Complex Network Theory

Lecture 2-1

Basic network concepts and metrics

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Thanks A. Rezvanian

A. Barabasi, L. Adamic and J. Leskovec

Outline

- Overview of class topics
  - Basic definitions
  - Basic concepts
  - Representation
  - Metrics
  - Measures
  - Centralities
- Next class:
  - Network centralities and metrics

Understanding large graphs

- What are the statistics of real life networks?
- In which terms we can describe the networks?
- How we can measure a large network?
- Can we explain how the networks were generated?
- Can we make models for network construction?
- To how much extent do the artificially
- constructed networks describe real networks?

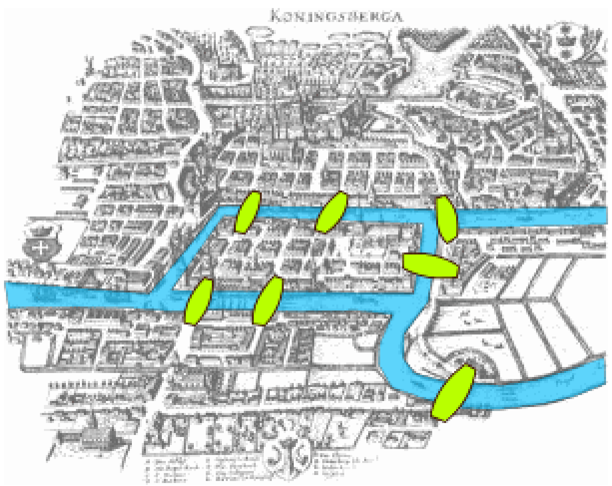
- **First step: Introducing network metrics**

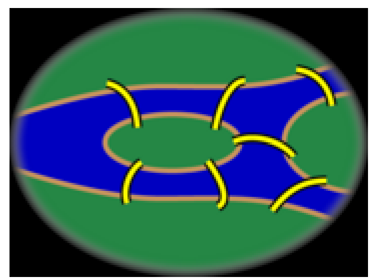
Networks became hot topic !

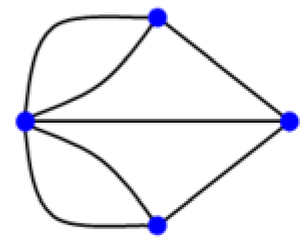
- Around 1999
  - Watts and Strogatz, Collective dynamics of small-world networks
  - Faloutsos[3], On power-law relationships of the Internet Topology
  - Kleinberg et al., The Web as a graph
  - Barabasi and Albert, The emergence of scaling in real networks

History: Graph theory

- Euler's **Seven Bridges of Königsberg** – one of the first problems in graph theory
- Is there a route that crosses each bridge only once and returns to the starting point?







Source: http://en.wikipedia.org/wiki/Seven_Bridges_of_Königsberg

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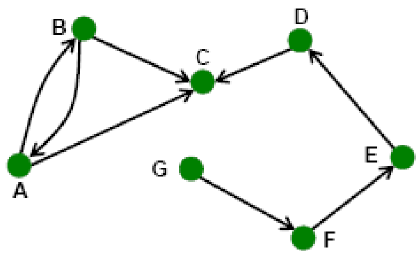
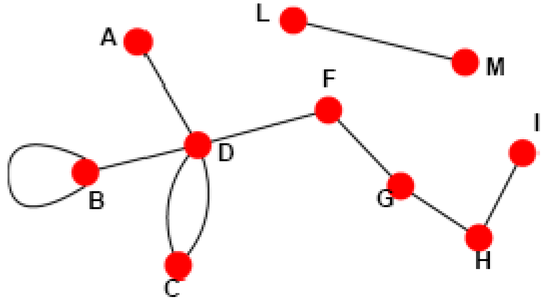
Network elements: edges

- Directed (also called arcs)-asymmetrical relations
  - A $\rightarrow$ B
    - A likes B, A gave a gift to B, A is B’s child,
    - A call B, A follows B



