives are substituted on the equation. For example, $y = \exp(2x)$ is a

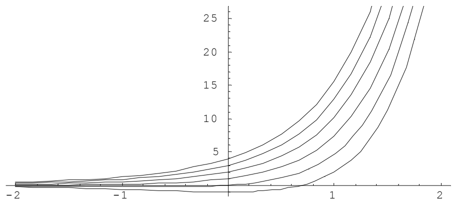

Figure 20:

solution to the differential equation $\frac{dy}{dx} - y = \exp(2x)$ on the interval $I = (-\infty, +\infty)$. However, this is not the only solution on I.

- The function $y = C\exp(x) + \exp(2x)$ is also a solution for every real value of the constant C. On a given interval I, a solution of a differential equation from which all solutions on I can be derived by substituting values for arbitrary constants is called a **general solution** of the equation on I.
- The general solution of an n-th order differential equation on an interval will contain n **arbitrary constants**, because n integrations are needed to recover a function from its n-th derivative, and each integration introduces an arbitrary constant.
- The graph of a solution of a differential equation is called the **integral curve** for the equation, so the general solution of a differential equation produces **a family of integral curves** (see figure 15) corresponding to the different possible choices for the arbitrary constants.
- When an applied problem leads to a differential equation, there are usually conditions in the problem that determine specific values for the arbitrary constants. For a first-order equation, the single arbitrary constant can be determined by specifying the value of the unknown function $y(x)$ at an arbitrary $x-$value x_0, say $y(x_0) = y_0$. This is called an **initial condition** and the problem of solving a f**irst-order initial-value problem.** Geometrically, the initial condition $y(x_0) = y_0$ has the effect of isolating the integral curve that passes through the point (x_0, y_0) from the family of integral curves. For example $y(0) = 3$ in the previous example yields $C = 2$.

4.3 Applications

- **Newton's second law**: an object's mass times its acceleration equals the total force acting on it. In the case of free fall, an object is released from a certain height above the ground and falls under the force of gravity. This leads to the equation $m\frac{d^2h}{dt^2} = -mg \Rightarrow \frac{d^2h}{dt^2} = -g \Longrightarrow \frac{dh}{dt} = -gt + c_1 \Longrightarrow h(t) = -gt^2 + c_1t + c_2$.

The constants of integration can be determined if we know the initial value and the initial velocity of the object.

- **Radioactive decay:**We begin from the premise that the rate of decay is proportional to the amount of radioactive substance present. This leads to the equation $\frac{dA}{dt} = -kA, k \succ 0$, where A is the unknown amount of radioactive substance present at time t and k is the proportionality constant.

  $\frac{dA}{dt} = -kA \Longrightarrow \frac{1}{A}dA = -kdt \Longrightarrow \int \frac{1}{A}dA = \int -kdt \Longrightarrow \ln A + C_1 = -kt + C_2 \Longrightarrow$

  $A = A(t) = \exp(\ln A) = \exp(-kt)\exp(C_2 - C_1) = C\exp(-kt)$. The value of C is determined if the initial amount of the radioactive substance is given.

4.4 More definitions

- A first order equation $\frac{dy}{dx} = f(x, y)$ specifies a slope at each point in the $xy-$plane where f is defined. In other words, it gives the direction that a solution to the equation must have at each point. a plot of short-line segments drawn at various points in the $xy-$plane is called a **direction field** for the equation.The direction field gives a flow of solutions and it facilitates the drawing of any particular solution such as the solution to an initial value problem.

- Equations of the form $\frac{dy}{dt} = f(y)$, for which the independent variable t does not appear explicitly are called **autonomous**.If t is interpreted as time, such equations are self-governing in the sense that the derivative $\acute{y}$ is steered by a function f determined solely by the current state y, and not by any external controller watching the clock. **Equilibrium points** are easily identified by their horizontal direction fields, that is points y where the slope f is zero: $f(y_1) = f(y_2) = ... = 0$. All solutions $y(t)$ that get sufficiently near an equilibrium point are compelled to approach it as $t \to +\infty$.

  - **Stable equilibrium:**If the equilibrium solution is somehow perturbed, it will asymptotically return to it.
    - **Sink:**solutions below the equilibrium are forced upwards,and solutions above are forced downwards.
  - **Unstable equilibrium:**If the equilibrium solution is somehow perturbed, it is driven away from it.
    - **Source:** Unstable equilibrium points that reper all neighboring solutions.
    - **Nodes:** Equilibria which are neither sinks or sources.

- **Phase line**:line on which equilibria are sketched together with arrows showing the sign of f (arrows point right if $f(y)$ is positive,arrows point right if $f(y)$ is negative).

4.5 Methods of solution

4.5.1 First- order linear differential equations

A first order linear differential equation generally takes the form:
$\frac{dy}{dx} + P(x)y(x) = Q(x)$

First- order linear differential equation with constant coefficient and constant term.

- **The homogenous case (reduced equation)**

  $\frac{dy}{dx} + ay(x) = 0$

  General solution: $y(x) = Ae^{-ax}$

  Definite solution: $y(x) = y(0)e^{-ax}$

  Particular solution: substituting any value of A.

