# Complex Network Theory

**Lecture 2-1**

**Basic network concepts and metrics**

Instructor: S. Mehdi Vahidipour

(Vahidipour@kashanu.ac.ir)

Spring 2018

Thanks A. Rezvanian

A. Barabasi, L. Adamic and J. Leskovec

# Outline

- Overview of class topics
  - Basic definitions
  - Basic concepts
  - Representation
  - Metrics
  - Measures
  - Centralities
- Next class:
  - Network centralities and metrics

# Understanding large graphs

- What are the statistics of real life networks?
- In which terms we can describe the networks?
- How we can measure a large network?
- Can we explain how the networks were generated?
- Can we make models for network construction?
- To how much extent do the artificially
- constructed networks describe real networks?

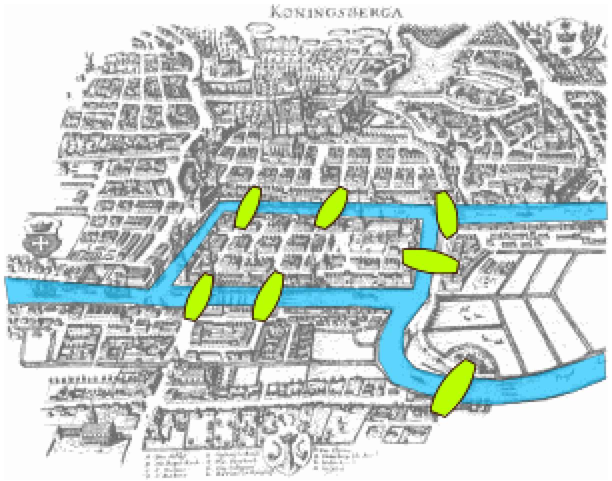
- **First step: Introducing network metrics**

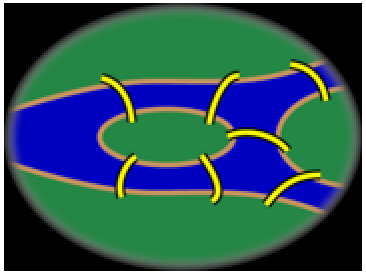
# Networks became hot topic !

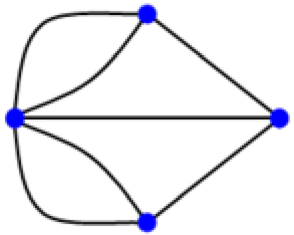
- Around 1999
  - Watts and Strogatz, Collective dynamics of small-world networks
  - Faloutsos$^3$, On power-law relationships of the Internet Topology
  - Kleinberg et al., The Web as a graph
  - Barabasi and Albert, The emergence of scaling in real networks

# History: Graph theory

- Euler's **Seven Bridges of Königsberg** – one of the first problems in graph theory
- Is there a route that crosses each bridge only once and returns to the starting point?



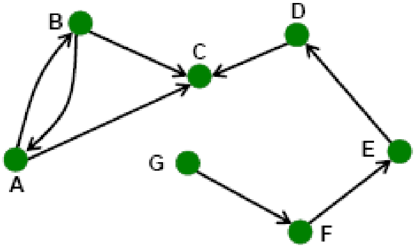




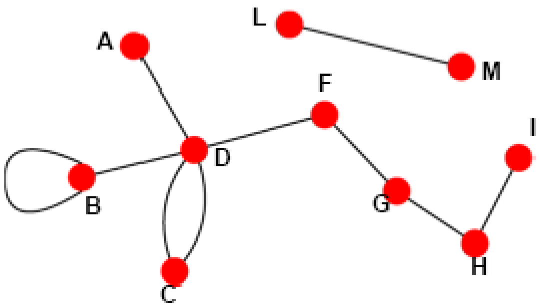
**Source: http://en.wikipedia.org/wiki/Seven_Bridges_of_Königsberg**
**Image 1 – GNU v1.2: Bogdan, Wikipedia; http://commons.wikimedia.org/wiki/Commons:GNU_Free_Documentation_License**
**Image 2 – GNU v1.2: Booyabazooka, Wikipedia; http://commons.wikimedia.org/wiki/Commons:GNU_Free_Documentation_License**
**Image 3 – GNU v1.2: Riojajar, Wikipedia; http://commons.wikimedia.org/wiki/Commons:GNU_Free_Documentation_License**