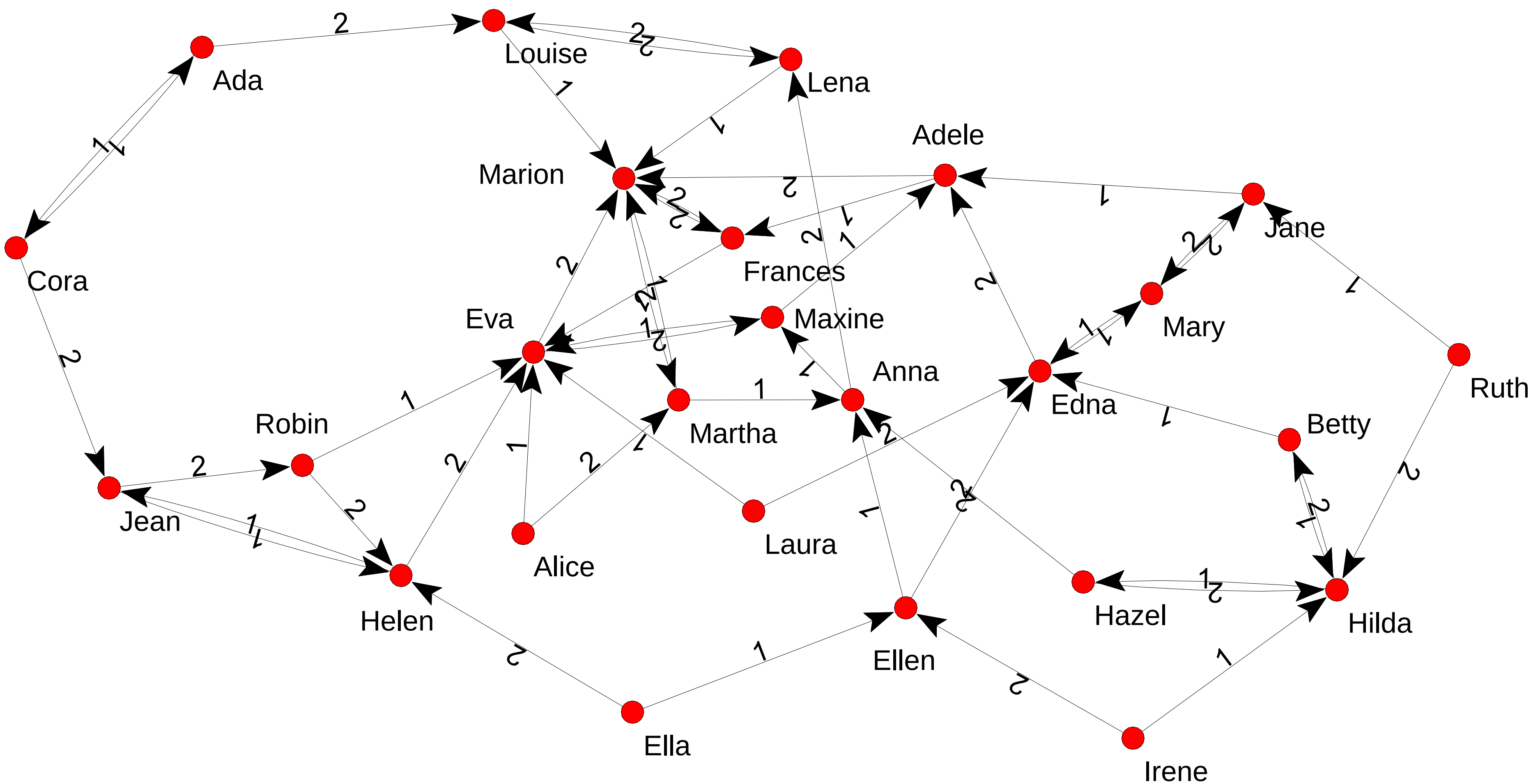
- Undirected (symmetrical, reciprocal relations)
  - A $\leftrightarrow$ B or A – B
    - A and B like each other
    - A and B are siblings
    - A and B are co-authors
    - A and B are friend



