Complex Network Theory

Lecture 3

Network centrality measures

Instructor: S. Mehdi Vahidipour

(Vahidipour@kashanu.ac.ir)

Thanks

A. Rezvanian

A. Barabasi, L.Adamic and J. Leskovec

March 2017

Outline

- Overview of class topics
  - Importance of nodes and links
  - Network centrality measures
  - Degree centrality
  - Closeness centrality
  - Betweenness centrality
  - Reach centrality
  - PageRank centrality
  - Vulnerability
  - Network entropy

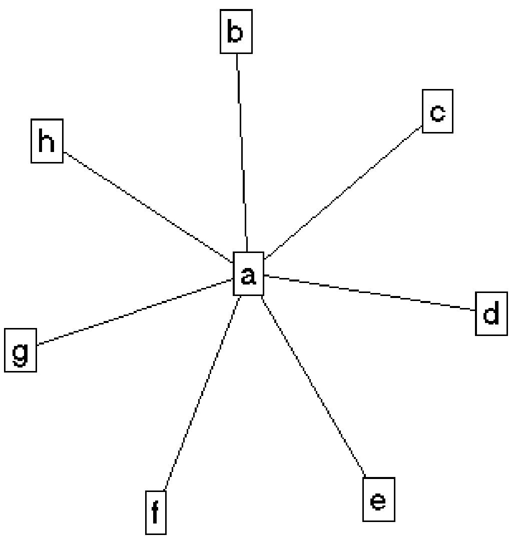
- Next class:
  - Network analysis

Centrality in Social Networks

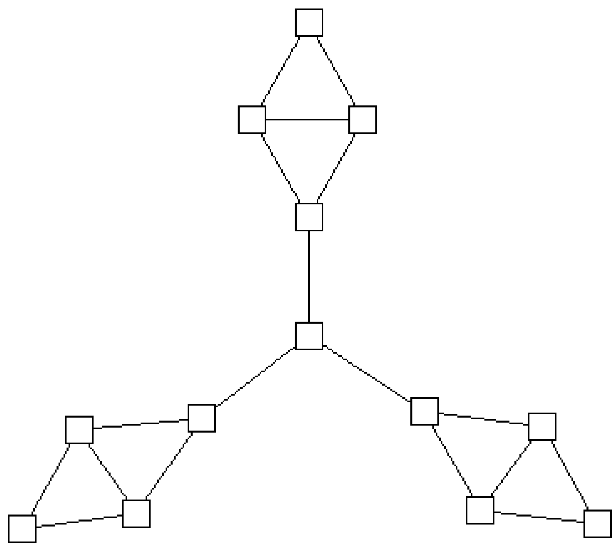
- Background: At the individual level, one dimension of position in the network can be captured through <u>centrality.</u>
- Conceptually, centrality is fairly straight forward: we want to identify which nodes are in the 'center' (**important**) of the network. In practice, identifying exactly what we mean by 'center' is somewhat complicated.
- Approaches:
    - Degree
    - Closeness
    - Betweenness
- Graph level measures: Centralization
- Central components may play critical role in network functions
    - Robustness
    - Collective behavior
    - Information spreading
    - Synchronization
    - Social dynamics

Central nodes

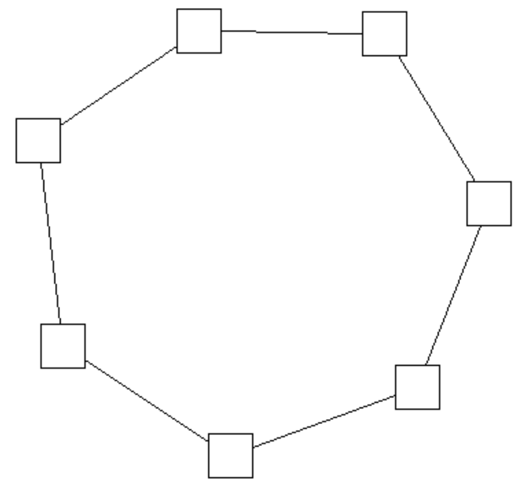
Intuitively, we want a method that allows us to distinguish "important" nodes (users, actors). Consider the following graphs:



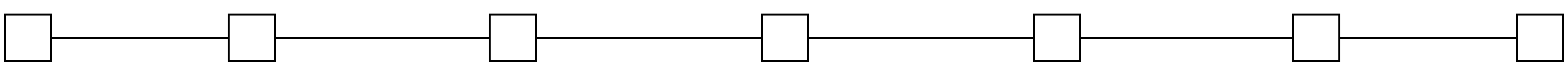
Star



Modular



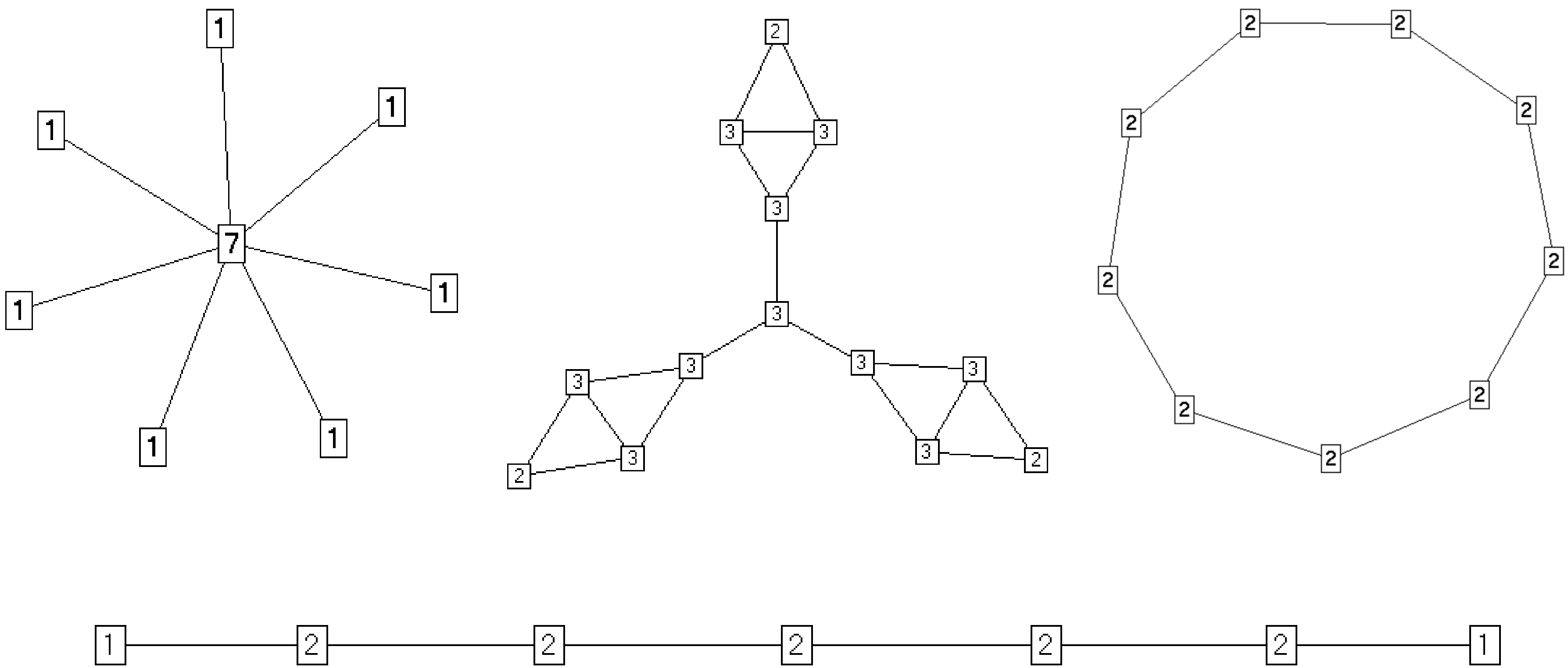
Circle



Line

Degree based centrality

The most intuitive notion of centrality focuses on degree: The node with the most ties is the most important:

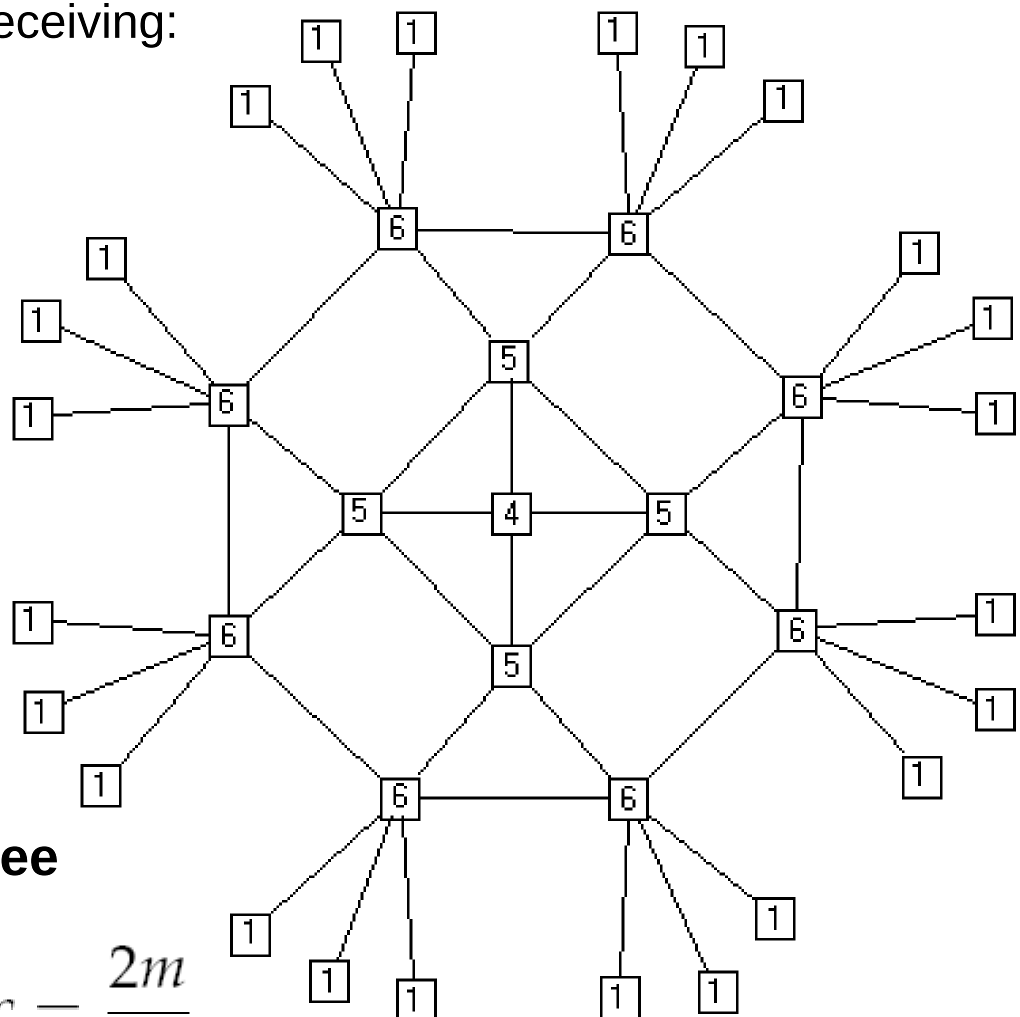


$$C_D(v_i) = d(v_i) = k_i = \sum_j a_{ij}$$

Degree based centrality

Degree centrality, however, can be deceiving:

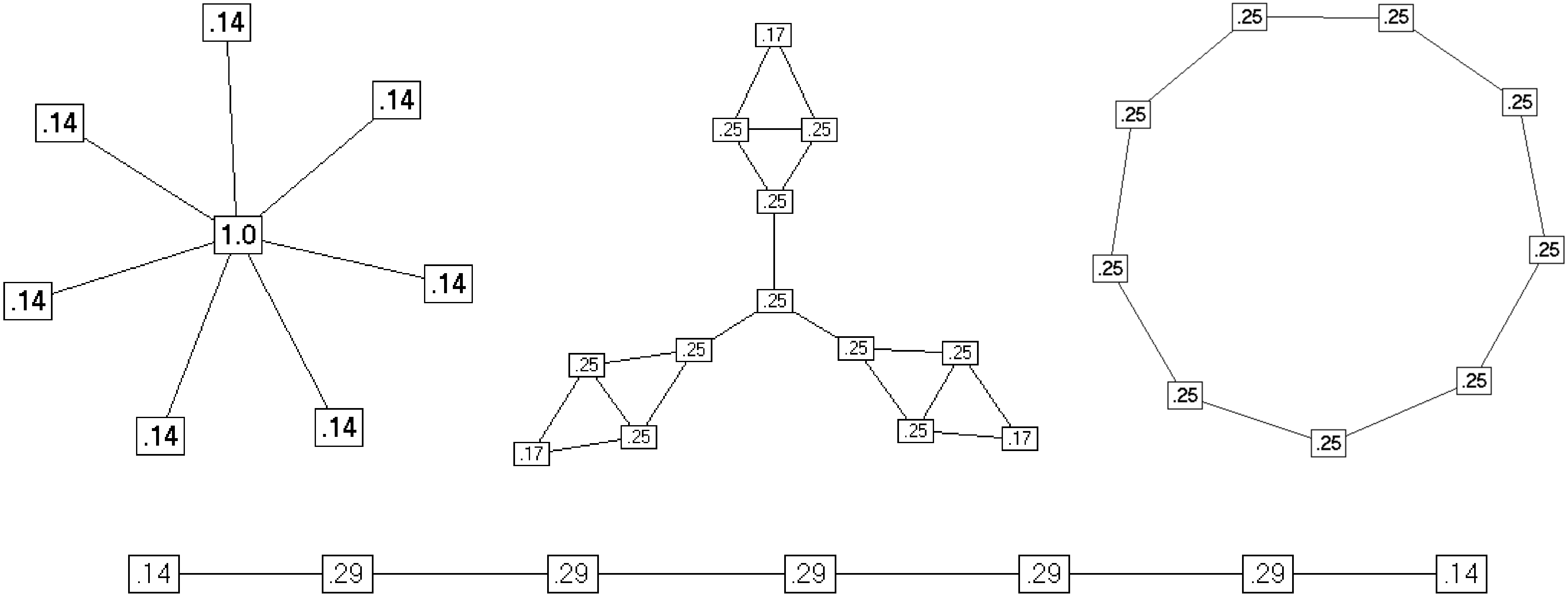
- In what ways does degree fail to capture centrality in the following graphs
    - ability to broker between groups
    - likelihood that information originating anywhere in the network reaches you



Other measures based on degree

- Max degree: k_{max}
- Mean (average) degree: <k> , $c = \frac{2m}{n}$
- Degree Distribution: P(k)
    - $P_{out}(k)$, $P_{in}(k)$

One often standardizes the degree distribution, by the maximum possible (n-1):



$$\overline{C_D}(i) = \bar{d}(v_i) = \bar{k}_i = \frac{\sum_j a_{ij}}{n-1} = \frac{k_i}{n-1}$$

Centralization (skew in distribution)

How much variation is there in the centrality scores among the nodes?