# Network elements: edges

- Directed (also called arcs)-asymmetrical relations
  - A → B
    - A likes B, A gave a gift to B, A is B's child,
    - A call B, A follows B



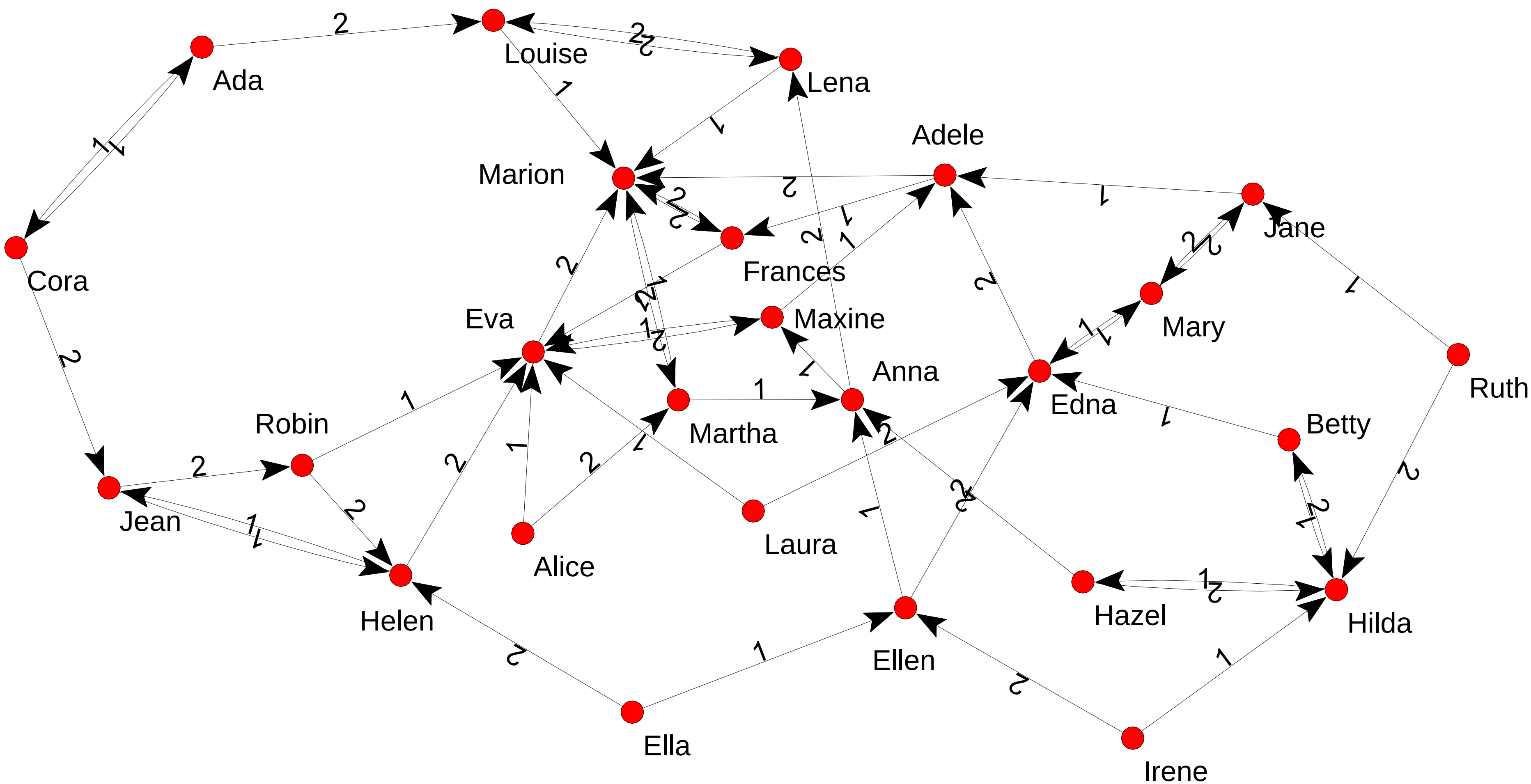

- Undirected (symmetrical, reciprocal relations)
  - A ↔ B or A – B
    - A and B like each other
    - A and B are siblings
    - A and B are co-authors
    - A and B are friend