- **The non-homogenous case (complete equation)**

  $\frac{dy}{dx} + ay(x) = b$

  General solution: $y(x) = y_c + y_p = Ae^{-ax} + \frac{b}{a}$

- - $y_c = Ae^{-ax}$ is called the complementary function and is the solution to the homogenous case (reduced equation). If $x = t$ (time), y_c reveals the deviation of the time path $y(t)$ from the equilibrium for each point of time.
  - $y_p = \frac{b}{a}$ is called the particular integral . The particular integral is any particular solution of the complete equation and provides us with the equilibrium value of the variable y. For example, if y is a constant function ($y = k$), then $\frac{dy}{dx} = 0 \Longrightarrow ay(x) = b \Longrightarrow y(x) = \frac{b}{a}$. In this case, the particular integral is $y_p = \frac{b}{a}$.

  Definite solution: $y(x) = [y(0) - \frac{b}{a}]e^{-ax} + \frac{b}{a}$
  Particular solution: substituting any value of A.

- **Examples:**
  - $\frac{dy}{dx} + 4y = 8, \quad y(0) = 2$
  - $\frac{dy}{dx} - 2y = 0, \quad y(0) = 3$
  - $\frac{dy}{dx} + 10y = 15, \quad y(0) = 0$
  - $2\frac{dy}{dx} + 4y = 6, \quad y(0) = 1$
  - $3\frac{dy}{dx} + 6y = 5, \quad y(0) = 0$

First- order linear differential equation with variable coefficient and variable term.

- **The homogenous case (reduced equation)**
  $\frac{dy}{dx} + P(x)y(x) = 0$
  General solution: $y(x) = Ae^{-\int P(x)dx}$
- **The non-homogenous case (complete equation)**
  $\frac{dy}{dx} + P(x)y(x) = Q(x)$
  Integrating factor: $\exp(\int P(x)dx)$

General solution: $y(x) = e^{-\int P(x)dx}(c + \int Q(x)e^{\int P(x)dx}dx)$

- **Examples:**
  - $\frac{dy}{dx} + 2xy = x, \quad y(0) = \frac{3}{2}$
  - $\frac{dy}{dx} + 2xy = 0, \quad y(0) = 3$
  - $\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x\cos x, \quad x \succ 0$
  - $\frac{dy}{dx} + \frac{4}{x}y = x^4$

4.5.2 Non-linear differential equations of the first order and first degree

Exact Differential Equations

- The equation of the form $M(x,y)dy + N(x,y)dx = 0$ is an exact equation if there is a function $F(x,y)$ such that $\frac{\partial F}{\partial y}(x,y) = M(x,y)$ and $\frac{\partial F}{\partial x}(x,y) = N(x,y)$ for all x, y, that is the total differential of $F(x,y)$ satisfies $dF(x,y) = M(x,y)dy + N(x,y)dx = 0$

  The solution to $M(x,y)dy + N(x,y)dx = 0$ is given implicitly by

  $F(x,y) = \int Mdy + \int Ndx - \int \left(\frac{\partial}{\partial x}\int Mdy\right)dx = c$

- Examples:

  - $2yxdy + y^2dx = 0$
  - $xdy + (y + 3x^2)dx = 0$

- **Integrating factors:** If the equation $M(x,y)dy + N(x,y)dx = 0$ is not exact, but the equation $\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$, then $\mu(x,y)$ is called an integrating factor of the equation.

- Example: consider the first order linear equation $\frac{dy}{dx} + P(x)y = Q(x) \Longrightarrow dy + [P(x)y - Q(x)]dx = 0 \Longrightarrow e^{\int P(x)dx}dy + e^{\int P(x)dx}[P(x)y - Q(x)]dx = 0$ is an exact equation and $\mu(x) = e^{\int P(x)dx}$ is the integrating factor.

- **Method for finding integrating factors**

  If $M(x,y)dy + N(x,y)dx = 0$ is neither separable nor linear, compute $\frac{\partial M}{\partial x}, \frac{\partial N}{\partial y}$. If $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$, the equation is exact. If the equation is not exact, consider $\frac{\frac{\partial N}{\partial y} - \frac{\partial M}{\partial x}}{M}$. If this is a function of x, then an integrating factor is given by $\mu(x) = \exp\int \frac{\frac{\partial N}{\partial y} - \frac{\partial M}{\partial x}}{M}dx$. If not consider $\frac{\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}}{N}$ .If this is a function of y, then an integrating factor is given by $\mu(y) = \exp\int \frac{\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}}{N}dx$.

- Example:$(2x^2y + y)dx + (x^2y - x)dy = 0$

Separable Variables Equations

- These equations take the form:

$f(x)dx = g(y)dy$,

in which x alone occurs on one side of the equation and y alone on the other side.

When f and g are continuous, a solution containing an arbitrary constant is readily obtained by integration.

- – Example:

  $dy + e^x y dx = e^x y^2 dx \Longrightarrow$

  $\frac{dy}{y^2-y} = e^x dx, \qquad y \neq 0,\, y \neq 1 \Longrightarrow$

  $\int \frac{dy}{y^2-y} = \int e^x dx \Longrightarrow$

  $\ln\left|\frac{y-1}{y}\right| = e^x + c....$

  – More Examples:
  $\frac{dy}{dx} = \frac{x-5}{y^2}$
  $\frac{dy}{dx} = \frac{y-1}{x+3}, y(-1) = 0$

- Equations reducible to separable form by change of variable.

  The equation $\frac{dy}{dx} = F(\frac{y}{x})$ suggests the substitution $u = \frac{y}{x} \Longrightarrow y = xu \Longrightarrow \frac{dy}{dx} = x\frac{du}{dx} + u = F(u) \Longrightarrow \frac{du}{F(u)-u} = \frac{dx}{x}$,which is separated.