If we want to measure the degree to which the graph as a whole is centralized, we look at the dispersion of centrality:

Simple: **variance** of the individual centrality scores.

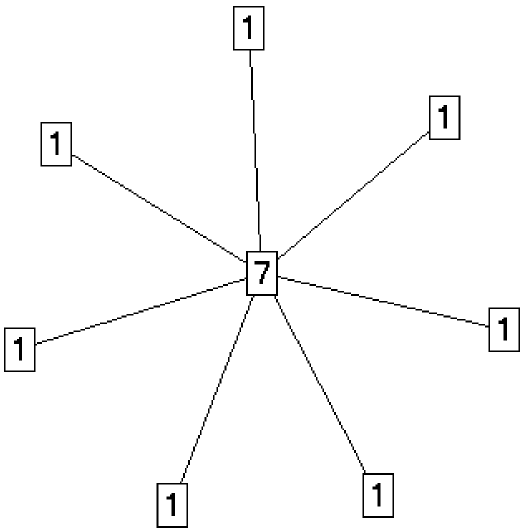
$$S_D^2 = \left[\sum_{i=1}^{n} (C_D(v_i) - \bar{C}_d)^2 \right] / n$$

Or, using Freeman's general formula for **centralization**:

maximum value in the network

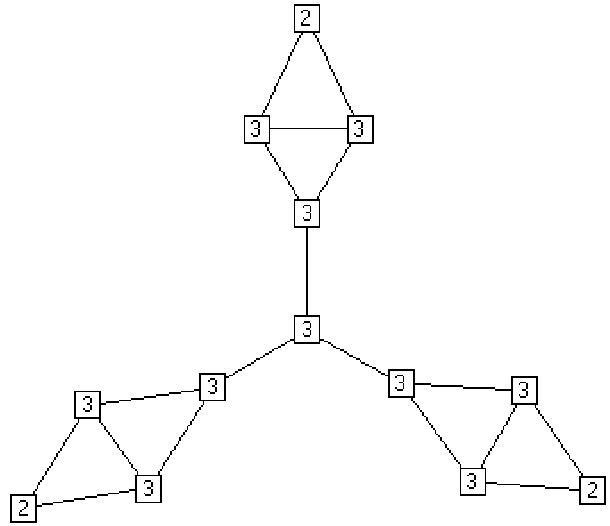
$$C_D = \frac{\sum_{i=1}^{n} \left[C_D(v^*) - C_D(v_i) \right]}{[(n-1)(n-2)]}$$

Degree Centralization Scores



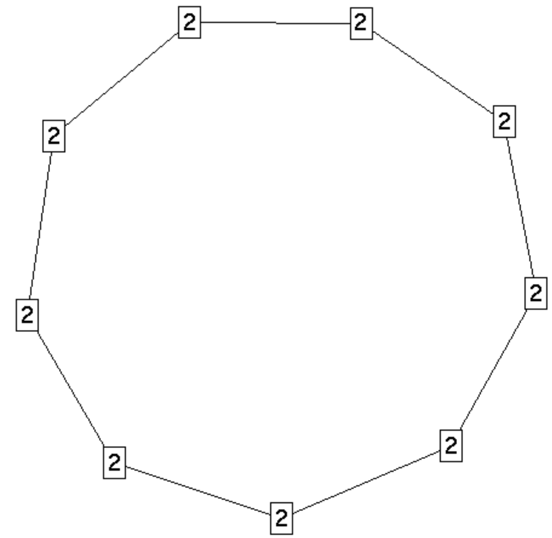
Freeman: 1.0

Variance: 3.9



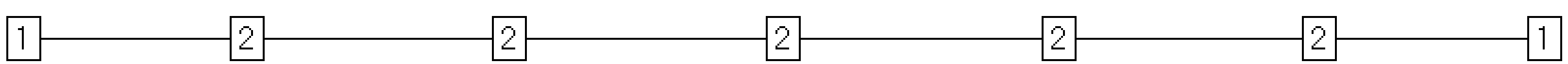
Freeman: .02

Variance: .17



Freeman: 0.0

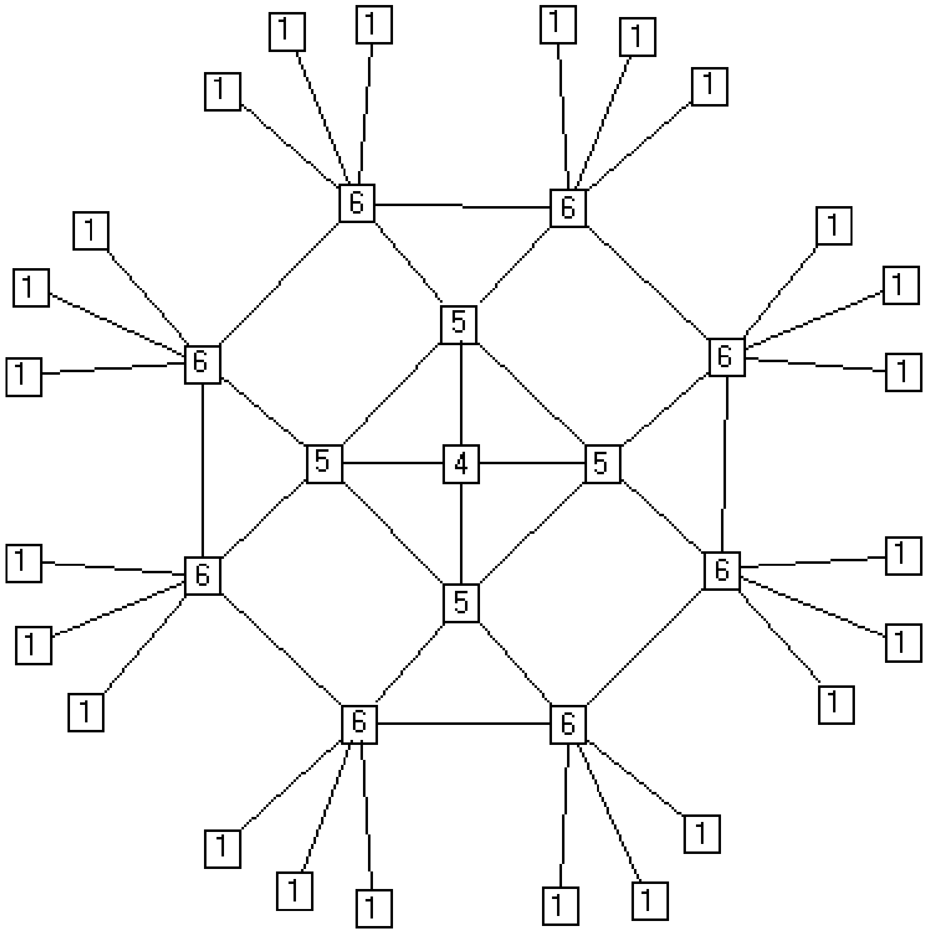
Variance: 0.0



Freeman: .07

Variance: .20

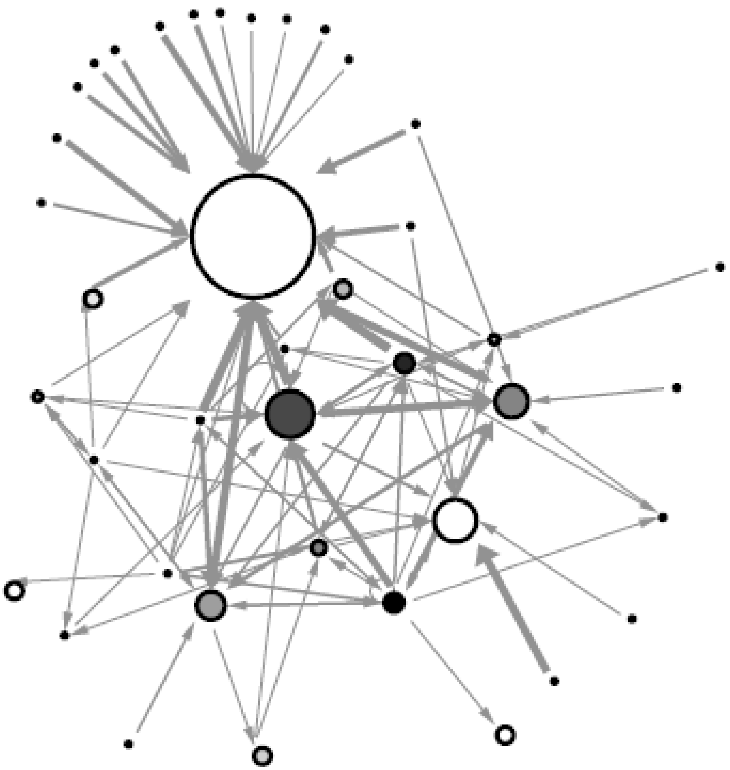
Degree Centralization Scores

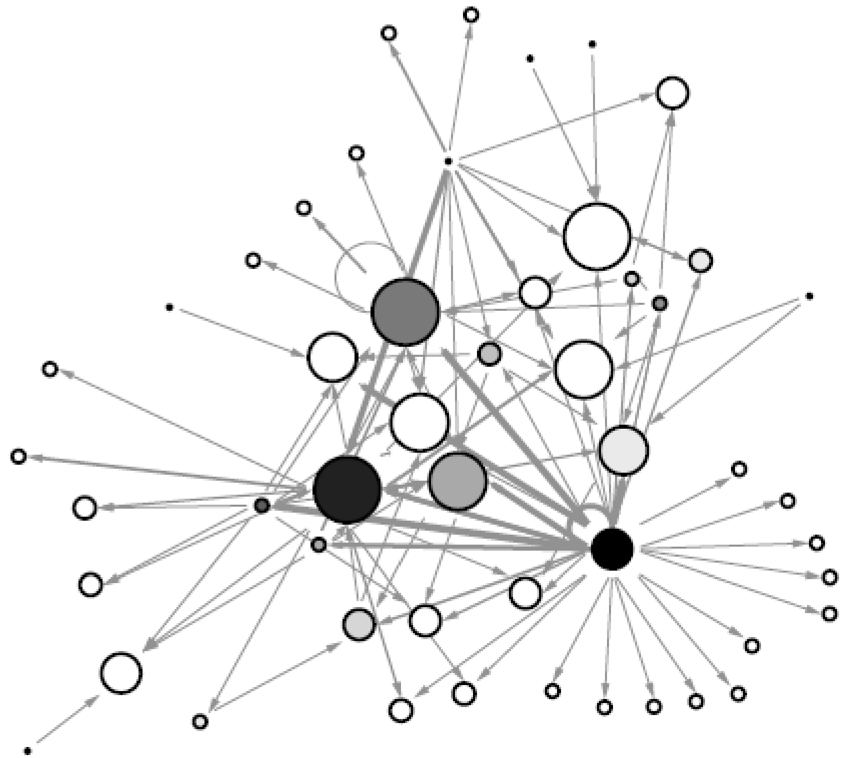


Freeman: 0.1

Variance: 4.84

Degree Centralization (example financial trading networks)

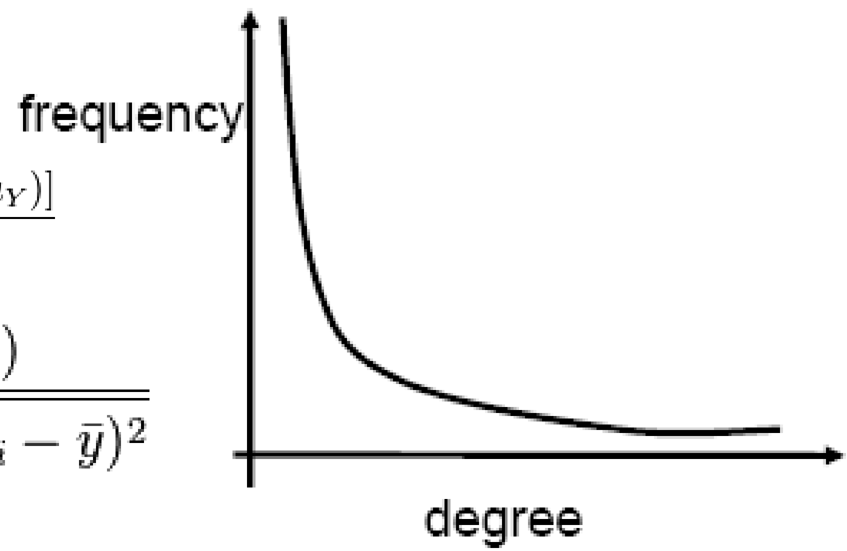




- high in-centralization: one node buying from many others
- low in-centralization: buying is more evenly distributed

Degree and degree distribution

- Degree k_i of node i is a measure of its centrality
- Nodes with high degrees are called **hubs**
- Maximum degree $k_{max} = \max_i(k_i)$ is also an important measure
- The variance of node-degrees can be an indicator of network heterogeneity, i.e. the more the variance the more the heterogeneity
- Degree distribution



$$\rho_{X,Y} = \frac{\mathrm{cov}(X,Y)}{\sigma_X \sigma_Y} \qquad \rho_{X,Y} = \frac{\mathrm{E}[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X \sigma_Y}$$

$$r = r_{xy} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

Degree-degree correlation

- It is important to know if the nodes with degree k are connected to nodes with degree k'.
- one way is to use the method proposed by Newman and compute the correlation coefficient
- Degree-degree correlation is computed as

$$r = \frac{\frac{1}{E}\sum_{j>i} k_i k_j a_{ij} - \left[\frac{1}{E}\sum_{j>i}\frac{1}{2}\left(k_i + k_j\right)a_{ij}\right]^2}{\frac{1}{E}\sum_{j>i}\frac{1}{2}\left(k_i^2 + k_j^2\right)a_{ij} - \left[\frac{1}{E}\sum_{j>i}\frac{1}{2}\left(k_i + k_j\right)a_{ij}\right]^2}$$