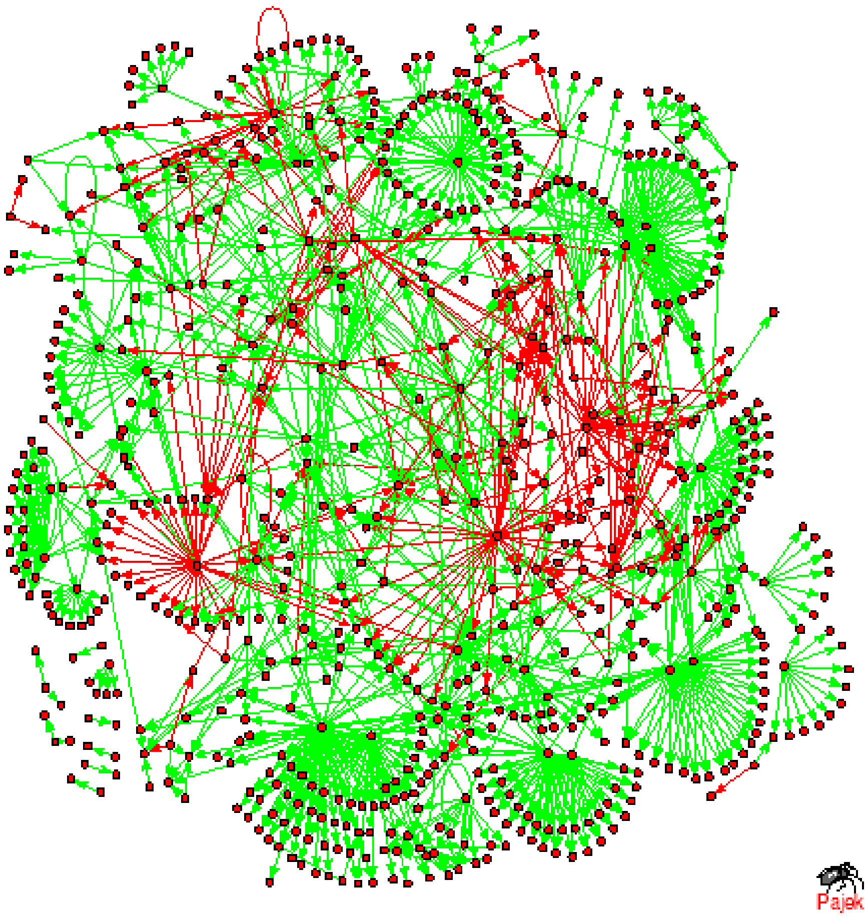

- Edge attributes
  - weight (e.g. frequency of communication)
  - ranking (best friend, second best friend…)
  - type (friend, relative, co-worker)
  - properties depending on the structure of the rest of the graph: e.g. betweenness

# Directed networks

- girls' school dormitory dining-table partners (Moreno, *The sociometry reader*, 1960)
- first and second choices shown



# Edge weights can have positive or negative values

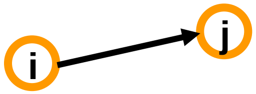


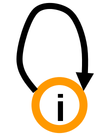

Transcription regulatory network in baker's yeast

- One gene activates/inhibits another
- One person trusting/distrusting another
  - Research challenge: How does one 'propagate' negative feelings in a social network? Is my enemy's enemy my friend?

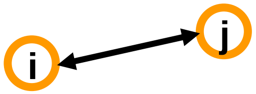
# Adjacency matrices

- Representing edges (who is adjacent to whom) as a matrix
  - $A_{ij}$ = 1 if node i has an edge to node j
    = 0 if node i does not have an edge to j



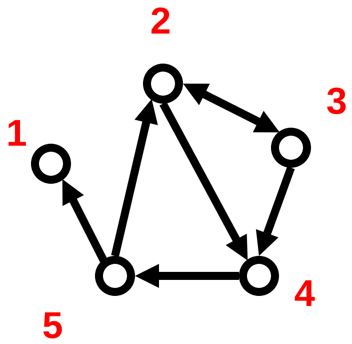




  - $A_{ii}$ = 0 unless the network has self-loops




  - $A_{ij} = A_{ji}$ if the network is undirected,
    or if i and j share a reciprocated edge