  – Example: $y^2 + x^2 y' = xyy' \Longrightarrow \frac{dy}{dx} = \frac{(y/x)^2}{y/x-1} \Longrightarrow F(u) = \frac{u^2}{u-1} \Longrightarrow \frac{u-1}{u}du = \frac{dx}{x} \Longrightarrow \int \frac{u-1}{u}du = \int \frac{dx}{x} \Longrightarrow u - \log|u| = \log|x| + c \Longrightarrow \frac{y}{x} - \log|y| = c$
  $\Longrightarrow y = x \log cy$.

  The equation $\frac{dy}{dx} = F(a + bx + y)$ suggests the substitution $u = a + bx + y \Longrightarrow \frac{du}{dx} = \frac{dy}{dx} + b = F(u) + b \Longrightarrow \frac{du}{F(u)+b} = dx$,which is separated.

  – Example:
  $dy - dx = xdx + ydx \Longrightarrow \frac{dy}{dx} = x + y + 1 \Longrightarrow \frac{du}{u+1} = dx$ where $u = x + y + 1 \Longrightarrow u + 1 = ce^x \Longrightarrow y = ce^x - x - 2$.

Equations with linear coefficients

- Equations of the form:

  $(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$

Bernoulli Equations

- Equations of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$.

  For $n \neq 0, 1$, the substitution $u = y^{1-n}$ transforms the Bernoulli equation into linear as follows: $\frac{dy}{dx} + P(x)y = Q(x)y^n \Longrightarrow y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$. (Taking $u = y^{1-n} \Longrightarrow \frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx}) \Longrightarrow \frac{1}{1-n}\frac{du}{dx} + P(x)u = Q(x)$

- Example:$\frac{dy}{dx} - 5y = -5/2xy^3$

4.5.3 Second-order linear differential equations

Second-order linear differential equations with constant coefficients and constant term. $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = b$, where a_1, a_2, b are all constants.

- If the term $b = 0$, then homogeous.
- If the term $b \neq 0$, then non-homogenous.
- General solution of the complete equation: $y(x) = y_c(x) + y_p(x)$ (complementary function + particular integral).

Homogenous case:

- Characteristic equation: $r^2 + a_1 r + a_2 = 0$ (or complete equation or auxilliary equation)
- Solve the characteristic equation and find two roots: r_1,r_2.
  - Distinct real roots: $y_c = A_1 e^{r_1 x} + A_2 e^{r_2 x}$.
  - Repeated real roots: $y_c = A_1 e^{rx} + A_2 x e^{rx}$.
  - Complex roots: $r_1 = a + ib, r_2 = a - ib \Longrightarrow y_c = A_1 e^{r_1 x} + A_2 e^{r_2 x} \Longrightarrow$
    $$y_c = B_1 e^{ax}\cos bx + B_2 e^{ax}\sin bx$$
    (Euler formulas: $e^{ibx} = \cos bx + i\sin bx$ and $e^{-ibx} = \cos bx - i\sin bx$)
- Examples:
  - $y' - y' - 2y = 0$
  - $y' + 4y' + 5y = 0$
  - $y' - 3y' + 4y = 0$
  - $y' + 4y' + 4y = 0$

Non-homogenous case:($b \neq 0$)

- The complementary solution is obtained by the homogenous equation.
- The particular integral is obtained as follows:
  - $y_p = \frac{b}{a_2}(a_2 \neq 0)$
  - $y_p = \frac{b}{a_1}x$ $(a_2 = 0, a_1 \neq 0)$: moving rather than stationary equilibrium
  - $y_p = \frac{b}{a_2}x^2$ $(a_2 = 0, a_1 = 0)$: moving rather than stationary equilibrium
- Examples:
  - $y' + y' - 2y = -10$
  - $y' + y' = -10$
  - $y' = -10$

4.5.4 n-order linear differential equations with constant coefficients and constant term.

- $\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + ... + a_{n-1}\frac{dy}{dx} + a_n y = b$, where $a_1, a_2, ..., a_n, b$ are all constants.

Homogenous case:

- Characteristic equation: $r^n + a_1 r^{n-1} + ... + a_n = 0$
- Solve the characteristic equation and find n roots: $r_1, r_2, ..., r_n$
  - Distinct real and complex roots: $y_c = A_1 e^{r_1 x} + A_2 e^{r_2 x} + ... + A_n e^{r_n x}$.
  - Repeated real and complex roots: $y_c = A_1 e^{rx} + A_2 x e^{rx} + ...$(The form depends on the multiplicity of each root....)
- Examples:
  - $y^{(4)} - 9y' - 20y = 0$
  - $y' - 6y' + 11y' - 6y = -10$
  - $y^{(5)} - y^{(4)} - 2y' + 2y' + y' - y = 0$

Non-homogenous case:$(b \neq 0)$

- The complementary solution is obtained by the homogenous equation.
- The particular integral is obtained as follows:
  - $y_p = \frac{b}{a_n}(a_n \neq 0)$
  - $y_p = \frac{b}{a_{n-1}}x$ $(a_n = 0, a_{n-1} \neq 0)$: moving rather than stationary equilibrium
  - $y_p = \frac{b}{a_{n-2}}x^2$ $(a_{n-2} \neq 0, a_n = 0, a_{n-1} = 0)$: moving rather than stationary equilibrium
  - ....
- Examples:
  - $y^{(5)} - y^{(4)} - 2y' + 2y' + y' - y = 24$

n-order linear differential equations with constant coefficients and variable term. $\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + ... + a_{n-1}\frac{dy}{dx} + a_n y = g(x)$, where $a_1, a_2, ..., a_n$ are all constants.