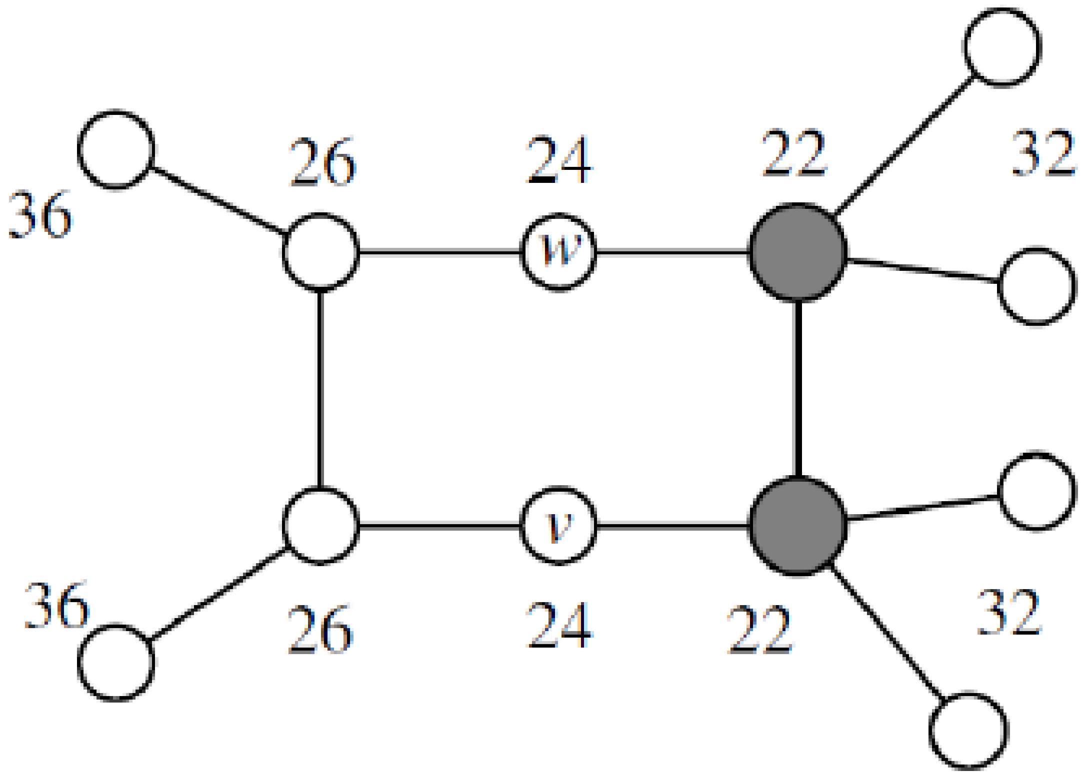
- E is the total number of edges
- a_{ij} is the entry (i,j) of the adjacency matrix
- k_i is the degree of node v_i

Degree-degree correlation

- **r > 0**: the network is called **assortative**
  - Node with large degree intent to connect to those with large degrees and nodes with low degrees intend to connect to those with low degrees (rich with rich and poor with poor)
- **r < 0:** the network is called **disassortative**
  - Node with large degree intent to connect to those with low degrees and nodes with low degrees intend to connect to those with high degrees (rich with poor)
- **r = 0:** no correlations
  - There is no specific intention in the connection between the nodes in the sense of their degrees

An example: shopping mall location

- The idea is that a node is central if it can quickly interact with all others (these nodes are called median of the graph)



Closeness-based centrality

A second measure of centrality is closeness centrality. A node is considered important if he/she is relatively close to all other actors.

Closeness is based on the inverse of the distance of each actor to every other actor in the network.

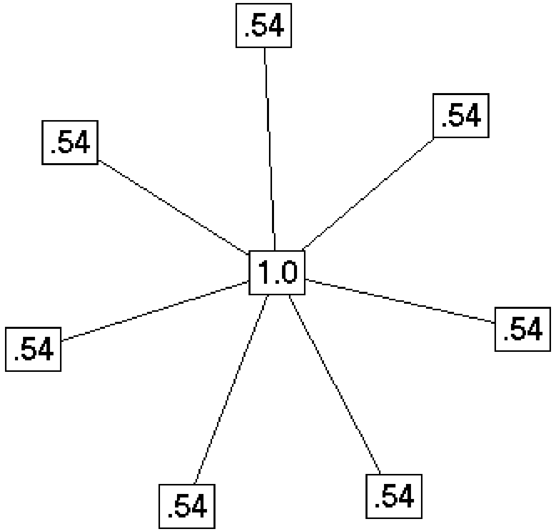
Closeness Centrality:

$$C_c(v_i) = \left[\sum_{j=1}^{n} d(v_i, v_j) \right]^{-1}$$

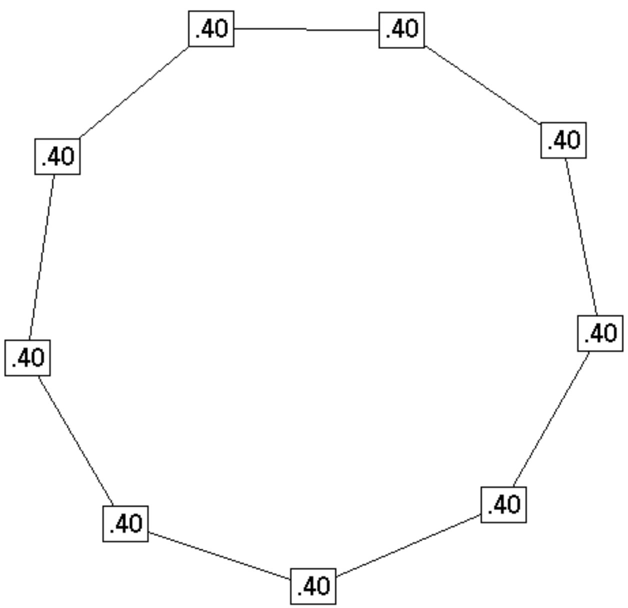
Normalized Closeness Centrality

$$\tilde{C}_c(v_i) = (C_C(v_i))(n-1) = \frac{(n-1)}{\sum_{j=1}^{n} d(v_i, v_j)}$$

Closeness Centrality in the examples



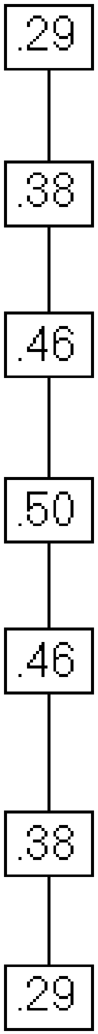
Distance	Closeness	normalized
0 1 1 1 1 1 1 1	.143	1.00
1 0 2 2 2 2 2 2	.077	.538
1 2 0 2 2 2 2 2	.077	.538
1 2 2 0 2 2 2 2	.077	.538
1 2 2 2 0 2 2 2	.077	.538
1 2 2 2 2 0 2 2	.077	.538
1 2 2 2 2 2 0 2	.077	.538
1 2 2 2 2 2 2 0	.077	.538



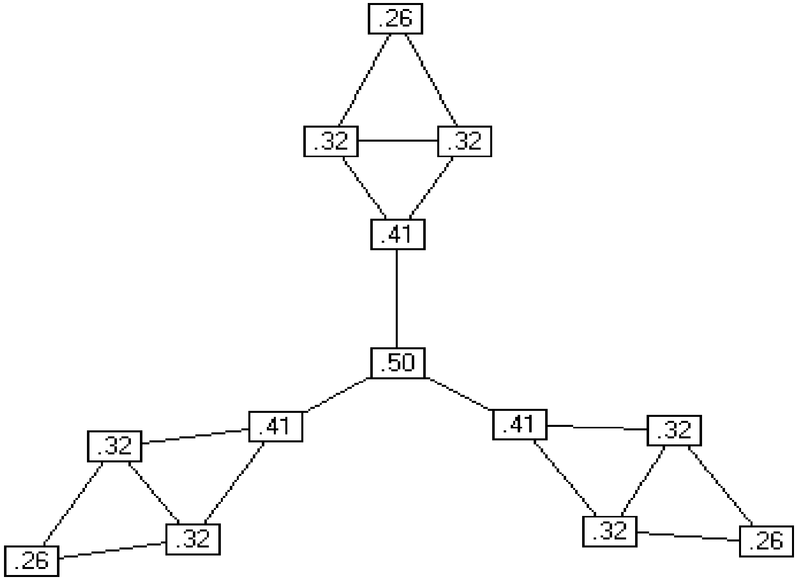
Distance	Closeness	normalized
0 1 2 3 4 4 3 2 1	.050	.400
1 0 1 2 3 4 4 3 2	.050	.400
2 1 0 1 2 3 4 4 3	.050	.400
3 2 1 0 1 2 3 4 4	.050	.400
4 3 2 1 0 1 2 3 4	.050	.400
4 4 3 2 1 0 1 2 3	.050	.400
3 4 4 3 2 1 0 1 2	.050	.400
2 3 4 4 3 2 1 0 1	.050	.400
1 2 3 4 4 3 2 1 0	.050	.400

Closeness centrality

Closeness Centrality in the examples



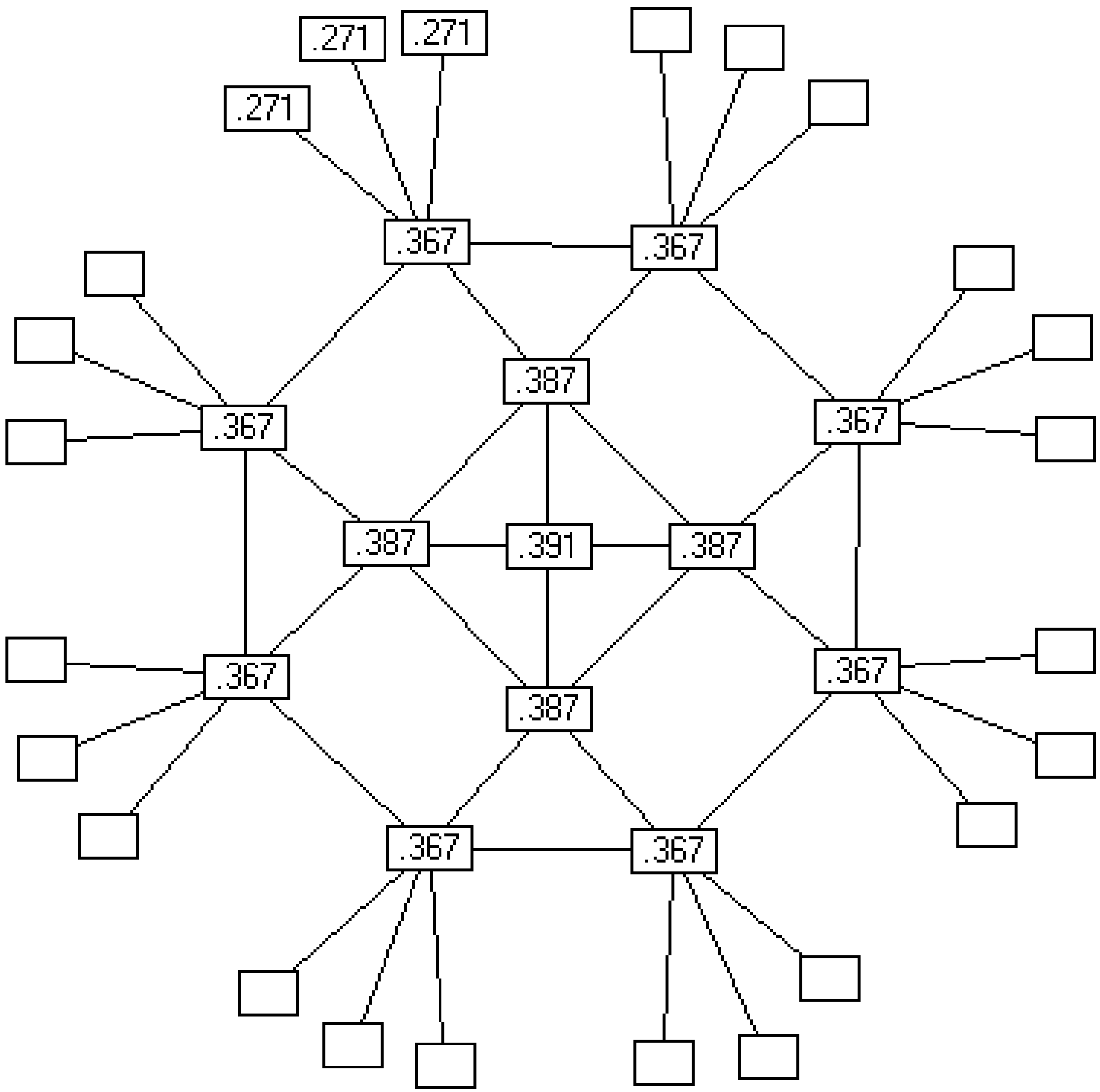
Distance							Closeness	normalized
0	1	2	3	4	5	6	.048	.286
1	0	1	2	3	4	5	.063	.375
2	1	0	1	2	3	4	.077	.462
3	2	1	0	1	2	3	.083	.500
4	3	2	1	0	1	2	.077	.462
5	4	3	2	1	0	1	.063	.375
6	5	4	3	2	1	0	.048	.286



Closeness Centrality in the example

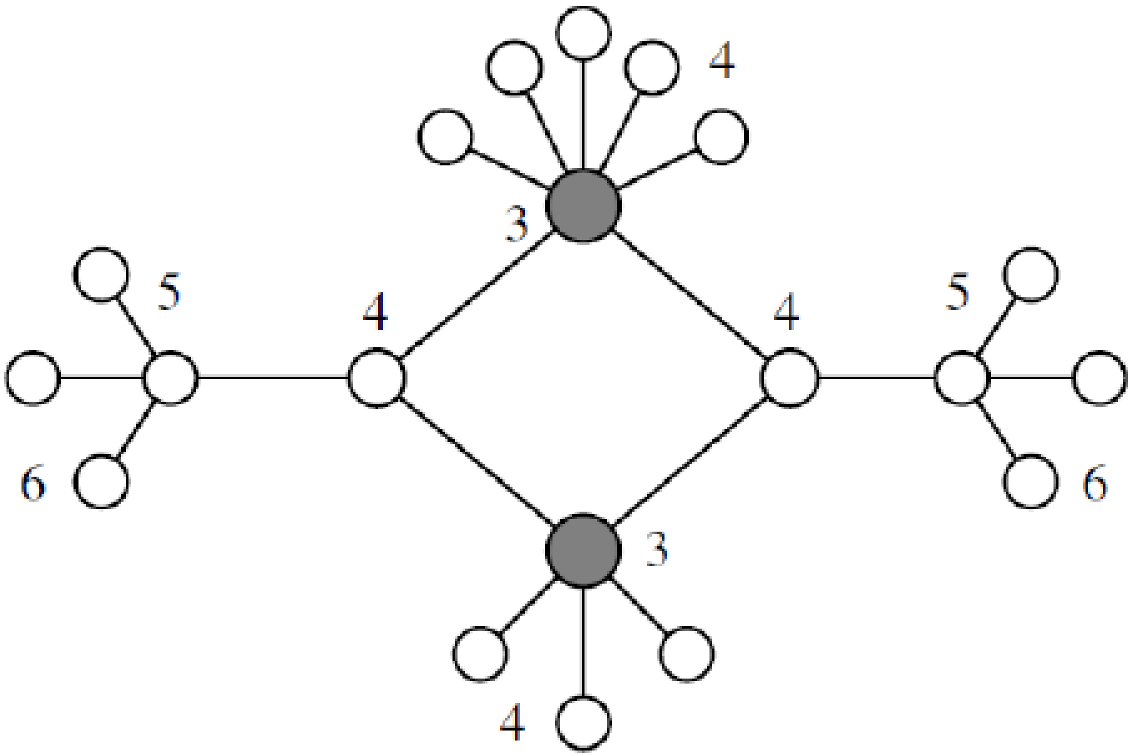
Distance													Closeness	normalized
0	1	1	2	3	4	4	5	5	6	5	5	6	.021	.255
1	0	1	1	2	3	3	4	4	5	4	4	5	.027	.324
1	1	0	1	2	3	3	4	4	5	4	4	5	.027	.324
2	1	1	0	1	2	2	3	3	4	3	3	4	.034	.414
3	2	2	1	0	1	1	2	2	3	2	2	3	.042	.500
4	3	3	2	1	0	2	3	3	4	1	1	2	.034	.414
4	3	3	2	1	2	0	1	1	2	3	3	4	.034	.414
5	4	4	3	2	3	1	0	1	1	4	4	5	.027	.324
5	4	4	3	2	3	1	1	0	1	4	4	5	.027	.324
6	5	5	4	3	4	2	1	1	0	5	5	6	.021	.255
5	4	4	3	2	1	3	4	4	5	0	1	1	.027	.324
5	4	4	3	2	1	3	4	4	5	1	0	1	.027	.324
6	5	5	4	3	2	4	5	5	6	1	1	0	.021	.255