Example:



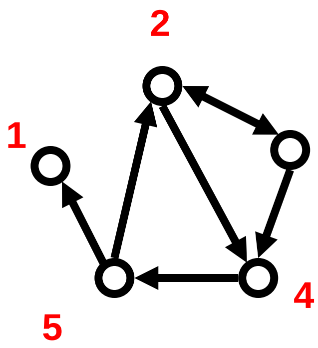

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

# Adjacency lists

- Edge list
  - 2 3
  - 2 4
  - 3 2
  - 3 4
  - 4 5
  - 5 2
  - 5 1

2,3
2,4
3,2
3,4
4,5
5,2
5,1

2;3
2;4
3;2
3;4
4;5
5;2
5;1

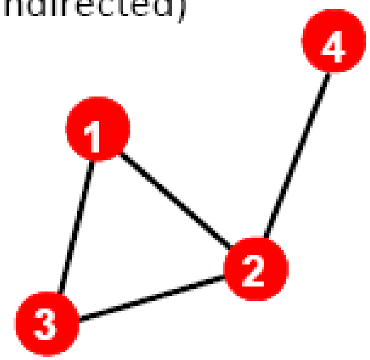



- Adjacency list
  - is easier to work with if network is
    - large
    - sparse
  - quickly retrieve all neighbors for a node
    - 1:
    - 2: 3 4
    - 3: 2 4
    - 4: 5
    - 5: 1 2

# More types of graphs

- **Unweighted**

  (undirected)




$$A_{ij} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

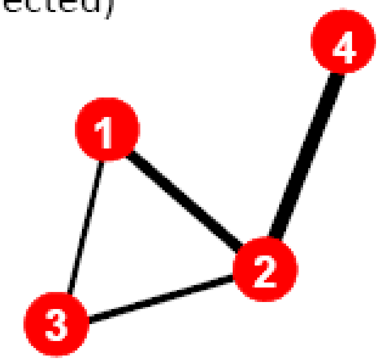
$$A_{ii} = 0 \qquad A_{ij} = A_{ji}$$

$$E = \frac{1}{2}\sum_{i,j=1}^{N} A_{ij} \qquad \overline{k} = \frac{2E}{N}$$

**Examples:** Friendship, Hyperlink

- **Weighted**

  (undirected)




$$A_{ij} = \begin{pmatrix} 0 & 2 & 0.5 & 0 \\ 2 & 0 & 1 & 4 \\ 0.5 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix}$$

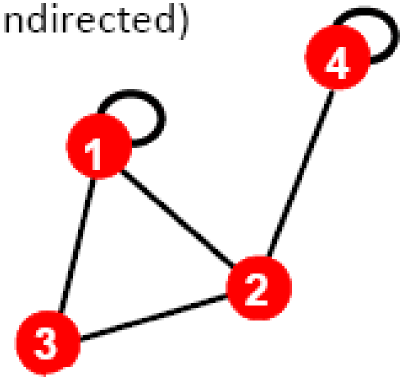
$$A_{ii} = 0 \qquad A_{ij} = A_{ji}$$

$$E = \frac{1}{2}\sum_{i,j=1}^{N} nonzero(A_{ij}) \qquad \overline{k} = \frac{2E}{N}$$

**Examples:** Collaboration, Internet, Roads

# More types of graphs

- **Self-edges (self-loops)**
(undirected)



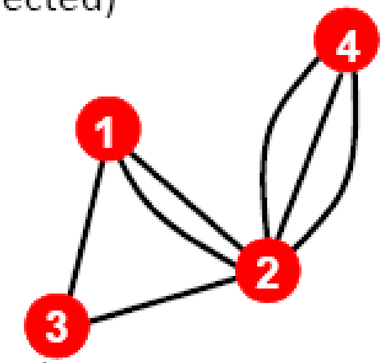

$$A_{ij} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$A_{ii} \neq 0 \qquad A_{ij} = A_{ji}$$

$$E = \frac{1}{2} \sum_{i,j=1, i \neq j}^{N} A_{ij} + \sum_{i=1}^{N} A_{ii} \qquad ?$$