- The complementary solution is obtained by the homogenous equation.
- The particular integral is found by the following methods.

Method of undetermined coefficients

- $f(x) = P_n(x)$(polynomial of order n),then $y_p = A_n x^n + A_{n-1}x^{n-1} + ... + A_1 x + A_0$
- $f(x) = e^{ax}P_n(x)$,then $y_p = e^{ax}(A_n x^n + A_{n-1}x^{n-1} + ... + A_1 x + A_0)$
- $f(x) = e^{ax}\sin bx P_n(x)$,then $y_p = e^{ax}\sin bx(A_n x^n + A_{n-1}x^{n-1} + ... + A_1 x + A_0) + e^{ax}\cos bx(B_n x^n + B_{n-1}x^{n-1} + ... + B_1 x + B_0)$
- $f(x) = e^{ax}\cos bx P_n(x)$,then $y_p = e^{ax}\sin bx(A_n x^n + A_{n-1}x^{n-1} + ... + A_1 x + A_0) + e^{ax}\cos bx(B_n x^n + B_{n-1}x^{n-1} + ... + B_1 x + B_0)$

- Examples:
  - $y' - y' - 2y = 4x^2$
  - $y' - y' - 2y = e^{3x}$
  - $y' - 6y' + 11y' - 6y = 2xe^{-x}$
  - $y' = 9x^2 + 2x - 1$

Method of variation of parameters (Lagrange) $y_p(x) = u_1(x)y_1(x) + u_2 y_2(x)$, where $y_1(x), y_2(x)$ are the solutions of the homogenous equation.

- Determine $u_1(x), u_2(x)$ by solving the system for $u_1'(x), u_2'(x)$:

  $y_1 u_1' + y_2 u_2' = 0$

  $y_1' u_1' + y_2' u_2' = g$
- $u_1(x) = \int \frac{-g(x)y_2(x)}{W[y_1,y_2](x)}dx$ and $u_2(x) = \int \frac{g(x)y_1(x)}{W[y_1,y_2](x)}dx$, where $W[y_1,y_2](x)$ is the Wronskian of the differential equation ($\neq 0$: linear independence of the solutions)
- Examples:
  - $y' + 4y' + 4y = e^{-2x}\ln x$
  - $y' - 6y' + 9y = x^{-3}e^{3x}$

4.5.5 Systems of linear ordinary differential equations with constant coefficients

- A system of n linear differential equations is in normal form if it is expressed as $x'(t) = A(t)x(t) + f(t)$,where $x(t) = col(x_1(t).....x_n(t)), f(t) = col(f_1(t).....f_n(t))$, $A(t) = [a_{ij}(t)]$ is an nxn matrix.
- If $f(t) = 0$, the system is called homogenous; otherwise it is called non-homogenous.
- When the elements of A are all constants the system is said to have constant coefficients.
- The initial value problem for a system is the problem of finding a differentiable vector function $x(t)$ that satisfies the system on an interval and satisfies the initial condition $x(t_0) = x_0$.

Homogenous linear system with constant coefficients

- Eigenvalues and eigenvectors: Let $A(t) = [a_{ij}(t)]$ be an nxn constant matrix. The eigenvalues of A are those real or complex numbers for which $(A - rI)\,u = 0$ has at least one nontrivial solution u. The corresponding nontrivial solutions are called the eigenvectors of A associated with r.
- If the nxn constant matrix A has n distinct eigenvalues $r_1, r_2, ...r_n$ and u_i is an eigenvector associated with r_i,then $\{e^{r_1 t}u_1, ..., e^{r_n t}u_n\}$ is a fundamental solution set for the homogenous system $x' = Ax$.
- If the real matrix A has complex conjugate eigenvalues $\alpha \pm i\beta$ with corresponding eigenvectors $a \pm ib$,then two linearly independent real vector solutions to $x'(t) = Ax(t)$ are $e^{\alpha t}\cos\beta ta - e^{\alpha t}\cos\beta tb$ and $e^{\alpha t}\cos\beta ta + e^{\alpha t}\cos\beta tb$.
- Examples:
  - $x'(t) = Ax(t)$, where $A = \begin{matrix} +2 & -3 \\ +1 & -2 \end{matrix}$
  - $x'(t) = Ax(t)$, where $A = \begin{matrix} +1 & -2 & +2 \\ -2 & +1 & +2 \\ +2 & +2 & +1 \end{matrix}$

Nonhomogenous linear system with constant coefficients

Method of undetermined coefficients

- If $f(t) = tg \Longrightarrow x_p(t) = ta + b$,where the constant vectors a and b are to be determined.

- If $f(t) = col(1, t, \sin t) \Longrightarrow x_p(t) = ta + b + (\sin t)\, c + (\cos t)\, d$,where the constant vectors a and b are to be determined.

- If $f(t) = col(t, e^t, t^2) \Longrightarrow x_p(t) = t^2 a + tb + c + e^t d$,where the constant vectors a, b, c and d are to be determined.