Closeness Centrality in the example



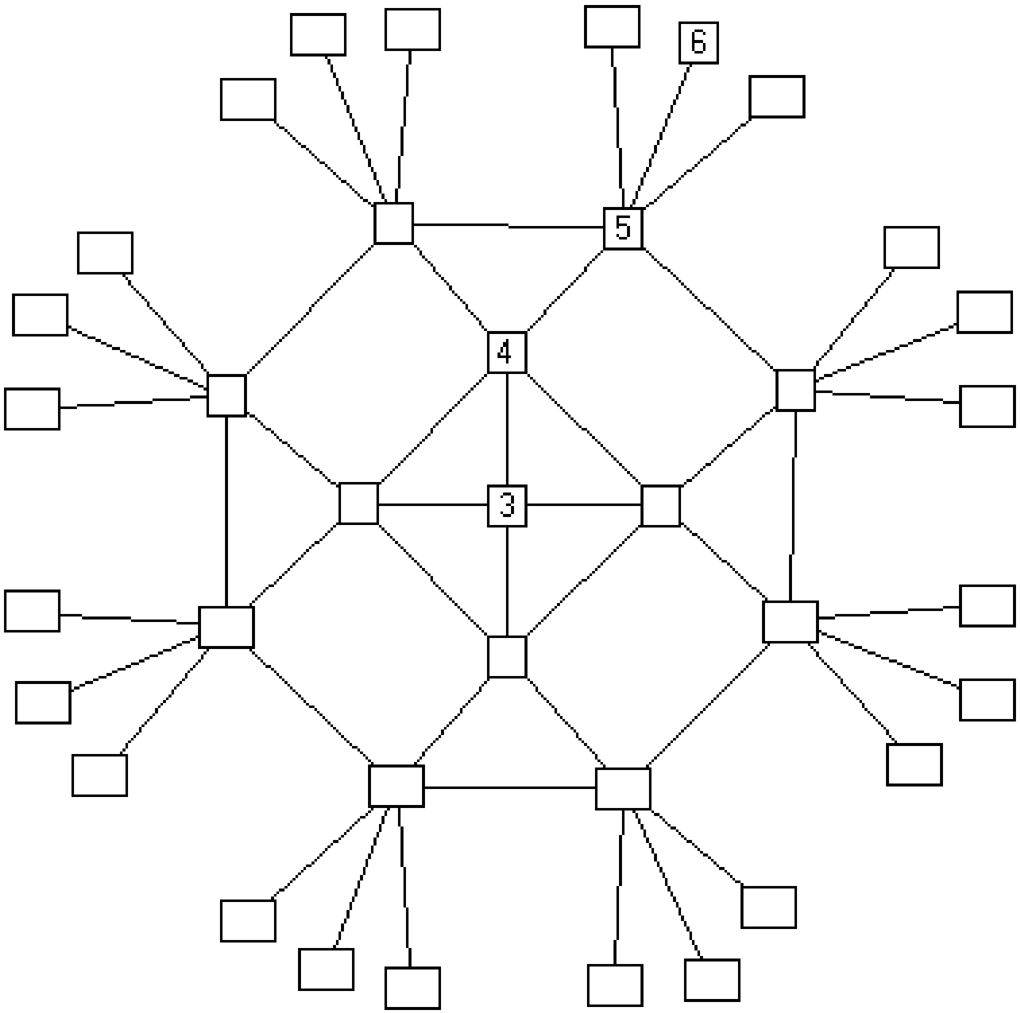
Example: Hospital location problem

- **Eccentricity**: Maximum distance of a node to all other nodes: $e(u) = \max\{d_{uv} : v \in V\}$
- **Radius of graph**: Minimum eccentricity



Graph Theoretic Center (Barry or Jordan Center).

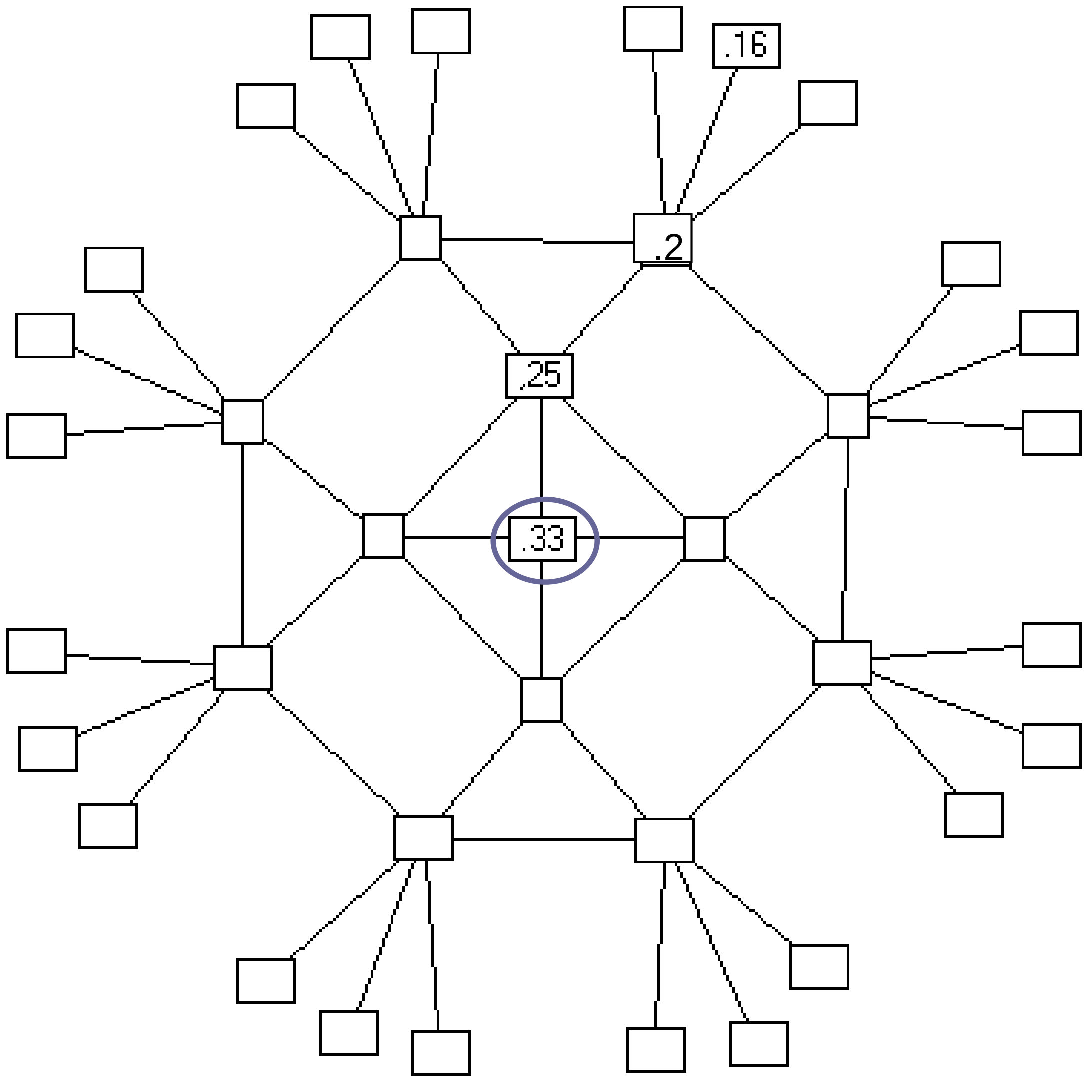
Identify the points with the smallest, maximum distance to all other points.



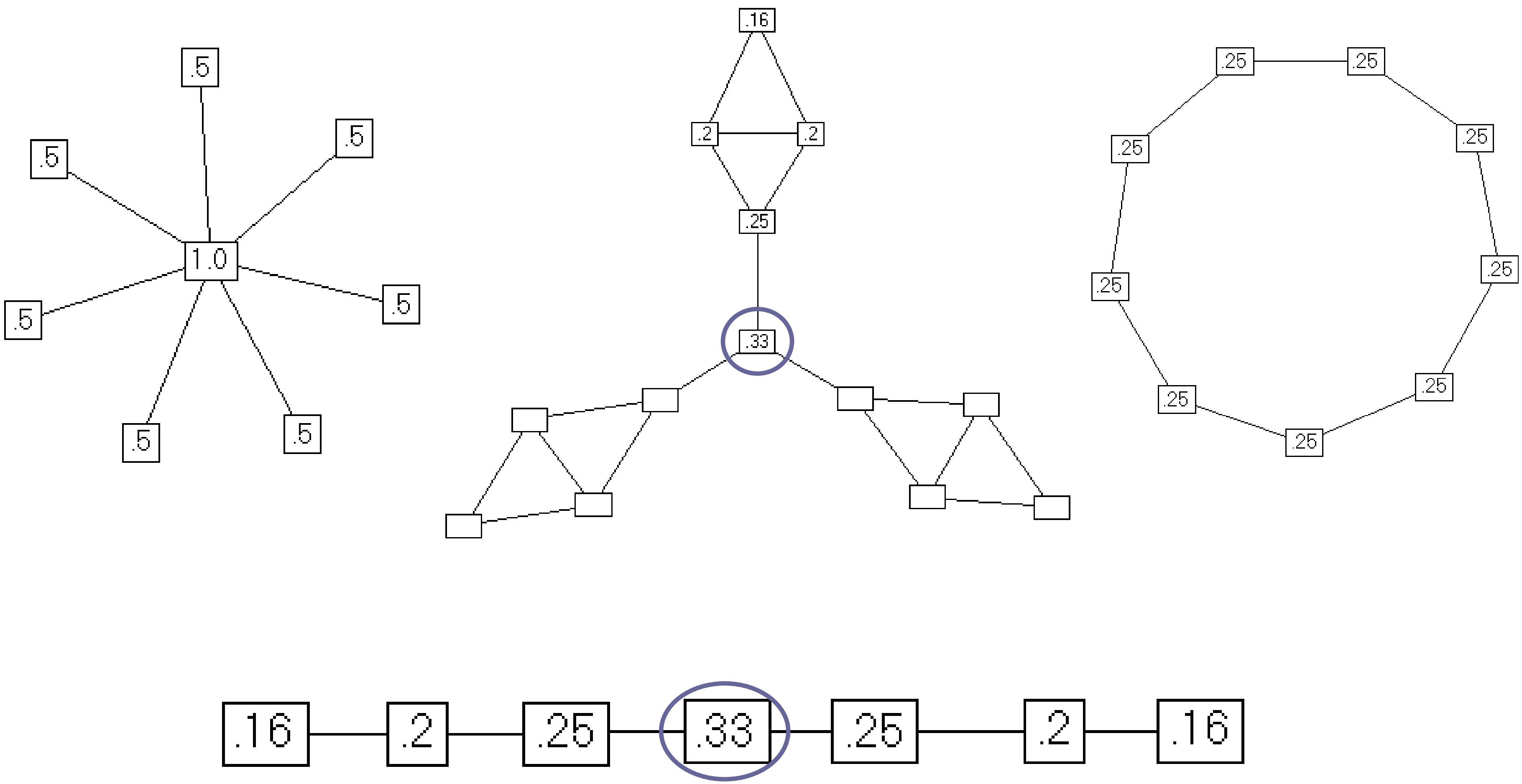
Value = longest distance to any other node.

The graph theoretic center is ‘3’, but you might also consider a continuous measure as the inverse of the maximum geodesic

Graph Theoretic Center (Barry or Jordan Center).

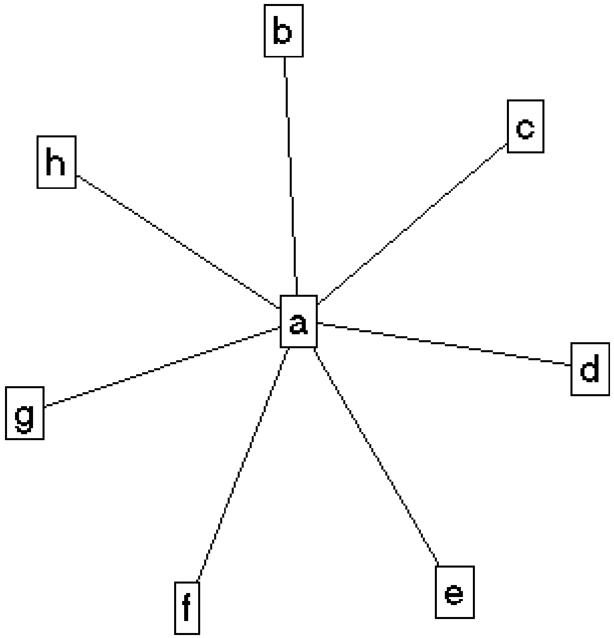


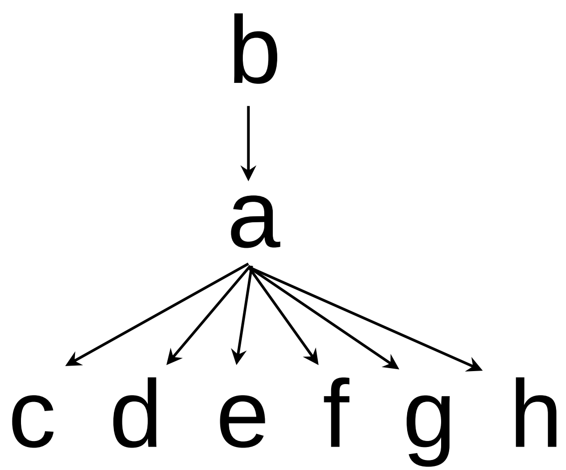
Graph Theoretic Center (Barry or Jordan Center).