**Examples:** Proteins, Hyperlink

- **Multigraph**
(undirected)




$$A_{ij} = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 3 \\ 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix}$$

$$A_{ii} = 0 \qquad A_{ij} = A_{ji}$$

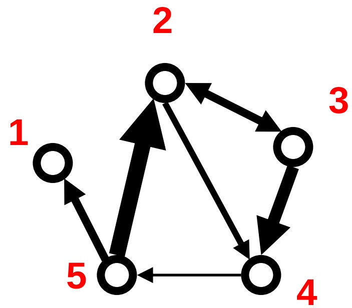
$$E = \frac{1}{2} \sum_{i,j=1}^{N} nonzero(A_{ij}) \qquad \bar{k} = \frac{2E}{N}$$

**Examples:** Communication, Collaboration

# Weighted Graph

- For weighted directed network the in-strength and out-strength are defined
- The strength distribution of the graph is also correspondingly defined

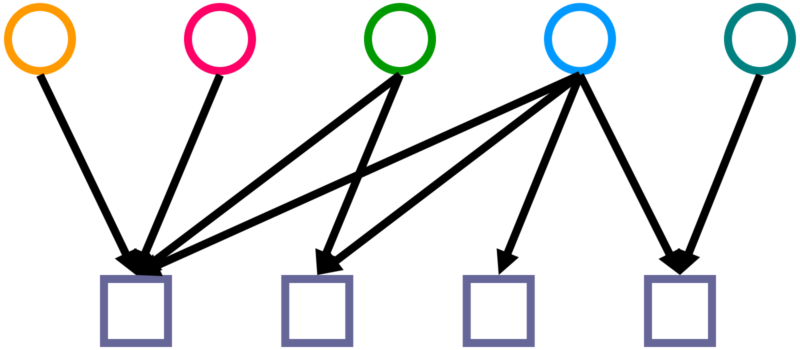
| | |
|---|---|
| 2 3  5 | 2;3; 5 |
| 2 4  5 | 2;4; 5 |
| 3 2  5 | 3;2; 5 |
| 3 4  7 | 3;4; 7 |
| 4 5  3 | 4;5; 3 |
| 5 2  9 | 5;2; 9 |
| 5 1  5 | 5;1; 5 |



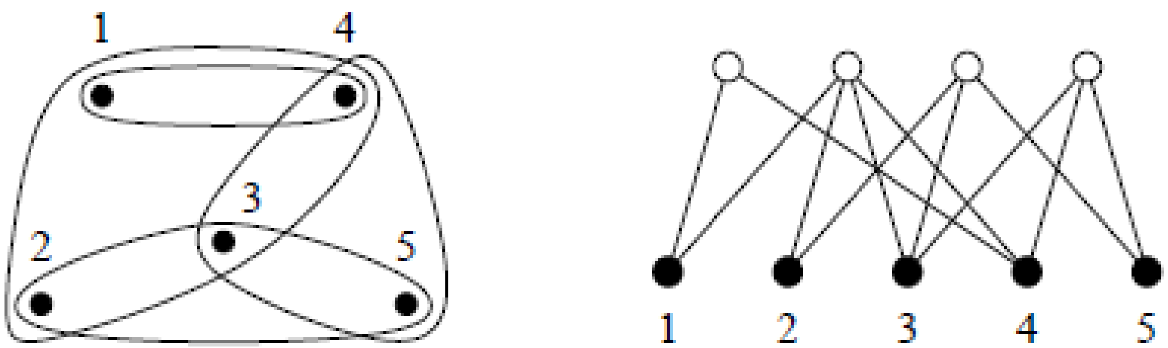
Weighted network

# bipartite (two-mode) networks

- edges occur only between two groups of nodes, not within those groups
- for example, we may have individuals and *events*
  - directors and boards of directors
  - customers and the items they purchase
  - metabolites and the reactions they participate in



# A hypergraph and corresponding bipartite graph




(a) (b)

