# Complex Network Theory

**Lecture 7**

## Scale free networks

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Spring 2018

Thanks A. Rezvanian
A. Barabasi, L.Adamic,

# Outline

- Heavy Tail distributions
- Power law distributions
- Scale free networks
- 20/80 rule
- What kinds of processes generate power laws?

- Next class:
  - Community structure

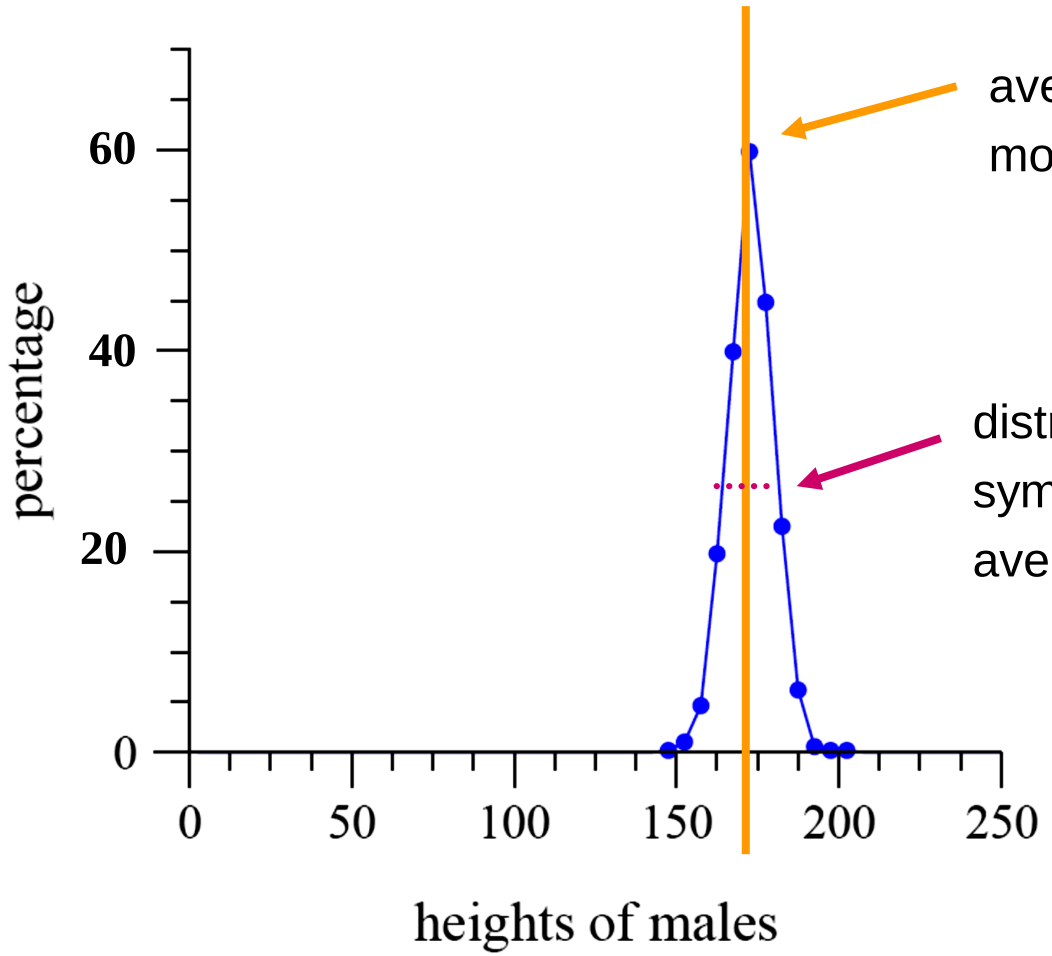
# What is a heavy tailed-distribution?