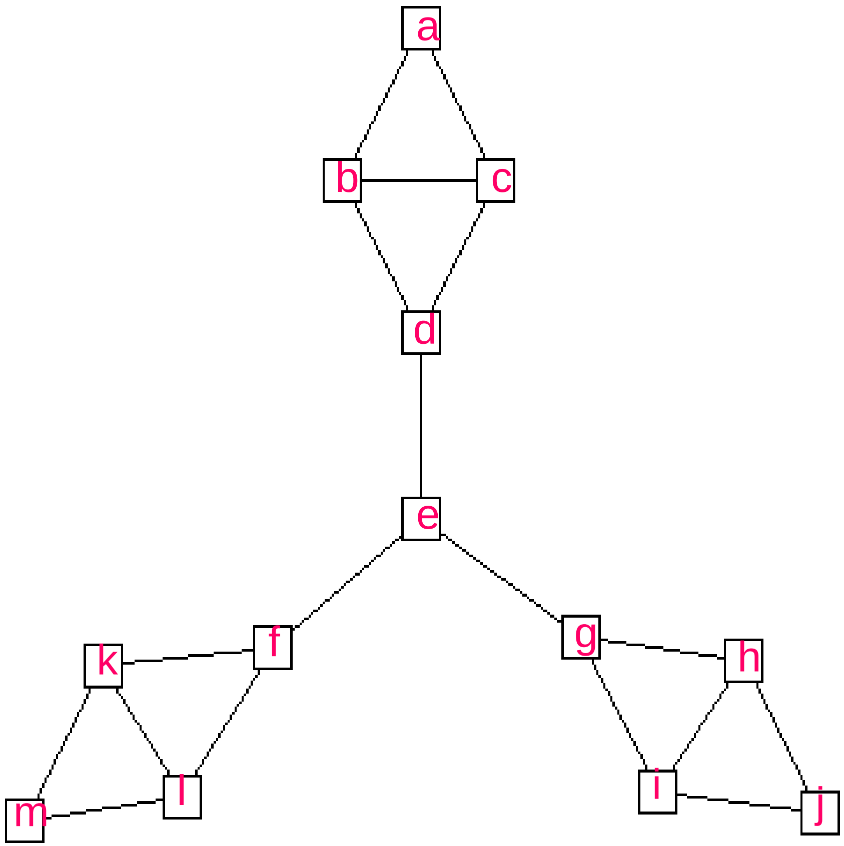
Betweenness Centrality

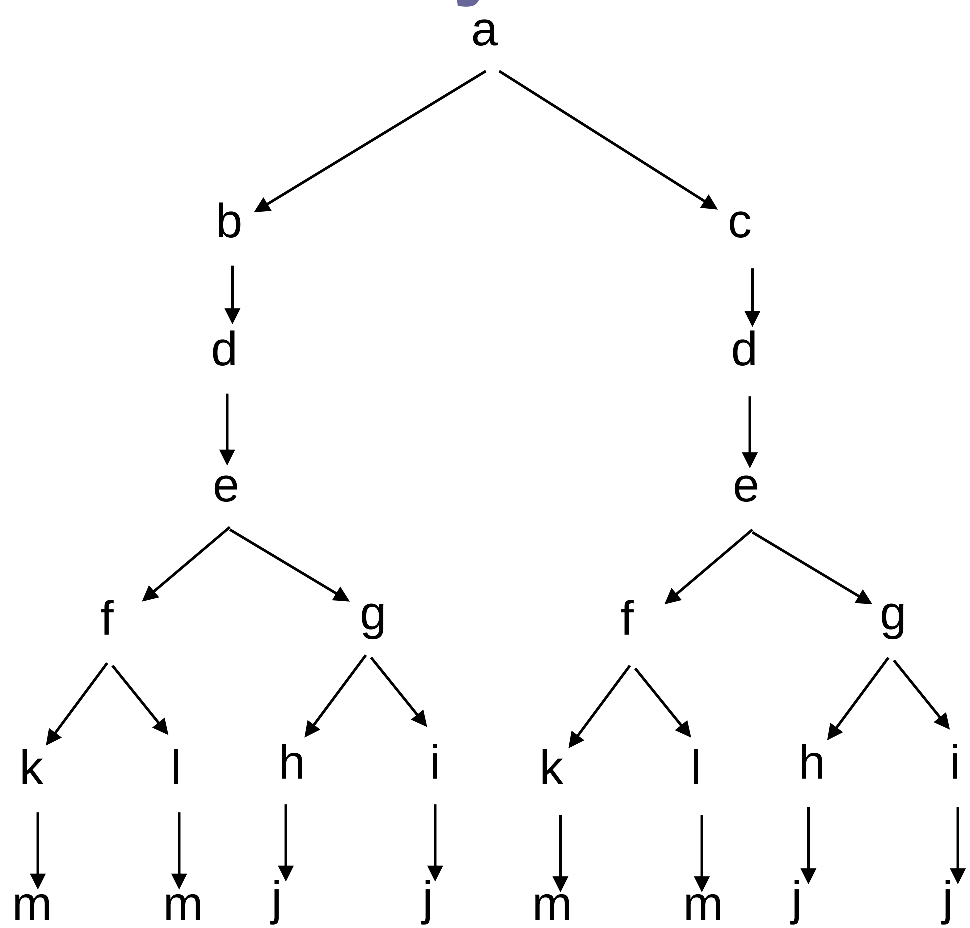
- Model based on communication flow: A person who lies on communication paths can control communication flow, and is thus important.
- Betweenness centrality counts the number of geodesic (shortest) paths between *i* and *k* that actor *j* resides on.





Betweenness Centrality





Betweenness Centrality

- **Node Betweenness**

$$C_B(v_i) = \frac{\sum_{j,k} g_{jk}(v_i)}{\sum_{j,k} g_{jk}}$$

g_{jk} = the number of shortest paths between v_j *and* v_k

$g_{jk}(v_i)$ = the number shortest paths between v_j *and* v_k that pass through node v_i

- **Edge Betweenness**: Its definition is similar to node betweenness

$$C_B(e_i) = \frac{\sum_{j,k} g_{jk}(e_i)}{\sum_{j,k} g_{jk}}$$

g_{jk} = the number of shortest paths between v_j *and* v_k

$g_{jk}(e_i)$ = the number shortest paths between v_j *and* v_k that pass through edge e_i

Usually normalized by: $$\tilde{C}_B(n_i) = \frac{C_B(v_i)}{E_{\max}}$$

Eigenvector centrality

- Bonacich eigenvector, 1972
- Idea: Connections to people who are themselves influential will lend a person more influence than connections to less influential people.
- x_i: centrality of node v_i is proportional to the average of the centralities of j's neighbors

$$C_{eig}(v_i) = \frac{1}{\lambda}\sum_j a_{ij} x_j$$

- Assuming centralities to be non-negative:
  - λ must be the largest eigenvalue of A and **x** the corresponding eigenvector (Perron–Frobenius theorem)

Bonacich power centrality

- Bonacich Power Centrality: Actor's centrality (prestige) is equal to a function of the prestige of those they are connected to. Thus, actors who are tied to very central actors should have higher prestige/ centrality than those who are not.

$$C(\alpha, \beta) = \alpha (I - \beta R)^{-1} R 1$$

- α is a scaling vector, which is set to normalize the score.
- β reflects the extent to which you *weight* the centrality of people ego is tied to.
- **R** is the adjacency matrix (can be valued)
- **I** is the identity matrix (1s down the diagonal)
- **1** is a matrix of all ones.

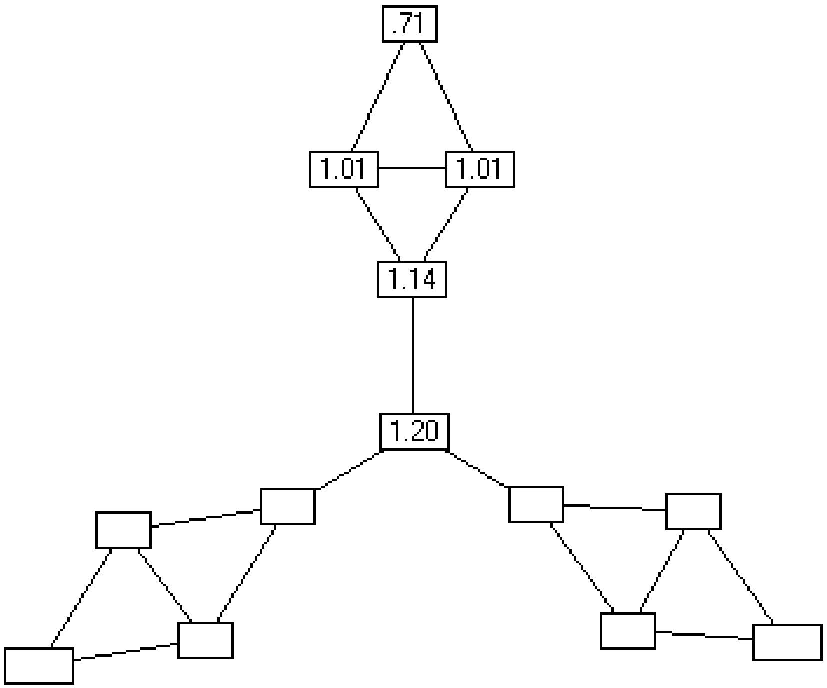
Bonacich power centrality

- The magnitude of β reflects the radius of power.
- Small values of β weight local structure.
- Large values of β weight global structure.

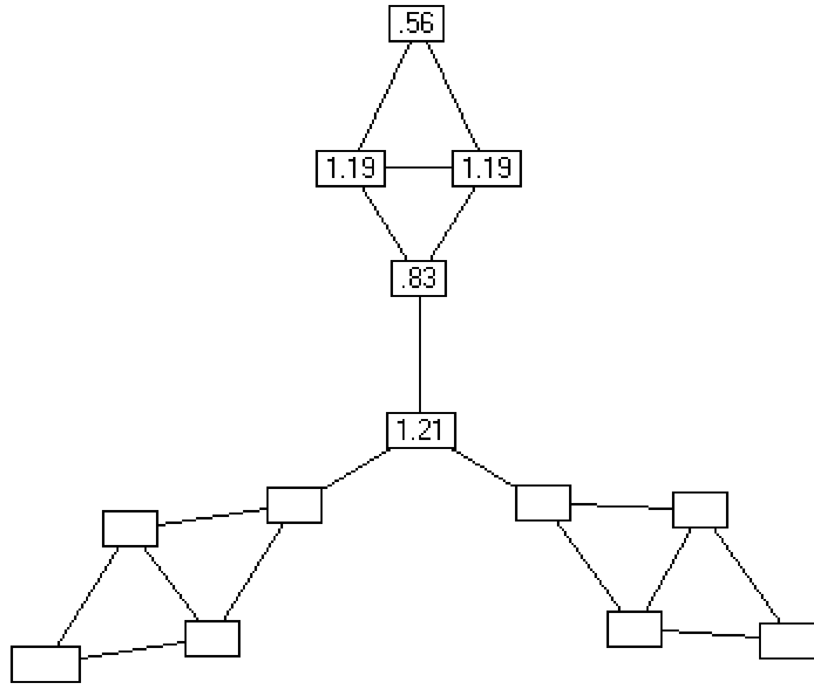
- If $\beta>0$, the node has higher centrality when tied to people who are central.
- If $\beta<0$, then node has higher centrality when tied to people who are not central
- $\beta=0$, It get degree centrality.