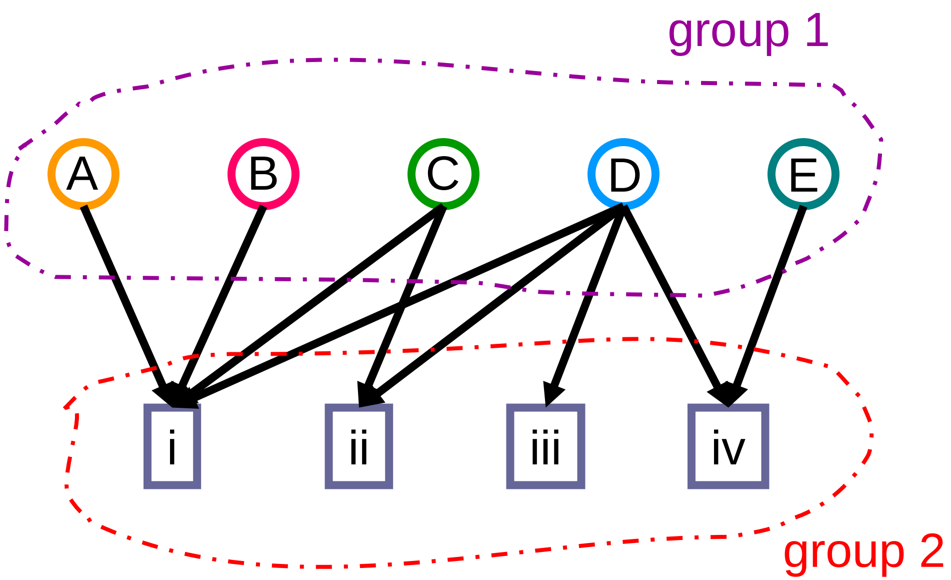
(a) And (b) show the same information

The membership of five vertices in four different groups.

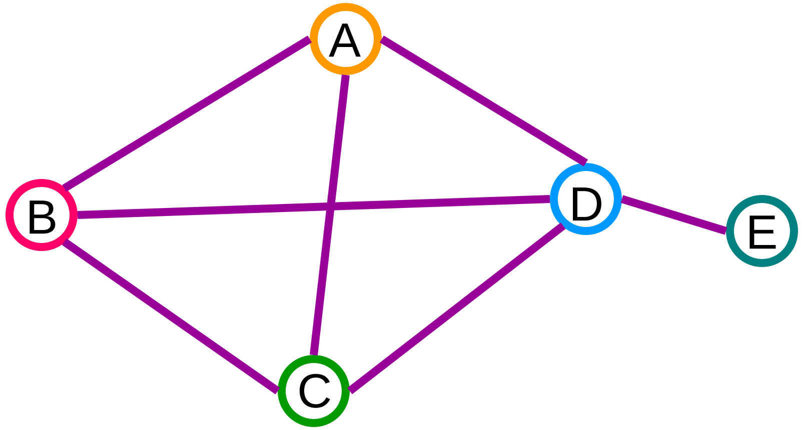
- (a) Hypergraph representation: groups are represented as hyper-edges (loops circling sets of vertices).
- (b) Bipartite representation

# going from a bipartite to a one-mode graph

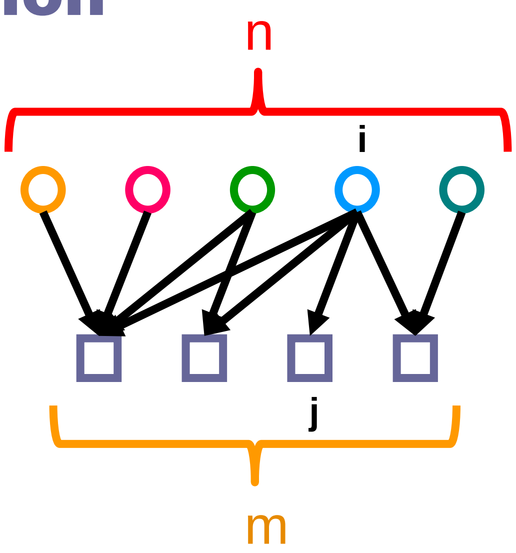
- Two-mode network




- One mode projection
  - two nodes from the first group are connected if they link to the same node in the second group
  - some loss of information
  - naturally high occurrence of cliques