- Example: $x^{\prime}(t) = \begin{matrix} +1 & -1 \\ -1 & +1 \end{matrix}\, x(t) + \begin{matrix} -3 \\ +1 \end{matrix}$

Method of variation of parameters

- If $x^{\prime}(t) = A(t)x(t) + f(t) \Longrightarrow x_p(t) = x(t)u(t) = x(t)\int x^{-1}(t)f(t)dt$ and given the initial value problem:$x(t_0) = x_0$, then $x(t) = x_c(t)c + x(t)\int_{t_0}^{t} x^{-1}(s)f(s)ds$,where $c = x(t)x^{-1}(t_0)x_0$.

- Example:$x^{\prime}(t) = \begin{matrix} +2 & -3 \\ +1 & -2 \end{matrix}\, x(t) + \begin{matrix} e^{2t} \\ +1 \end{matrix}, x_0 = \begin{matrix} -1 \\ 0 \end{matrix}$.

The Matrix exponential function

- $e^{At} = I + At + A^2\frac{t^2}{2!} + ... + A^n\frac{t^n}{n!} + ...$
- $x^{\prime}(t) = Ax(t) \Longrightarrow x(t) = e^{At}K$
- $x^{\prime}(t) = Ax(t) + f(t) \Longrightarrow x(t) = e^{At}K + e^{At}\int e^{-At}f(t)dt$
- $x^{\prime}(t) = Ax(t) + f(t), x(t_0) = c \Longrightarrow x(t) = e^{A(t-t_0)}c + e^{At}\int_{t_0}^{t} e^{-As}f(s)ds$
- A special case: when the characteristic polynomial for A has the form $p(r) = (r_1 - r)^n$ that is when A has an eigenvalue r_1 of multiplicity n , $(r_1 I - A)^n = 0$ (hence $A - r_1 I$ is nilpotent)

  and $e^{At} = e^{r_1 t}\left\{I + (A - r_1 I)t + ... + (A - r_1 I)^{n-1}\frac{t^{n-1}}{(n-1)!}\right\}$
- Example

  - $x^{\prime} = Ax, A = \begin{matrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ -2 & -2 & -1 \end{matrix}$

4.5.6 Phase Plane Analysis - Stability of autonomous systems (linear systems in the plane)

- An autonomous system in the plane has the form:

  $\frac{dx}{dt} = f(x, y)$

  $\frac{dy}{dt} = g(x, y)$

- Phase plane equation: $\frac{dx}{dy} = \frac{f(x,y)}{g(x,y)}$

- A solution to the system is a pair of functions of t : $(x(t), y(t))$ that satisfies the equations for all t in some interval I. If we plot the points $(x(t), y(t))$ in the $xy-$plane as t varies, the resulting curve is known as the **trajectory** of the solution pair $(x(t), y(t))$ and the $xy-$plane is called the **phase plane**.

- A point (x_0,y_0) where $f(x_0,y_0) = 0$ and $g(x_0,y_0) = 0$ is called a **critical point** or equilibrium point of the system, and the corresponding constant solution $x(t) = x_0, y(t) = y_0$ is called an **equilibrium solution**. The set of all critical points is called the critical point set.

- A linear autonomous system in the plane has the form:

  $x'(t) = a_{11}x + a_{12}y + b_1$

  $y'(t) = a_{21}x + a_{22}y + b_2$,where a_{ij},b_{ij} are constants. We can always transform a given linear system to the one of the form:

  $x'(t) = ax + by$

  $y'(t) = cx+dy$, where the origin $(0, 0)$ is now tha critical point. We analyse this system under the assumption that $ad - bc \neq 0$, which makes $(0, 0)$ an isolated critical point.

- Characteristic equation: $r^2 - (a + d)r + (ad - bc) = 0$

- The asymptotic (long-term) behavior of the solutions is linked to the nature of the roots r_1,r_2 of the characteristic equation.

  - r_1,r_2 real , distinct and positive: $x(t) = A_1e^{r_1t} + A_2e^{r_2t}, y(t) = B_1e^{r_1t} + B_2e^{r_2t}$. The origin is an unstable improper node (unstable because the trajectories move away from the origin and improper because almost all the trajectories have the same tangent line at the origin). Example: $\frac{dx}{dt} = x$, $\frac{dy}{dt} = 3y$
  - r_1,r_2 real , distinct and negative: $x(t) = A_1e^{r_1t} + A_2e^{r_2t}, y(t) = B_1e^{r_1t}+B_2e^{r_2t}$. The origin is an asymptotically stable improper node (stable because the trajectories approach the origin and improper because almost all the trajectories have the same tangent line at the origin).Example: $\frac{dx}{dt} = -2x$, $\frac{dy}{dt} = -y$