Bonacich power centrality

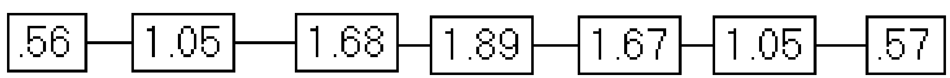
$\beta=0.35$



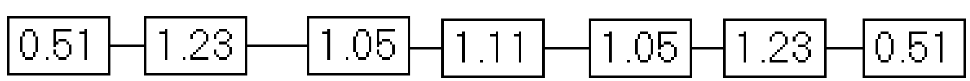
$\beta= -0.35$



$\beta=0.23$

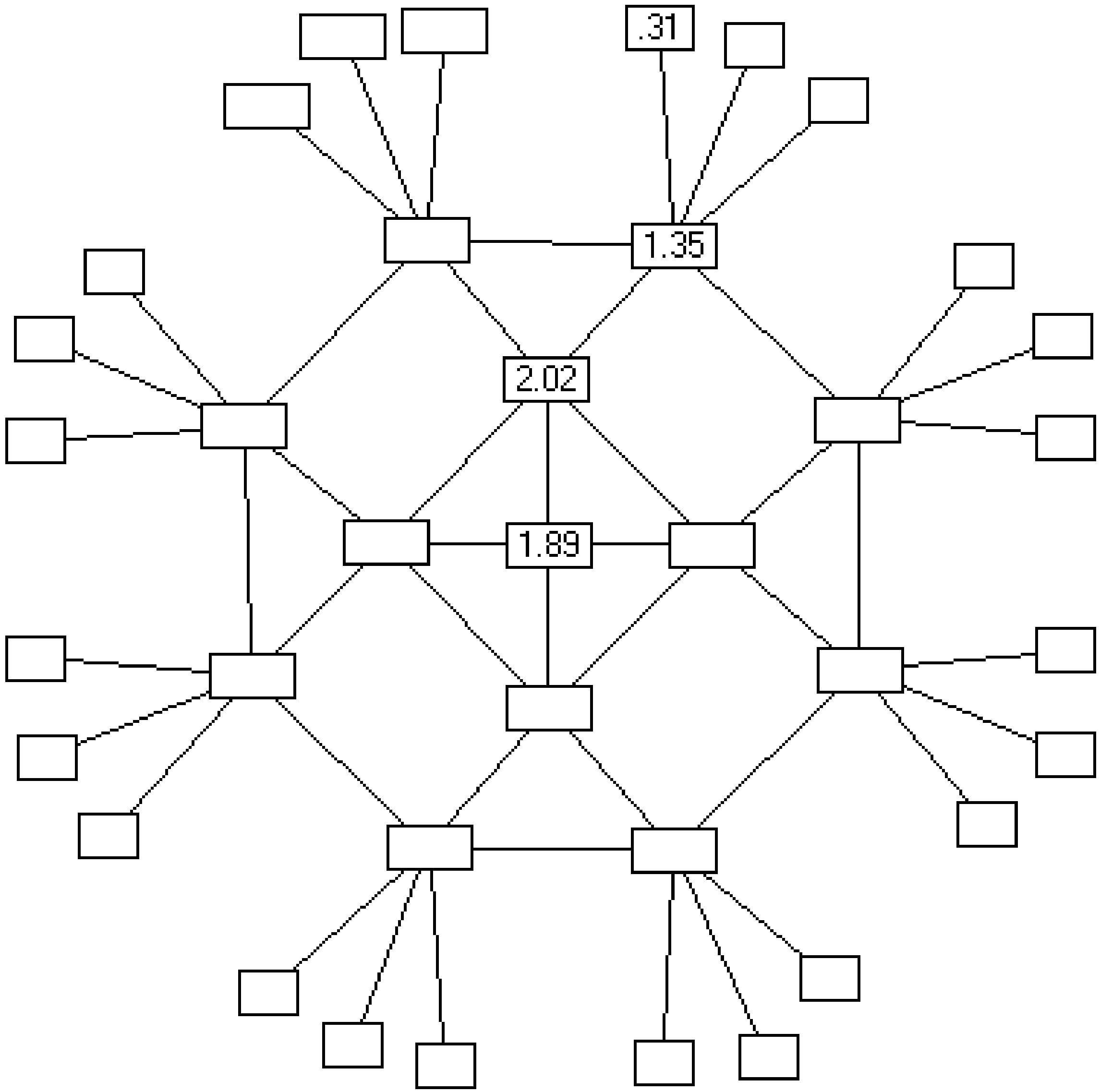


$\beta= -0.23$



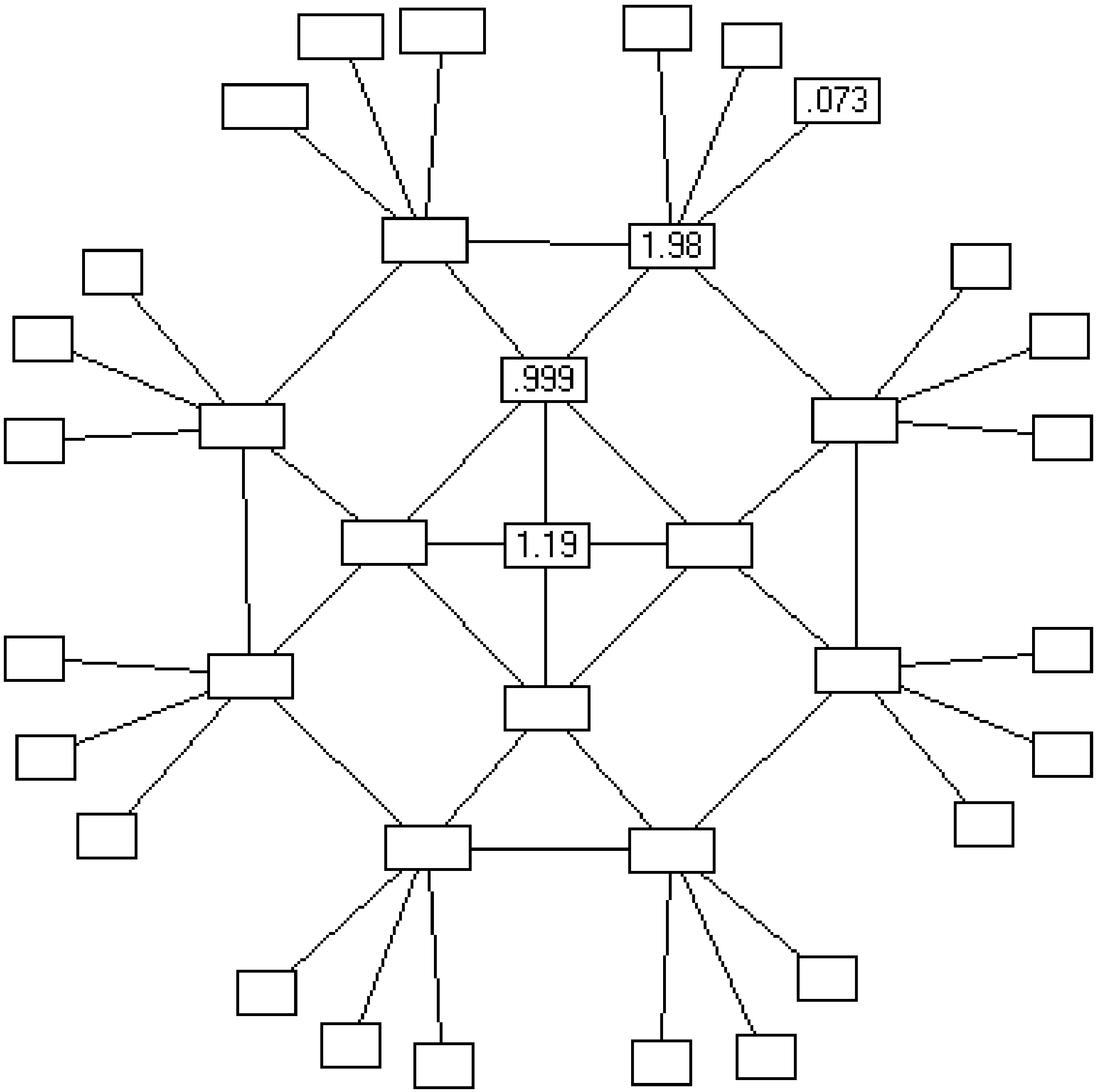
Bonacich power centrality

β=.23



Bonacich power centrality

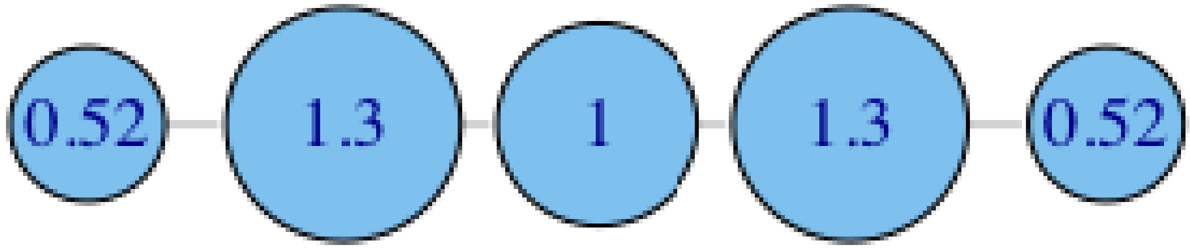
β=-.23



Bonacich power centrality

- Example
  - $\beta=0.25$

  - $\beta=-0.25$



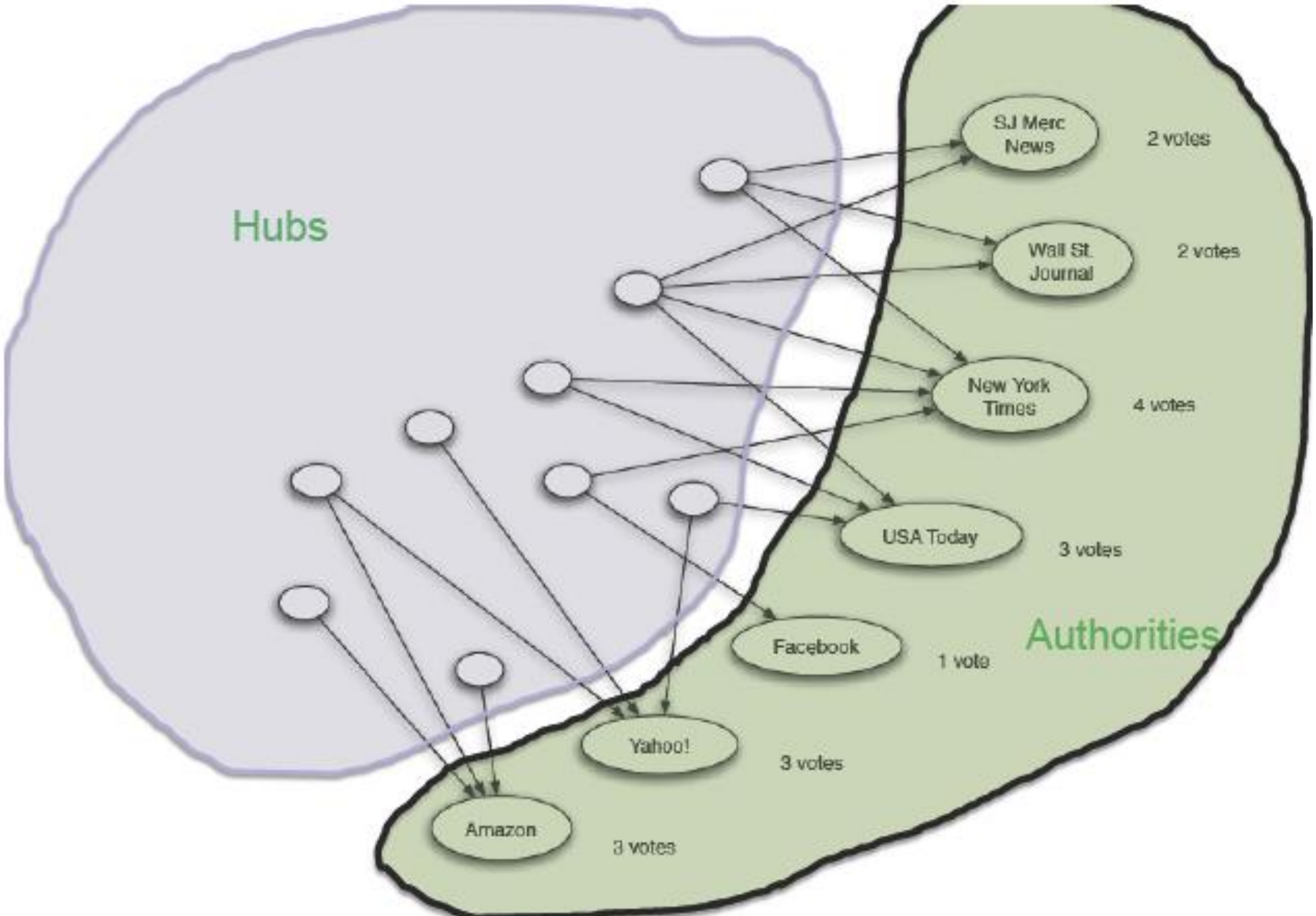
HITS centrality

- Hyperlink-Induced Topic Search
- also known as Hubs and authorities
- Developed by Jon Kleinberg
- Precursor to Page Rank
- Certain web pages, known as hubs, serve as large directories

Hubs vs. Authorities

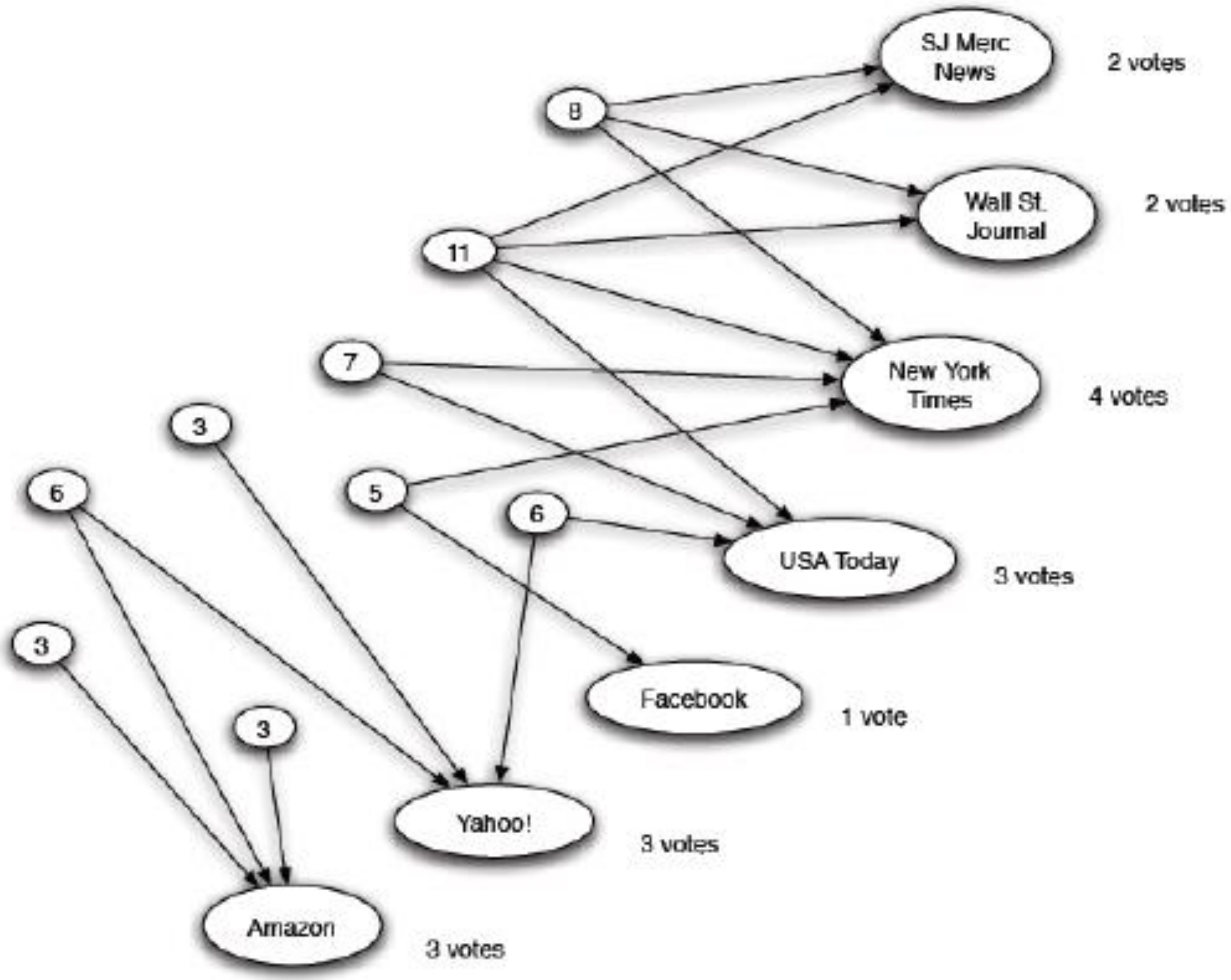
- Hubs are not actually authoritative in the information that it hold, but are used as compilations of a broad catalog of information that lead users directly to other authoritative pages.
- a good hub represents a page that points to many other pages, and
- a good authority represents a page that is linked by many different hubs.