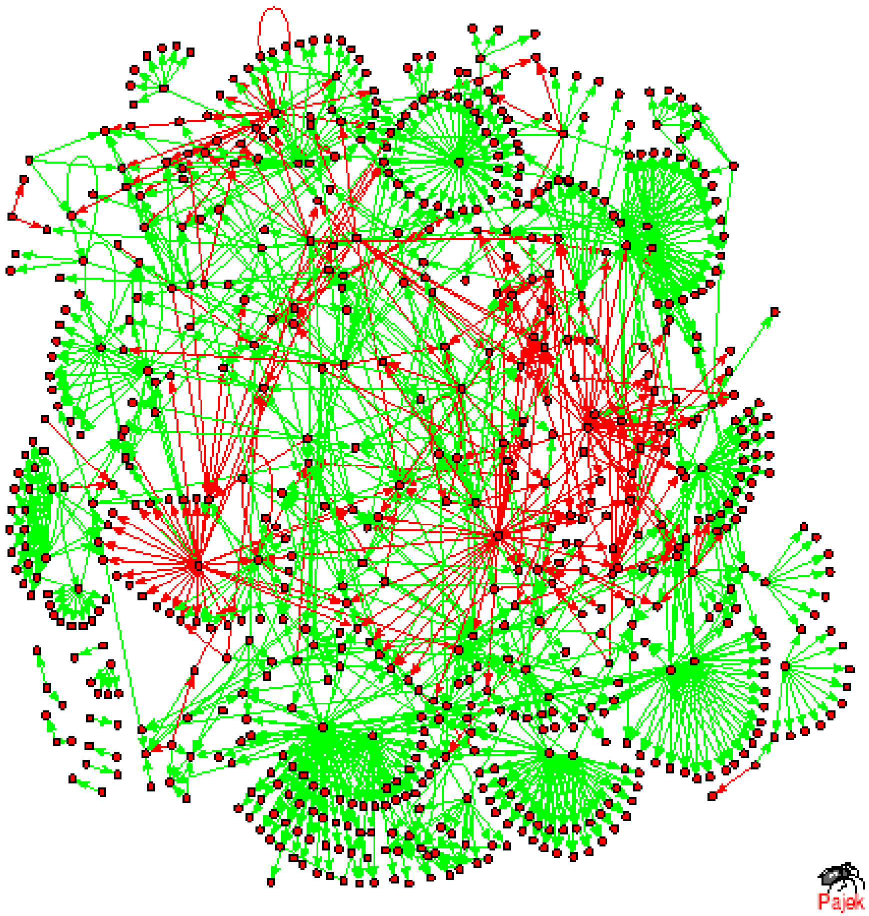
- Edge attributes
  - weight (e.g. frequency of communication)
  - ranking (best friend, second best friend…)
  - type (friend, relative, co-worker)
  - properties depending on the structure of the rest of the graph: e.g. betweenness

Directed networks

- girls' school dormitory dining-table partners (Moreno, *The sociometry reader*, 1960)
- first and second choices shown



Edge weights can have positive or negative values

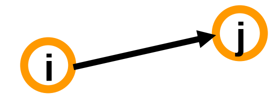


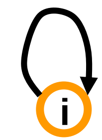
Transcription regulatory network in baker's yeast

- One gene activates/inhibits another
- One person trusting/distrusting another
  - Research challenge: How does one 'propagate' negative feelings in a social network? Is my enemy's enemy my friend?

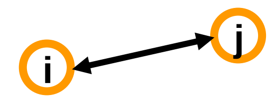
Adjacency matrices

- Representing edges (who is adjacent to whom) as a matrix
  - $\mathbf{A_{ij}}$ = 1 if node i has an edge to node j
    = 0 if node i does not have an edge to j



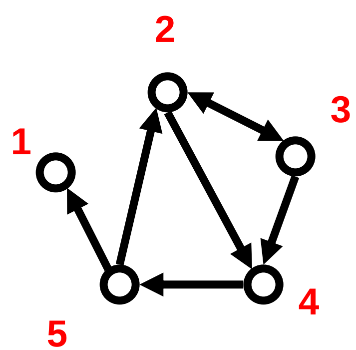


  - $\mathbf{A_{ii}}$ = 0 unless the network has self-loops



  - $\mathbf{A_{ij}} = \mathbf{A_{ji}}$ if the network is undirected,
    or if i and j share a reciprocated edge