# Now in matrix notation

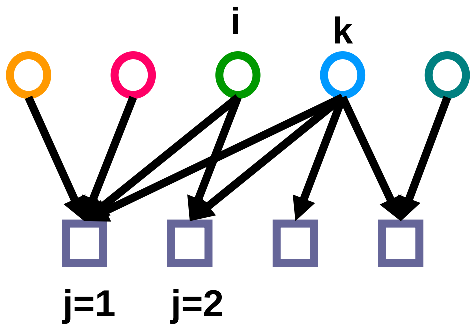



- $B_{ij}$
  - = 1 if node i from the first group
    links to node j from the second group
  - = 0 otherwise
- B is usually not a square matrix!
  - for example: we have n customers and m products

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

# Collapsing to a one-mode network

- *i* and *k* are linked if they both link to *j*
- $A_{ik} = \sum_j B_{ij} B_{kj}$ ➔ $A = B.B^T$
  - $B^T$ swaps $B_{xy}$ and $B_{yx}$
  - if B is an $n \times m$, $B^T$ is an $m \times n$
- $A_{ii}$ is equal to the number of groups to which vertex *i* belongs
- $A' = B^T B$ ?




$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

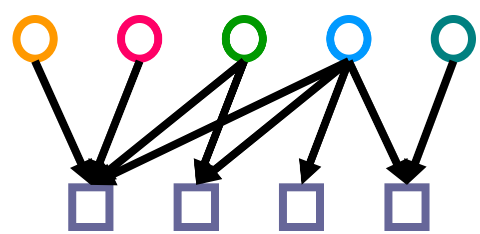
$$B^T = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

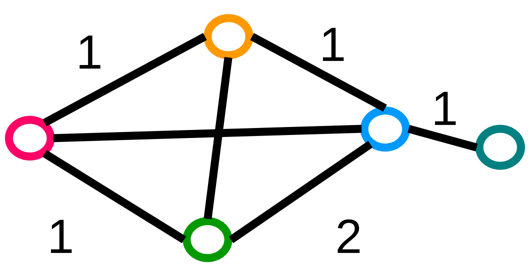
# Matrix multiplication

- general formula for matrix multiplication $Z_{ij} = \sum_k X_{ik} Y_{kj}$
- let $Z = A$, $X = B$, $Y = B^T$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 2 & 0 \\ 1 & 1 & 2 & 4 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 1*1 + 1*1 + 1*0 + 1*0 = 2$$

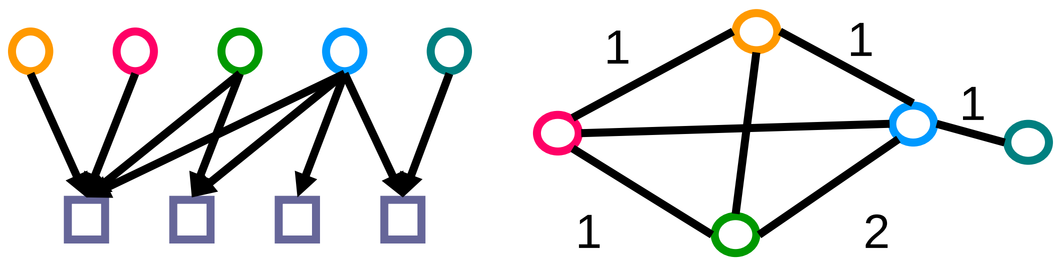




## Collapsing a two-mode network to a one mode-network

- Assume the nodes in group 1 are people and the nodes in group 2 are movies
- The diagonal entries of A give the number of movies each person has seen
- The off-diagonal elements of A give the number of movies that both people have seen
- A is symmetric

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 2 & 0 \\ 1 & 1 & 2 & 4 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$



# Readings

- Easley, David, and Jon Kleinberg. **Networks, crowds, and markets: Reasoning about a highly connected world**. Cambridge University Press, 2010. (Ch.1-2)
- Newman, Mark. **Networks: an introduction**. Oxford University Press, 2010. (Ch. 6)
- L. da F. Costa, F. A. Rodrigues, G. Travieso, and P. R. Villas Boas. **Characterization of complex networks: A survey of measurements**. Advances in Physics, 56(1):167 – 242, 2007.