HITS

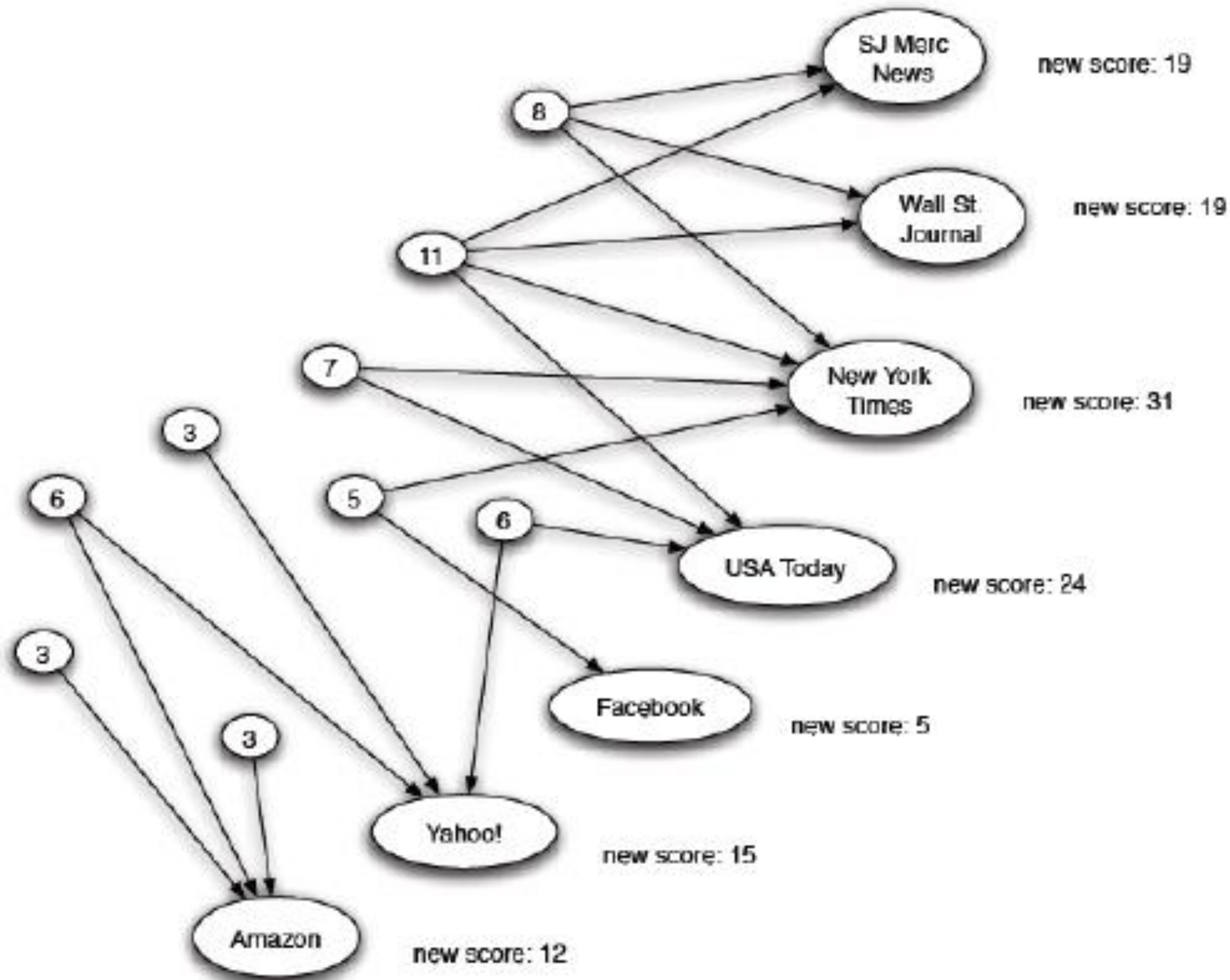


Hubs
SJ Merc News
2 votes
Wall St. Journal
2 votes
New York Times
4 votes
USA Today
3 votes
Facebook
1 vote
Authorities
Yahoo!
3 votes
Amazon
3 votes

HITS

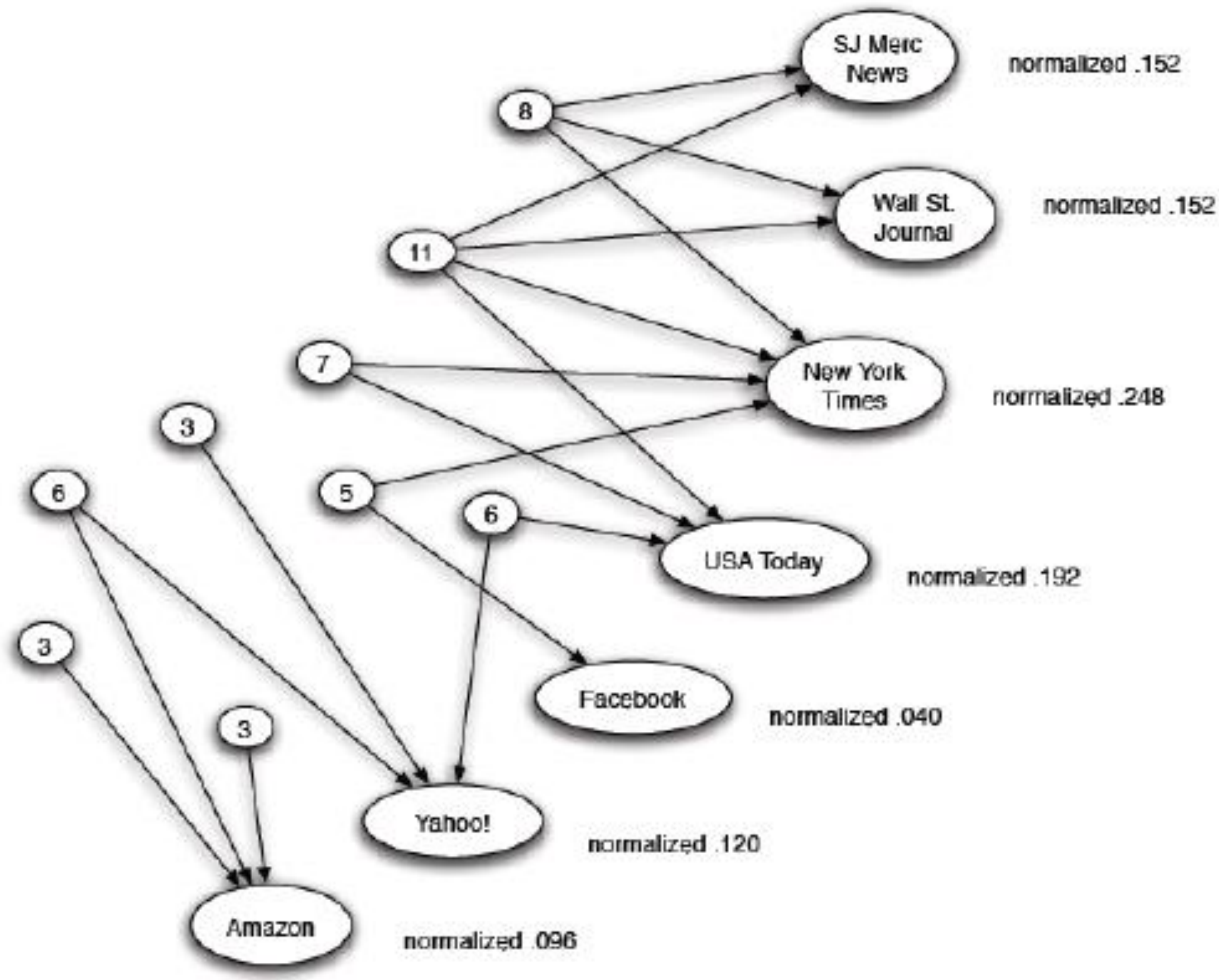

SJ Merc News
2 votes
8
Wall St. Journal
2 votes
11
New York Times
4 votes
7
3
5
6
6
USA Today
3 votes
3
Facebook
1 vote
3
Yahoo!
3 votes
Amazon
3 votes

HITS



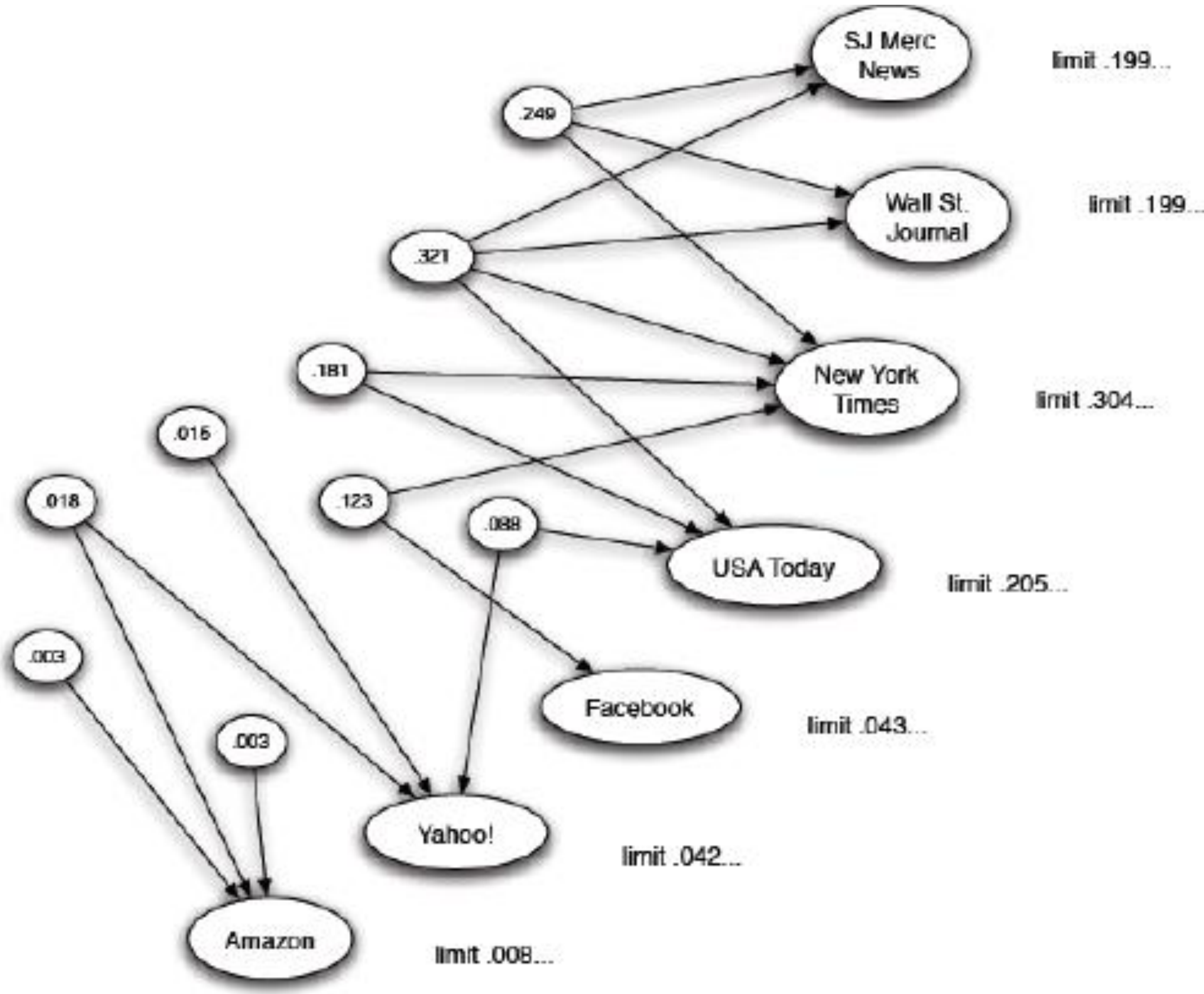
SJ Merc News
new score: 19
8
Wall St. Journal
new score: 19
11
New York Times
new score: 31
7
3
6
5
6
USA Today
new score: 24
3
Facebook
new score: 5
3
Yahoo!
new score: 15
Amazon
new score: 12

HITS



SJ Merc News
normalized .152
8
Wall St. Journal
normalized .152
11
New York Times
normalized .248
7
3
6
5
6
USA Today
normalized .192
3
Facebook
normalized .040
3
Yahoo!
normalized .120
Amazon
normalized .096

HITS



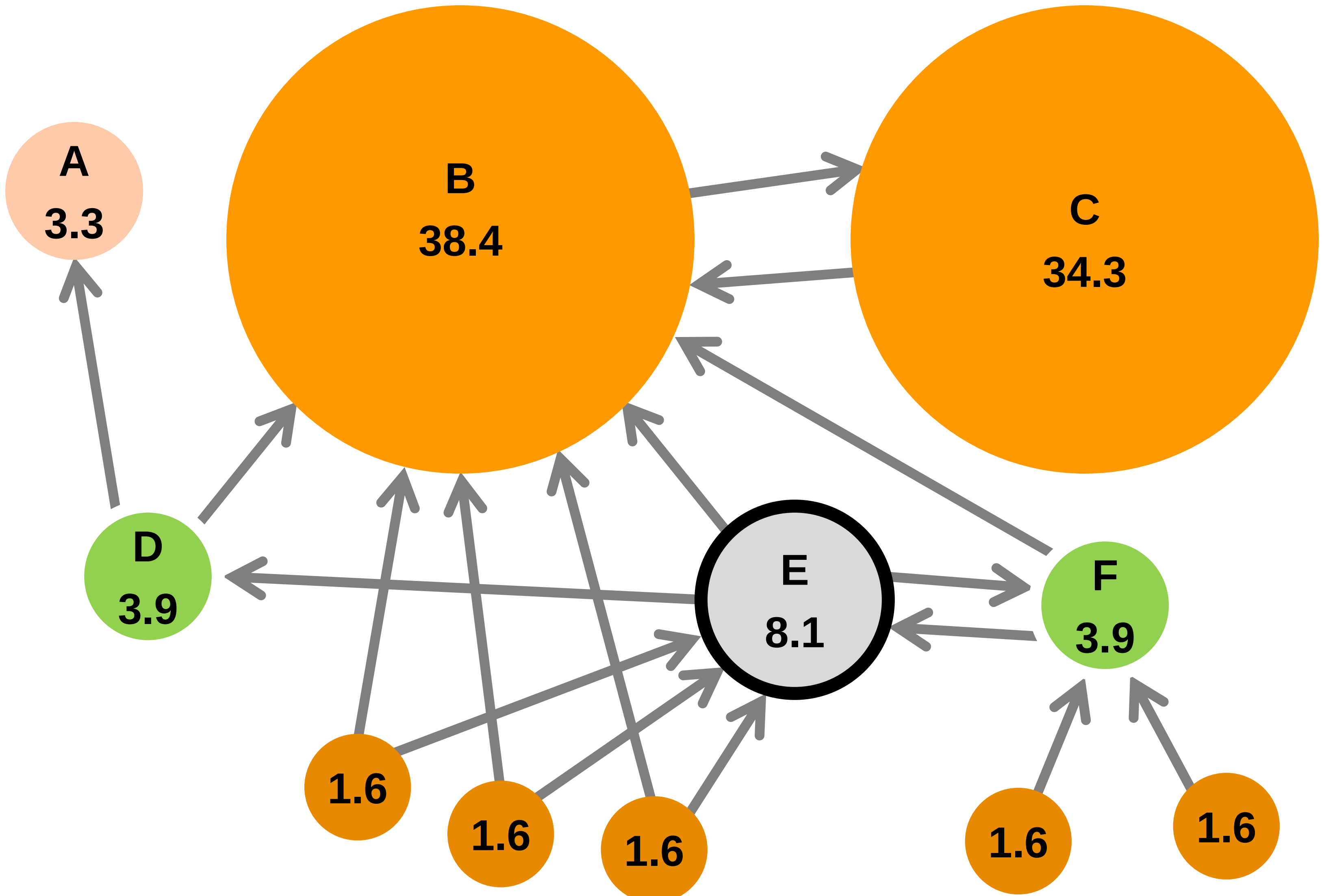
SJ Merc News
limit .199...
.249
Wall St. Journal
limit .199...
.321
.181
New York Times
limit .304...
.015
.018
.123
.088
USA Today
limit .205...
.003
Facebook
limit .043...
.003
Yahoo!
limit .042...
Amazon
limit .008...