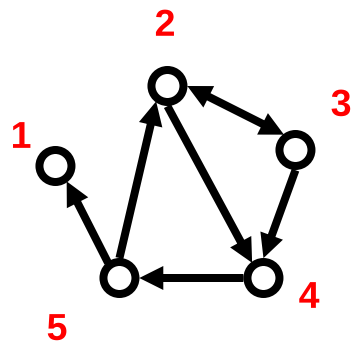
Example:



$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Adjacency lists

- Edge list

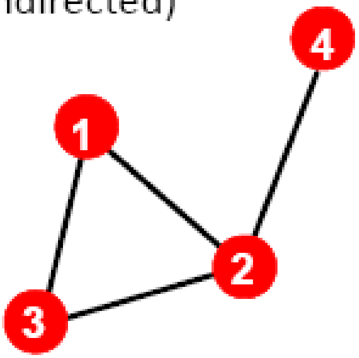


2 3	2,3	2;3
2 4	2,4	2;4
3 2	3,2	3;2
3 4	3,4	3;4
4 5	4,5	4;5
5 2	5,2	5;2
5 1	5,1	5;1

- Adjacency list
  - is easier to work with if network is
    - large
    - sparse
  - quickly retrieve all neighbors for a node
    - 1:
    - 2: 3 4
    - 3: 2 4
    - 4: 5
    - 5: 1 2

More types of graphs

- **Unweighted**
  (undirected)



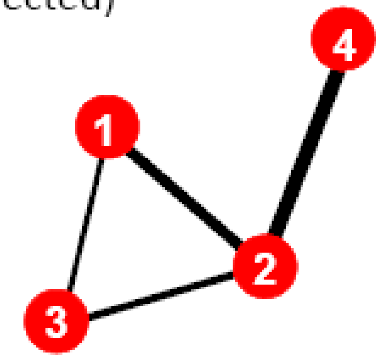
$$A_{ij} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$A_{ii} = 0 \qquad A_{ij} = A_{ji}$$

$$E = \frac{1}{2}\sum_{i,j=1}^{N} A_{ij} \qquad \bar{k} = \frac{2E}{N}$$

Examples: Friendship, Hyperlink

- **Weighted**
  (undirected)



$$A_{ij} = \begin{pmatrix} 0 & 2 & 0.5 & 0 \\ 2 & 0 & 1 & 4 \\ 0.5 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix}$$

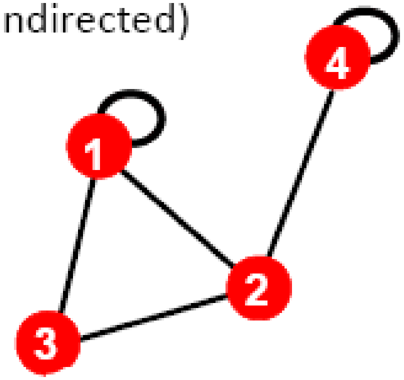
$$A_{ii} = 0 \qquad A_{ij} = A_{ji}$$

$$E = \frac{1}{2}\sum_{i,j=1}^{N} nonzero(A_{ij}) \qquad \bar{k} = \frac{2E}{N}$$

Examples: Collaboration, Internet, Roads

More types of graphs

- **Self-edges (self-loops)**
  (undirected)



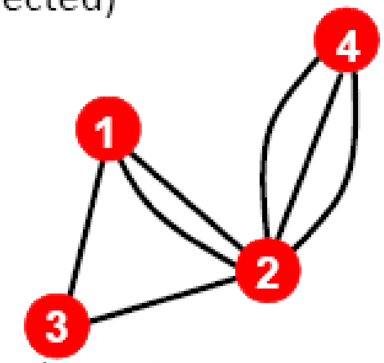
$$A_{ij} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$A_{ii} \neq 0 \qquad A_{ij} = A_{ji}$$

$$E = \frac{1}{2} \sum_{i,j=1, i\neq j}^{N} A_{ij} + \sum_{i=1}^{N} A_{ii} \qquad ?$$