- r_1,r_2 real opposite signs: $x(t) = A_1e^{r_1t} + A_2e^{r_2t}, y(t) = B_1e^{r_1t} + B_2e^{r_2t}$. The origin is an unstable saddle point (unstable because there are trajectories that pass arbitrarily near the origin but then eventually move away). Example: $\frac{dx}{dt} = 5x - 4y, \frac{dy}{dt} = 4x - 3y$
- $r_1 = r_2$ equal roots:$\frac{dx}{dt} = rx, \frac{dy}{dt} = ry \Longrightarrow x(t) = Ae^{rt}, y(t) = Be^{rt}$. The trajectories lie on the integral curves $y = (B/A)x$.
  * When $r > 0$, these trajectories move away from the origin, so the origin is unstable.
  * When $r < 0$, these trajectories approach the origin, so the origin is stable.
  * In either case, the trajectories lie on lines passing through the origin. Because every direction through the origin defines a trajectory, the origin is called a proper node.
- Complex roots $r = a \pm ib$ $(a \neq 0, b \neq 0)$ $x(t) = e^{at}[A_1 \cos bt + B_1 \sin bt], y(t) = e^{at}[A_2 \cos bt + B_2 \sin bt]$.
  * When $a > 0$ the trajectories travel away from the origin and the origin is an unstable one. The solution spiral away from the origin.
  * When $a < 0$ the trajectories approach the origin and the origin is a stable one. The solution spirals in toward the origin.
  * Example: $\frac{dx}{dt} = x - 4y, \frac{dy}{dt} = 4x + y$
- Pure imaginary roots $r = \pm ib$ $x(t) = A_1 \cos bt + B_1 \sin bt, y(t) = A_2 \cos bt + B_2 \sin bt$.

  The trajectories are concentric circles about the origin, the origin is called a center and is a stable one. Example: $\frac{dx}{dt} = -by, \frac{dy}{dt} = bx$.

5 Appendix A

5.1 Trigonometric functions

5.1.1 D*efinitions*

- *Conversion of rads to* ^{o} *and vice versa*: $1^o = \frac{\pi}{180} rad$ & $1\ rad = \left(\frac{180}{\pi}\right)^o$
- *Arc length:* $\theta = \frac{s}{r} \Rightarrow s = \theta * r$ (θ measured in rads)
- *Area of a sector:* $A = \frac{1}{2} r^2 * \theta$ (θ measured in rads)
- *Trigonometric functions (defined for a positive acute angle* θ of a right triangle)

$$
\begin{aligned}
\sin\theta &= \frac{\text{side opposite } \theta}{\text{hypotenuse}} &= \frac{y}{r} \\
\cos\theta &= \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} &= \frac{x}{r} \\
\tan\theta &= \frac{\text{side opposite } \theta}{\text{side adjacent to } \theta} &= \frac{y}{x} \\
\cot\theta &= \frac{\text{side adjacent to } \theta}{\text{side opposite } \theta} &= \frac{x}{y} \\
\sec\theta &= \frac{\text{hypotenuse}}{\text{side adjacent to } \theta} &= \frac{r}{x} \\
\csc\theta &= \frac{\text{hypotenuse}}{\text{side opposite } \theta} &= \frac{r}{y}
\end{aligned}
$$

5.1.2 *Relationships:*

$$
\begin{array}{lll}
\csc\theta = \frac{1}{\sin\theta} & \sec\partial = \frac{1}{\cos\theta} & \cot\theta = \frac{1}{\tan\theta} \\
\tan\theta = \frac{\sin\theta}{\cos\theta} & \cot\theta = \frac{\cos\theta}{\sin\theta} &
\end{array}
$$

5.1.3 *Trigonometric identities*

$$
\begin{array}{cc}
\begin{gathered}
\sin^2\theta + \cos^2\theta = 1 \\
\tan^2\theta + 1 = \sec^2\theta \\
1 + \cot^2\theta = \csc^2\theta
\end{gathered}
&
\begin{gathered}
\sin(\pi - \theta) = \sin\theta \\
\cos(\pi - \theta) = -\cos\theta \\
\tan(\pi - \theta) = -\tan\theta \\
\cot(\pi - \theta) = -\cot\theta
\end{gathered}
\end{array}
$$

$$
\begin{array}{cc}
\begin{gathered}
\sin(\pi + \theta) = -\sin\theta \\
\cos(\pi + \theta) = -\cos\theta \\
\tan(\pi + \theta) = \tan\theta \\
\cot(\pi + \theta) = \cot\theta
\end{gathered}
&
\begin{gathered}
\sin(-\theta) = -\sin\theta \\
\cos(-\theta) = \cos\theta \\
\tan(-\theta) = -\tan\theta \\
\cot(-\theta) = -\cot\theta
\end{gathered}
\end{array}
$$

$$
\begin{gathered}
\sin(\tfrac{\pi}{2} - \theta) = \cos\theta \\
\cos(\tfrac{\pi}{2} - \theta) = \sin\theta \\
\tan(\tfrac{\pi}{2} - \theta) = \cot\theta
\end{gathered}
$$

$$
\begin{gathered}
\sin\theta = \sin(\theta \pm 2n\pi), n = 0, 1, 2, \cdots \\
\cos\theta = \cos(\theta \pm 2n\pi), n = 0, 1, 2, \cdots \\
\tan\theta = \tan(\theta \pm n\pi), n = 0, 1, 2, \cdots
\end{gathered}
$$

5.1.4 *Law of cosines*

If the sides of a triangle have lengths a, b, and c and if θ is the angle between the sides with lengths a and b, then
$c^2 = a^2 + b^2 - 2ab\cos\theta$

5.1.5 *Formulas*

- *Addition formulas:*

$$\begin{aligned}
\sin(a+b) &= \sin a\cos b + \cos a\sin b \\
\cos(a+b) &= \cos a\cos b - \sin a\sin b \\
\sin(a-b) &= \sin a\cos b - \cos a\sin b \\
\cos(a-b) &= \cos a\cos b + \sin a\sin b \\
\tan(a+b) &= \tfrac{\tan a+\tan b}{1-\tan a\tan b} \\
\tan(a-b) &= \tfrac{\tan a-\tan b}{1+\tan a\tan b}
\end{aligned}$$