History !

How to organize the Web?

- **First try:** Human curated **Web directories**
  - Yahoo, DMOZ, LookSmart
- **Second try: Web Search**
  - **Information Retrieval** investigates: Find relevant docs in a small and trusted set
    - Newspaper articles, Patents, etc.
  - <u>**But:**</u> Web is **huge**, full of untrusted documents, random things, web spam, etc.

Example: PageRank Scores

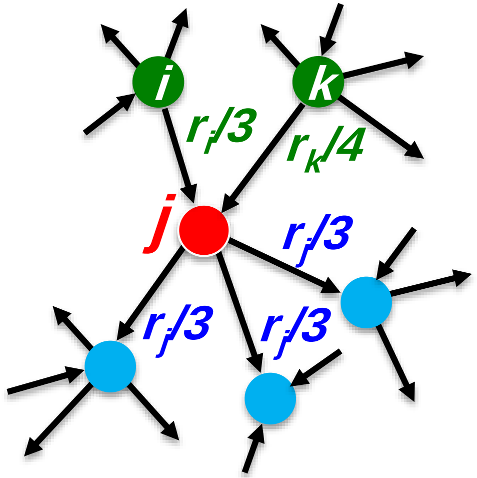


55

Simple Recursive Formulation

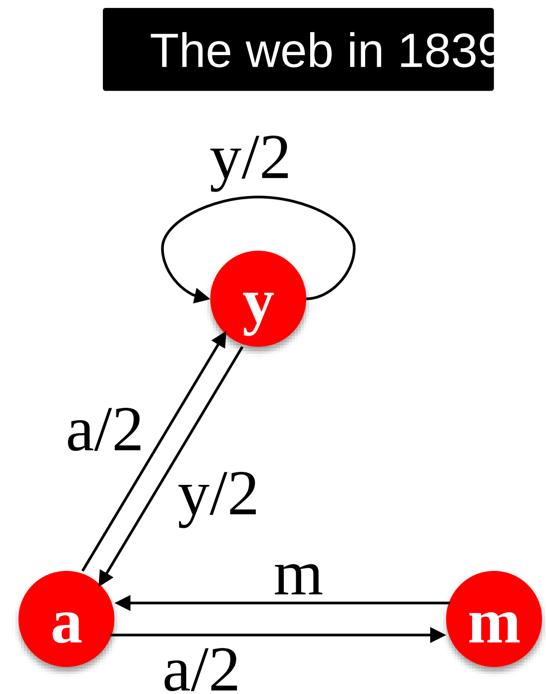
- Each link's vote is proportional to the **importance** of its source page
- If page j with importance r_j has n out-links, each link gets r_j / n votes
- Page j's own importance is the sum of the votes on its in-links

$$r_j = r_i/3 + r_k/4$$



PageRank: The "Flow" Model

- A "vote" from an important page is worth more
- A page is important if it is pointed to by other important pages
- Define a "rank" r_j for page j



$$r_j = \sum_{i \to j} \frac{r_i}{\mathrm{d_i}}$$

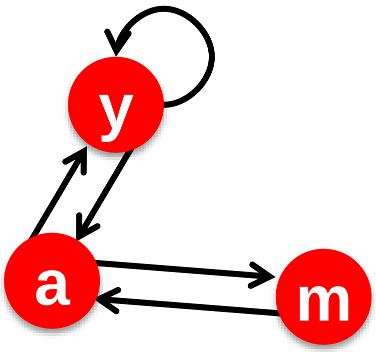
"Flow" equations:

$$r_y = r_y/2 + r_a/2$$
$$r_a = r_y/2 + r_m$$
$$r_m = r_a/2$$

d_i … out-degree of node i

PageRank: How to solve?

- **Power Iteration:**
  - Set $r_j = 1/N$
  - **1:** $r'_j = \sum_{i \to j} \frac{r_i}{d_i}$
  - **2:** $r = r'$
  - Goto **1**
- **Example:**



	y	a	m
y	½	½	0
a	½	0	1
m	0	½	0

$$r_y = r_y/2 + r_a/2$$
$$r_a = r_y/2 + r_m$$
$$r_m = r_a/2$$

$$\begin{bmatrix} r_y \\ r_a \\ r_m \end{bmatrix} = \begin{matrix} 1/3 & 1/3 & 5/12 & 9/24 & & 6/15 \\ 1/3 & 3/6 & 1/3 & 11/24 & \ldots & 6/15 \\ 1/3 & 1/6 & 3/12 & 1/6 & & 3/15 \end{matrix}$$

Iteration 0, 1, 2, …

The Google Matrix

- **PageRank equation** [Brin-Page, '98]

$$r_j = \sum_{i \to j} \beta \frac{r_i}{d_i} + (1-\beta)\frac{1}{N}$$

- **What is β?**
  - In practice $\beta = 0.85$

$[1/N]_{NxN}$…N by N matrix where all entries are 1/N

- **The Google Matrix *R*:**

$$R = \beta\, \boldsymbol{\ell} R + (1-\beta)\left[\frac{1}{N}\right]_{N\times N}$$

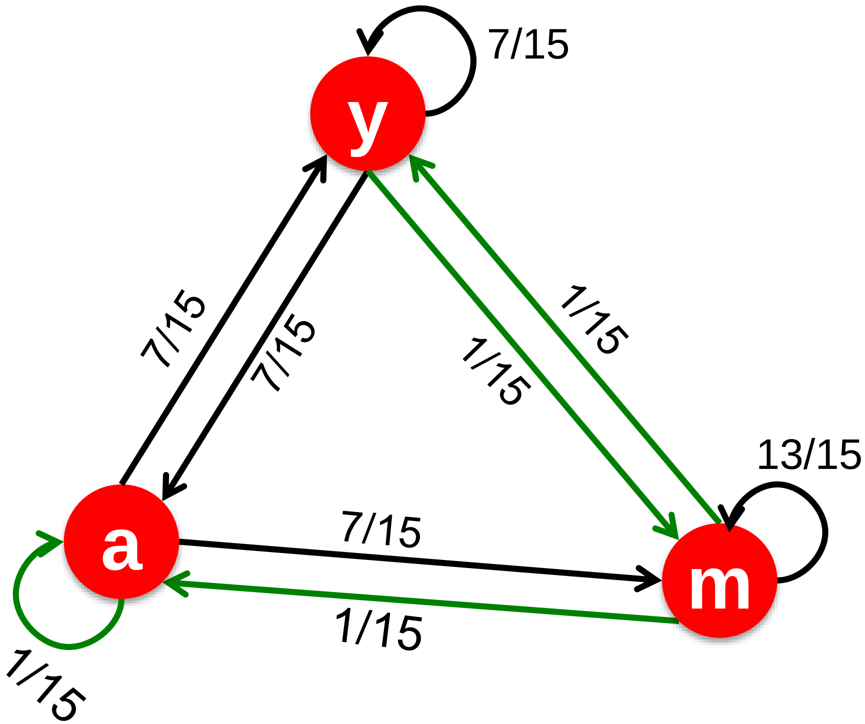
- **We have a recursive problem**

$$\mathbf{R} = \begin{bmatrix} PR(p_1) \\ PR(p_2) \\ \vdots \\ PR(p_N) \end{bmatrix} \qquad \begin{bmatrix} \ell(p_1,p_1) & \ell(p_1,p_2) & \cdots & \ell(p_1,p_N) \\ \ell(p_2,p_1) & \ddots & & \vdots \\ \vdots & & \ell(p_i,p_j) & \\ \ell(p_N,p_1) & \cdots & & \ell(p_N,p_N) \end{bmatrix}$$

$$\sum_{j=1}^{N} \boldsymbol{\ell}(p_i, p_j) = 1$$

If $\boldsymbol{\ell}(p_i, p_j) > 0$, then p_i links to p_j

Random Teleports ($\beta = 0.8$)



$$0.8 \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} + 0.2 \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

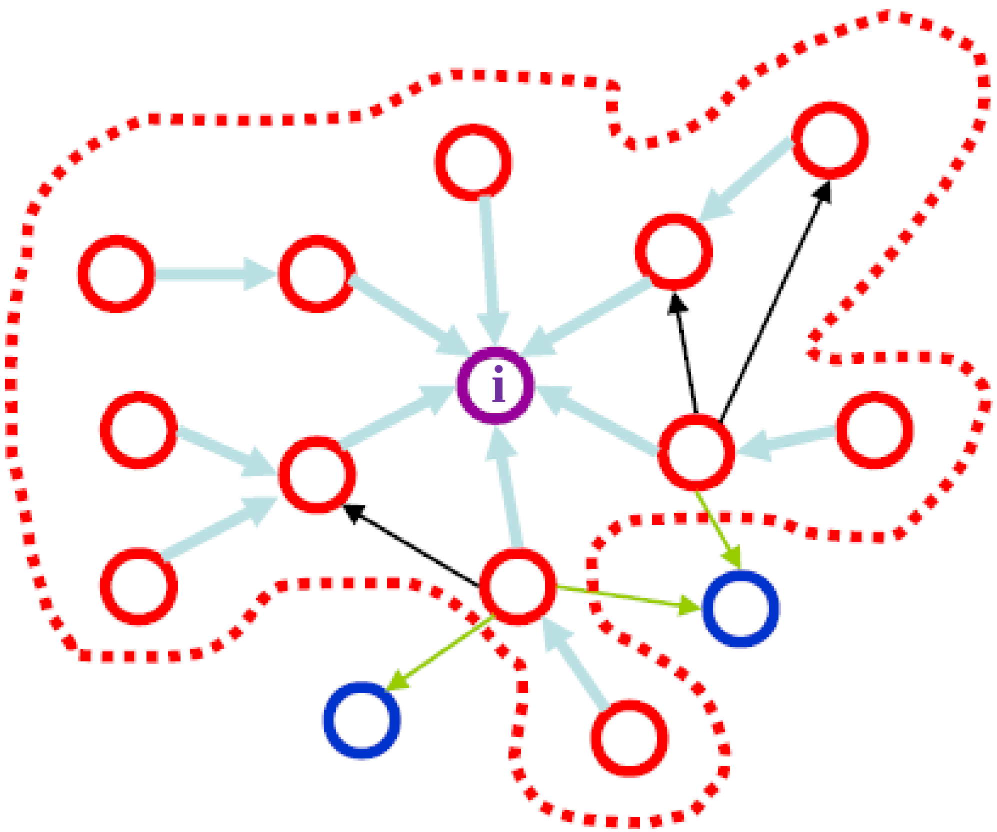
$$\begin{array}{c} y \\ a \\ m \end{array} \begin{bmatrix} 7/15 & 7/15 & 1/15 \\ 7/15 & 1/15 & 1/15 \\ 1/15 & 7/15 & 13/15 \end{bmatrix}$$

A

$$\begin{array}{c} y \\ a \\ m \end{array} = \begin{array}{c} 1/3 \\ 1/3 \\ 1/3 \end{array} \quad \begin{array}{c} 0.33 \\ 0.20 \\ 0.46 \end{array} \quad \begin{array}{c} 0.24 \\ 0.20 \\ 0.52 \end{array} \quad \begin{array}{c} 0.26 \\ 0.18 \\ 0.56 \end{array} \quad \cdots \quad \begin{array}{c} 7/33 \\ 5/33 \\ 21/33 \end{array}$$

Influence range in directed networks

- The influence range of i is the set of vertices who are reachable from the node i
- Alternatively, we can also consider the influential range of node i as a set of the nodes with a path to i



Network entropy (of degree distribution)

- Entropy is an indicator of disorder
- The more the disorder, the more the entropy
- The entropy of the degree distribution provides an average measurement of the heterogeneity of the network

$$H = -\sum_{k} P(k) \log P(k)$$

- P(k) the probability network
- The maximum value of entropy is obtained for a uniform degree distribution
- The minimum value $H_{min} = 0$ is achieved whenever all nodes have the same degree

Vulnerability

- It is important to know which component (nodes or edges) are crucial to the best performance
- The more the drop in the efficiency by removing a component the more crucial that component
- Degree (hub node) might be a criterion
- Only degree is not enough, e.g. all vertices of a binary tree network have equal degree, i.e. no hub, but disconnection of vertices closer to the root and the root itself have a greater impact than of those near the leaves.
- The amount of change in the efficiency (or other network properties) as a component is removed can be an indicator of the vulnerability

Vulnerability

$$V_i = \frac{E - E_i}{E}$$

- where V_i is the vulnerability of component i and E_i is the efficiency the networks by removed that component.

$$V = \max_i V_i$$

- V can be regarded as the vulnerability of the network
- the ordered distribution of nodes with respect to their vulnerability V_i is related to the network hierarchy
- The most vulnerable (critical) node occupies the highest position in the network hierarchy
- The same is also true for the edges

Disconnecting and cut sets

- How many edges or nodes must be removed in order to disconnect an originally connected graph?
- If a node is removed then all edges joining it will also be removed
- But, the converse may not be true, i.e. an edge may removed without necessarily removing the nodes tipping to that
- **Disconnecting set:** a set of edges $E_o(G)$, after it is being removed, the graph G will be become disconnected
- **Cut set**: the smallest disconnecting set, i.e. No proper subset of which is a disconnecting set

Readings

- Newman, Mark. ***Networks: an introduction***. Oxford University Press, 2010. (Ch. 7)
- Easley, David, and Jon Kleinberg. **Networks, crowds, and markets: Reasoning about a highly connected world**. Cambridge University Press, 2010. (Ch. 3)
- L. da F. Costa, F. A. Rodrigues, G. Travieso, and P. R. Villas Boas. **Characterization of complex networks: A survey of measurements**. Advances in Physics, 56(1):167 – 242, 2007.