Examples: Proteins, Hyperlink

- **Multigraph**
  (undirected)



$$A_{ij} = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 3 \\ 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix}$$

$$A_{ii} = 0 \qquad A_{ij} = A_{ji}$$

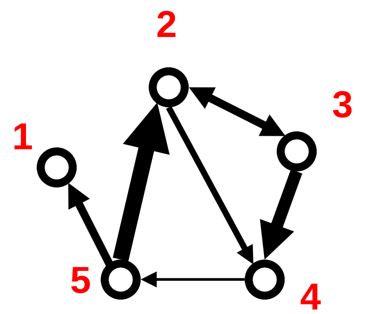
$$E = \frac{1}{2} \sum_{i,j=1}^{N} nonzero(A_{ij}) \qquad \bar{k} = \frac{2E}{N}$$

Examples: Communication, Collaboration

Weighted Graph

- For weighted directed network the in-strength and out-strength are defined
- The strength distribution of the graph is also correspondingly defined

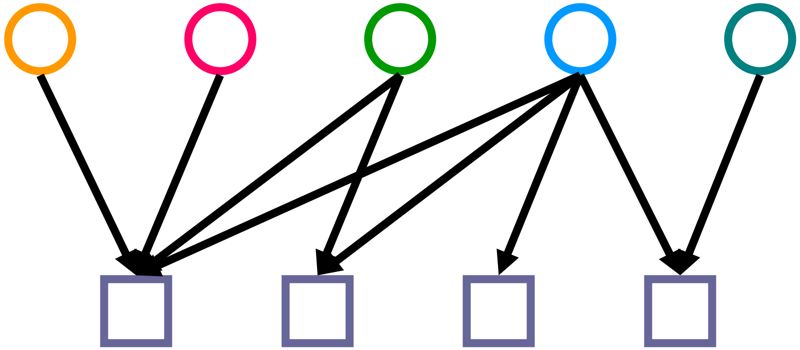
2 3 5	2;3; 5
2 4 5	2;4; 5
3 2 5	3;2; 5
3 4 7	3;4; 7
4 5 3	4;5; 3
5 2 9	5;2; 9
5 1 5	5;1; 5



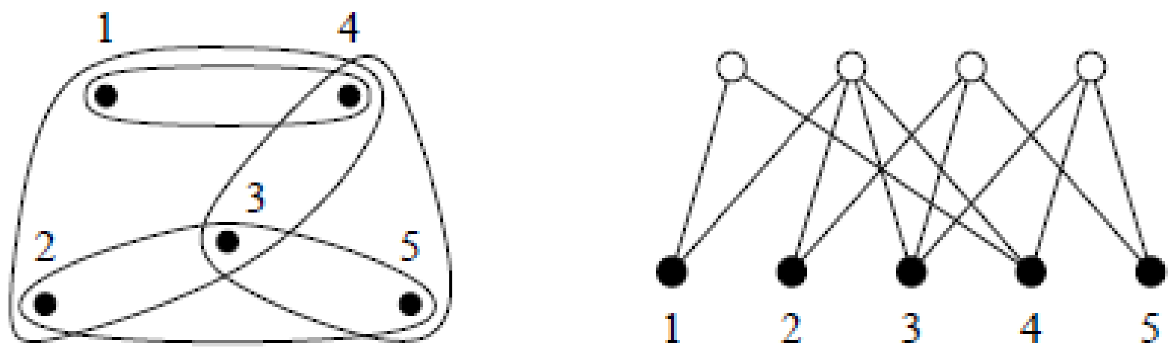
Weighted network

bipartite (two-mode) networks

- edges occur only between two groups of nodes, not within those groups
- for example, we may have individuals and *events*
  - directors and boards of directors
  - customers and the items they purchase
  - metabolites and the reactions they participate in



A hypergraph and corresponding bipartite graph



(a) (b)