- *Double-angle formulas:*

$$\begin{aligned}
\sin 2a &= 2\sin a\cos a \\
\cos 2a &= \cos^2 a - \sin^2 a \\
&= 2\cos^2 a - 1 \\
&= 1 - 2\sin^2 a \\
\tan 2a &= \tfrac{2\tan a}{1-\tan^2 a}
\end{aligned}$$

- *Half-angle formulas:*

$$\begin{aligned}
\cos^2\tfrac{a}{2} &= \tfrac{1+\cos a}{2} \\
\sin^2\tfrac{a}{2} &= \tfrac{1-\cos a}{2}
\end{aligned}$$

- *Product-to-sum formulas:*

$$\begin{aligned}
\sin a\cos b &= \tfrac{1}{2}\left[\sin(a-b) + \sin(a+b)\right] \\
\sin a\sin b &= \tfrac{1}{2}\left[\cos(a-b) + \cos(a+b)\right] \\
\cos a\cos b &= \tfrac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]
\end{aligned}$$

- *Sum-to-product formulas:*

$$\begin{aligned}
\sin a + \sin b &= 2\sin\tfrac{a+b}{2}\cos\tfrac{a-b}{2} \\
\sin a - \sin b &= 2\cos\tfrac{a+b}{2}\sin\tfrac{a-b}{2} \\
\cos a + \cos b &= 2\cos\tfrac{a+b}{2}\cos\tfrac{a-b}{2} \\
\cos a - \cos b &= -2\sin\tfrac{a+b}{2}\sin\tfrac{a-b}{2}
\end{aligned}$$

5.1.6 *Amplitude and period*

- *Periodic function:* $f(x+p) = f(p), p \succ 0$. The smallest value of p is called the fundamental period of f.
- The functions $a\sin bx$ and $a\cos bx$ have fundamental period $2\pi/|b|$ and their graphs oscillate between $-a$ and a. The number $|a|$

is called the amplitude of $a \sin bx$ and $a \cos bx$.The function $\tan bx$ has fundamental period $\pi / |b|$.

5.2 Inverse trigonometric functions

The basic trigonometric functions are periodic, thus they can not have inverses, but if we impose restrictions on their domains, we can have their inverses.

- *Inverse sine:* For each x in the interval $[-1, 1]$, we define $\arcsin x$ to be that number y in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, such that $\sin y = x$.

$$\begin{array}{ll} \arcsin(\sin x) = x & x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \sin(\arcsin x) = x & x \in [-1, 1] \end{array}$$

- *Inverse cosine:* For each x in the interval $[-1, 1]$, we define $\arccos x$ to be that number y in the interval $[0, \pi]$, such that $\cos y = x$.

$$\begin{array}{ll} \arccos(\sin x) = x & x \in [0, \pi] \\ \cos(\arccos x) = x & x \in [-1, 1] \end{array}$$

- *Inverse tangent:* For each x in the interval $(-\infty, +\infty)$, we define $\arctan x$ to be that number y in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, such that $\tan y = x$.

$$\begin{array}{ll} \arctan(\tan x) = x & x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \tan(\arctan x) = x & x \in (-\infty, +\infty) \end{array}$$

- *Inverse cesant:* For each x in the set $(-\infty, -1] + \cup[1, +\infty)$, we define $\operatorname{arcsec} x$ to be that number y in the set $\left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right]$ such that $\sec y = x$.

$$\begin{array}{ll} \operatorname{arcsec}(\sec x) = x & x \in \left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right] \\ \sec(\operatorname{arcsec} x) = x & x \in (-\infty, -1] + \cup[1, +\infty) \end{array}$$

- The inverse cotangent and cosecant are of lesser importance.

5.2.1 Derivatives

$$\begin{array}{ll} \frac{d}{dx}[\arcsin(x)] = & \frac{1}{\sqrt[2]{1-x^2}} \\ \frac{d}{dx}[\arccos(x)] = & -\frac{1}{\sqrt[2]{1-x^2}} \\ \frac{d}{dx}[\arctan(x)] = & \frac{1}{1+x^2} \\ \frac{d}{dx}[\operatorname{arccot}(x)] = & -\frac{1}{1+x^2} \\ \frac{d}{dx}[\operatorname{arcsec}(x)] = & \frac{1}{x\sqrt[2]{x^2-1}} \\ \frac{d}{dx}[\operatorname{arccsc}(x)] = & -\frac{1}{x\sqrt[2]{x^2-1}} \end{array}$$

5.3 Exponentials

- $f(x) = b^x, b \succ 0$
- If $b \succ 1$ then $f(x)$ is an increasing function, while if $0 \prec b \prec 1$ is a decreasing and a constant one if $b = 1$.