- Right skew
  - Normal distribution (not heavy tailed)
    - e.g. heights of human males: centered around 180cm
  - Zipf's or power-law distribution (heavy tailed)
    - e.g. city population sizes: Tehran 12 million, but many, many small towns
- High ratio of max to min
  - human heights
    - tallest man: 272cm, shortest man: 56 cm *ratio: 4.8*
      from the Guinness Book of world records
  - city sizes
    - Tehran: pop. 12 million, a village pop. 78, *ratio: 150,000*

# The Heavy Tail

- The power law distribution implies an “infinite variance”
  - The “area” of “big ks” in an exponential distribution tend to zero with $k \rightarrow \infty$
  - This is not true for the power law distribution, implying an infinite variance
- In other words, the power law implies that
  - The probability to have elements very far from the average is not negligible
- Using an exponential distribution
  - The probability for a Web page to have more than 100 incoming links, considering the average number of links for page, would be less in the order of $1^{-20}$
  - which contradicts the fact that we know a lot of “well linked” sites

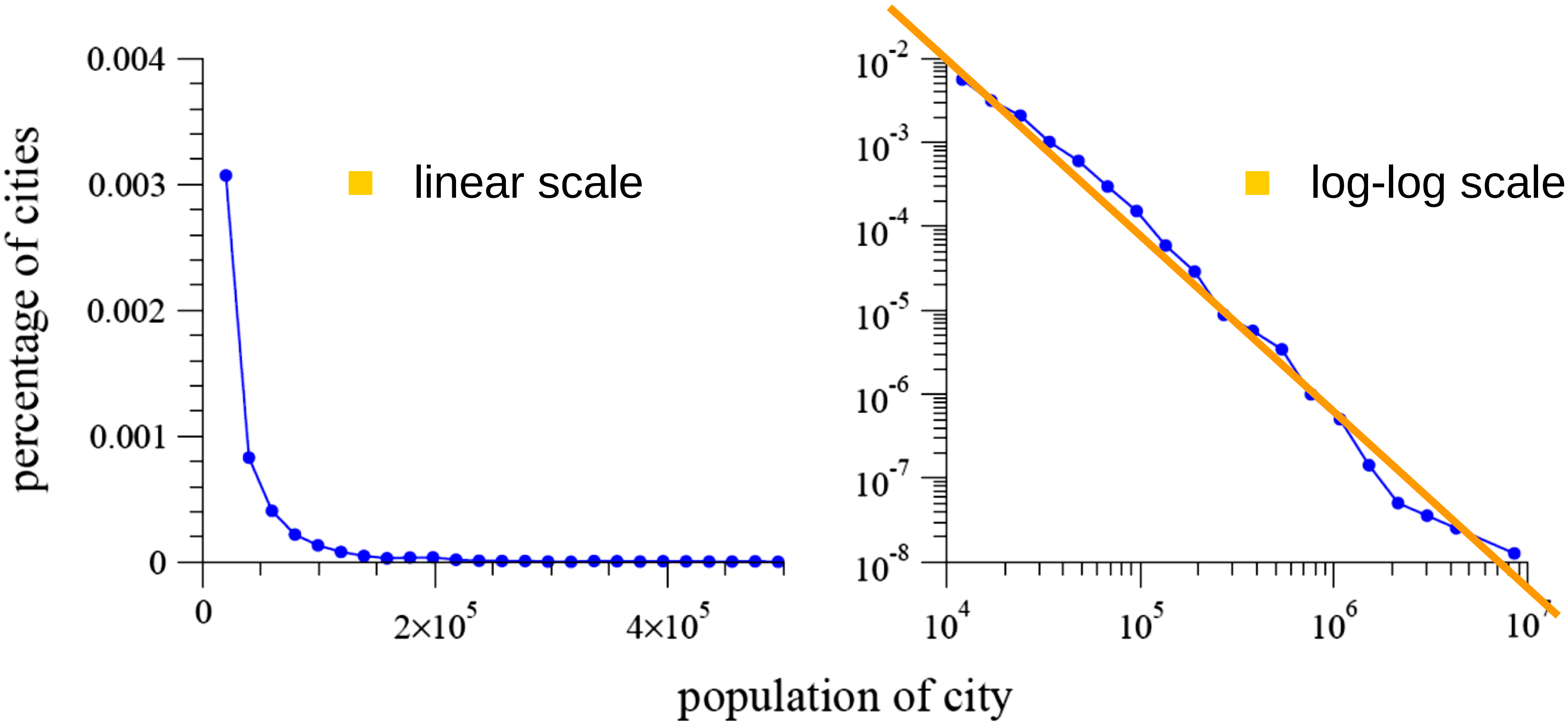
# Normal (also called Gaussian) distribution of human heights