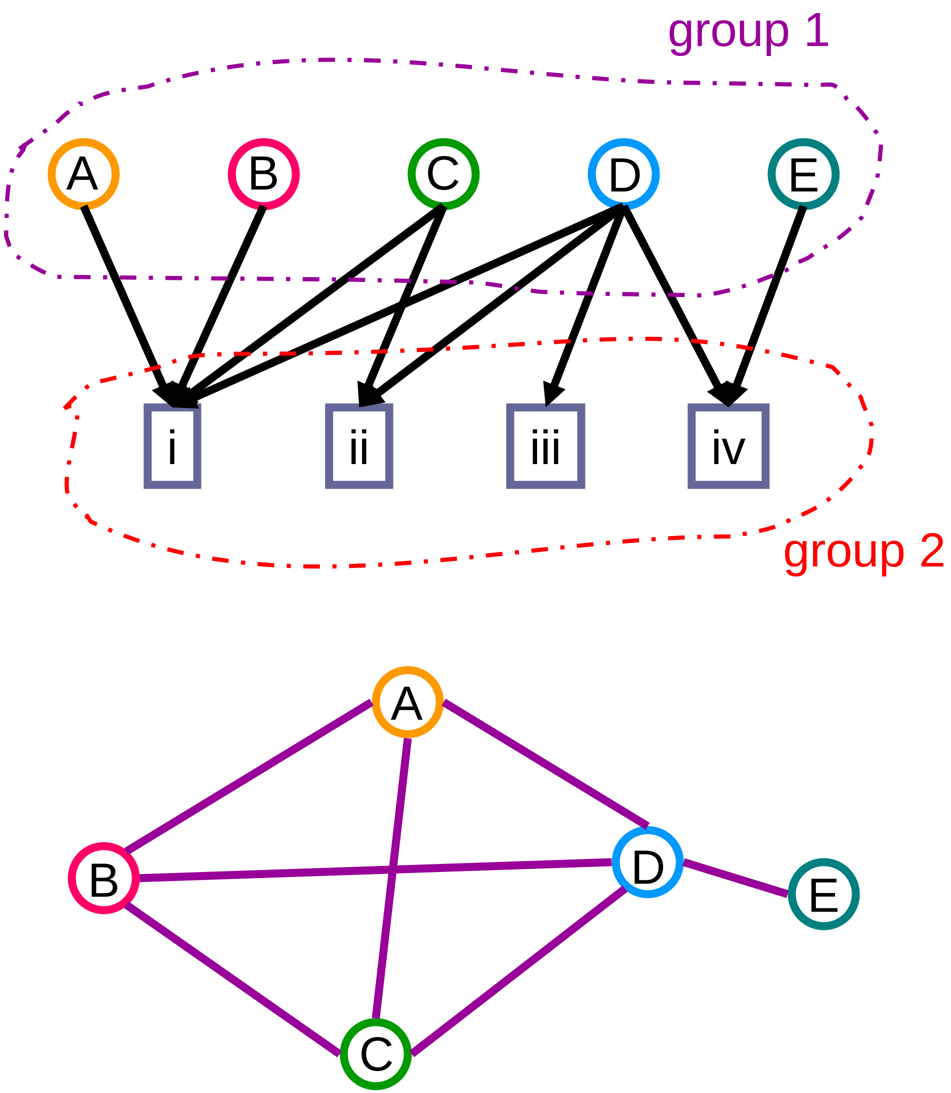
(a) And (b) show the same information

The membership of five vertices in four different groups.

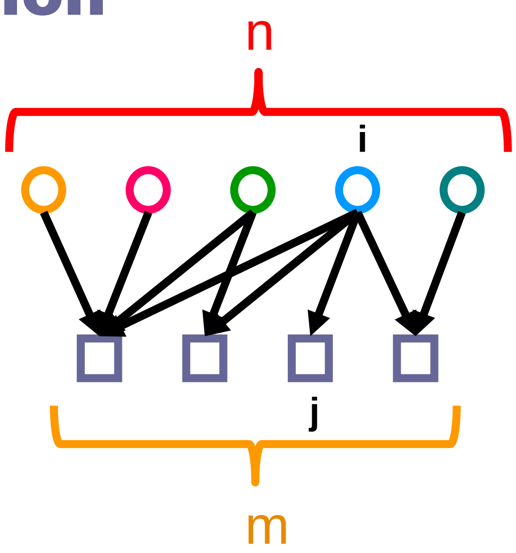
- (a) Hypergraph representation: groups are represented as hyper-edges (loops circling sets of vertices).
- (b) Bipartite representation

going from a bipartite to a one-mode graph

- Two-mode network
- One mode projection
  - two nodes from the first group are connected if they link to the same node in the second group
  - some loss of information
  - naturally high occurrence of cliques