5.4 Logarithms

- $f(x) = \log_b x, b \succ 0, b \neq 1, x \succ 0$, represents that power to which b must be raised in order to produce x.
- *Properties of the logarithmic function*

$\log_b \ln 1 = 0$	$\log_b ac = \log_b a + \log_b c$
$\log_b \frac{a}{c} = \log_b a - \log_b c$	$\log_b a^r = r \log_b a$
$\log_b \frac{1}{c} = -\log_b c$	$\log_b b = 1$

- *Common logarithms:* the ones that have base 10.
- *Natural logarithms:*the ones that have base e $(e = \lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x \Longleftrightarrow e = \lim_{x \to 0} (1 + x)^{\frac{1}{x}})$.
- *Properties of the natural logarithm*

  –

$\ln 1 = 0$	$\ln ac = \ln a + \ln c$
$\ln \frac{a}{c} = \ln a - \ln c$	$\ln a^r = r \ln a$
$\ln \frac{1}{c} = -\ln c$	

5.5 Hyperbolic functions

Hyperbolic functions are certain combinations of e^x and e^{-x}.They have many applications in engineering and many properties in common with the trigonometric functions.

5.5.1 Definitions

hyperbolic sine	$\sinh x =$	$\frac{e^x - e^{-x}}{2}$	
hyperbolic cosine	$\cos hx =$	$\frac{e^x + e^{-x}}{2}$	
hyperbolic tangent	$\tanh x =$	$\frac{\sinh x}{\cos hx}$	$\frac{e^x - e^{-x}}{e^x + e^{-x}}$
hyperbolic cotangent	$\coth x =$	$\frac{\cosh x}{\sinh x}$	$\frac{e^x + e^{-x}}{e^x - e^{-x}}$
hyperbolic cesant	$\sec hx =$	$\frac{1}{\cosh x}$	$\frac{2}{e^x + e^{-x}}$
hyperbolic cosecant	$\csc hx =$	$\frac{1}{\sinh x}$	$\frac{2}{e^x - e^{-x}}$

5.5.2 Hyperbolic identities

$$\cosh^2 x - \sinh^2 x = 1$$
$$1 - \tanh^2 x = \sec h^2 x$$
$$\coth^2 x - 1 = \csc h^2 x$$

5.5.3 Why are they called hyperbolic?

For any real number t, the point $(\cosh t, \sinh t)$ lies on the curve $x^2 - y^2 = 1$ (this curve is called *hyperbola*) because $\cosh^2 t - \sinh^2 t = 1$

5.5.4 Derivatives

$$\begin{aligned}
\tfrac{d}{dx}[\sinh x] &= \cosh x \\
\tfrac{d}{dx}[\cosh x] &= \sinh x \\
\tfrac{d}{dx}[\tanh x] &= \sec h^2 x \\
\tfrac{d}{dx}[\coth x] &= -\csc h^2 x \\
\tfrac{d}{dx}[\sec hx] &= -\sec hx \tanh x \\
\tfrac{d}{dx}[\csc hx] &= -\csc hx \coth x
\end{aligned}$$

5.6 Inverse hyperbolic functions

They are particularly useful in integration.

5.6.1 Definitions

$$\begin{array}{lll}
y = \operatorname{arcsin} hx \iff & x = \sinh y & \text{for all } x, y \\
y = \operatorname{arccos} hx \iff & x = \cosh y & x \succeq 1, y \succeq 0 \\
y = \operatorname{arctan} hx \iff & x = \tanh y & -1 \prec x \prec 1, and - \infty \prec x \prec +\infty \\
y = \operatorname{arccot} hx \iff & x = \coth y & \mid x \mid \succ 1, y \neq 0 \\
y = \operatorname{arcsec} hx \iff & x = \sec hy & 0 \prec x \preceq 1, y \succeq 0 \\
y = \operatorname{arccsc} hx \iff & x = \csc hy & x \neq 0, y \neq 0
\end{array}$$

5.6.2 Formulas

$$\begin{array}{lll}
\operatorname{arcsin} hx = & \ln(x + \sqrt[2]{x^2+1}) & -\infty \prec x \prec \infty \\
\operatorname{arccos} hx = & \ln(x + \sqrt[2]{x^2-1}) & x \succeq 1 \\
\operatorname{arctan} hx = & \frac{1}{2} \ln \frac{1+x}{1-x} & -1 \prec x \prec 1 \\
\operatorname{arccot} hx = & \frac{1}{2} \ln \frac{x+1}{x-1} & |x| \succ 1 \\
\operatorname{arcsec} hx = & \ln(\frac{1+\sqrt[2]{1-x^2}}{x}) & 0 \prec x \preceq 1 \\
\operatorname{arccsc} hx = & \ln(\frac{1}{x} + \frac{\sqrt[2]{1+x^2}}{|x|}) & x \neq 0
\end{array}$$

5.6.3 Derivatives

$\frac{d}{dx}\left[\operatorname{arcsin} hx\right] =$	$\frac{1}{\sqrt[2]{x^2+1)}}$	$-\infty \prec x \prec \infty$
$\frac{d}{dx}\left[\operatorname{arccos} hx\right] =$	$\frac{1}{\sqrt[2]{x^2-1)}}$	$x \succ 1$
$\frac{d}{dx}\left[\operatorname{arctan} hx\right] =$	$\frac{1}{1-x^2}$	$\lvert x\rvert \prec 1$
$\frac{d}{dx}\left[\operatorname{arccot} hx\right] =$	$\frac{1}{1-x^2}$	$\lvert x\rvert \succ 1$
$\frac{d}{dx}\left[\operatorname{arcsec} hx\right] =$	$-\frac{1}{x\sqrt[2]{1-x^2}}$	$0 \prec x \prec 1$
$\frac{d}{dx}\left[\operatorname{arccsc} hx\right] =$	$-\frac{1}{\lvert x\rvert\sqrt[2]{1-x^2}}$	$x \neq 0$