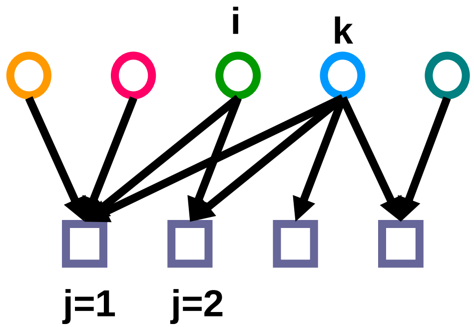
Now in matrix notation



- B_{ij}
  - = 1 if node i from the first group links to node j from the second group
  - = 0 otherwise
- B is usually not a square matrix!
  - for example: we have n customers and m products

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Collapsing to a one-mode network



- *i* and *k* are linked if they both link to *j*
- $A_{ik} = \sum_j B_{ij} B_{kj}$ ➔ $A = B.B^T$
  - B^T swaps B_{xy} and B_{yx}
  - if B is an $n \times m$, B^T is an $m \times n$
- A_{ii} is equal to the number of groups to which vertex *i* belongs
- $A' = B^T B$?

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

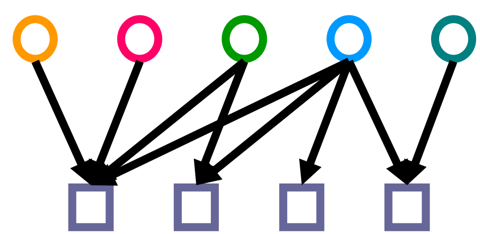
$$B^T = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

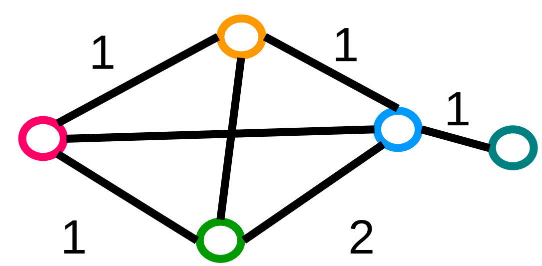
Matrix multiplication

- general formula for matrix multiplication $Z_{ij} = \sum_k X_{ik} Y_{kj}$
- let $Z = A$, $X = B$, $Y = B^T$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 2 & 0 \\ 1 & 1 & 2 & 4 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 1*1 + 1*1 + 1*0 + 1*0 = 2$$

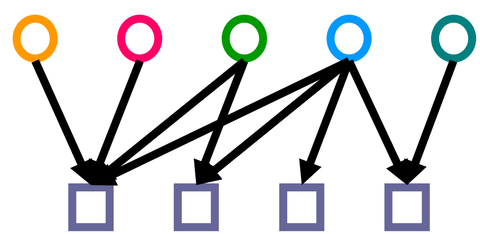


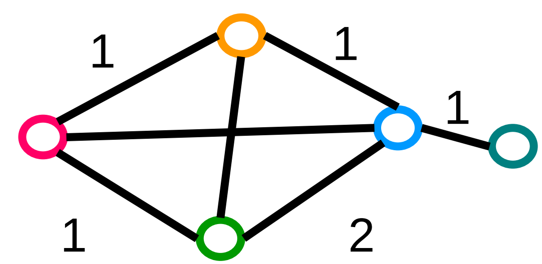


Collapsing a two-mode network to a one mode-network

- Assume the nodes in group 1 are people and the nodes in group 2 are movies
- The diagonal entries of A give the number of movies each person has seen
- The off-diagonal elements of A give the number of movies that both people have seen
- A is symmetric

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 2 & 0 \\ 1 & 1 & 2 & 4 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$





Readings

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- Newman, Mark. **Networks: an introduction**. Oxford University Press, 2010. (Ch. 6)

- L. da F. Costa, F. A. Rodrigues, G. Travieso, and P. R. Villas Boas. **Characterization of complex networks: A survey of measurements**. Advances in Physics, 56(1):167 – 242, 2007.