average value close to most typical

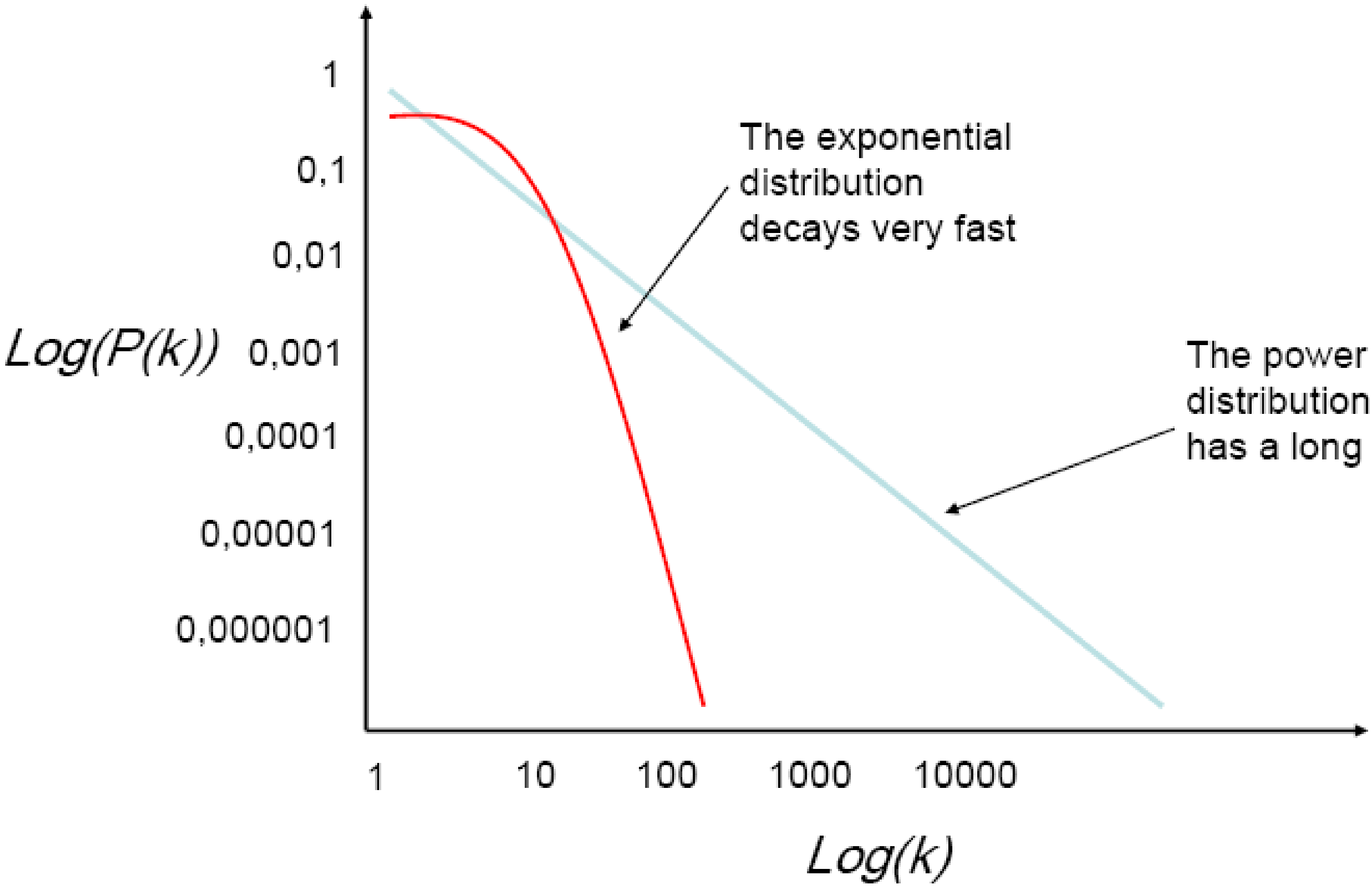
distribution close to symmetric around average value

# Power-law distribution



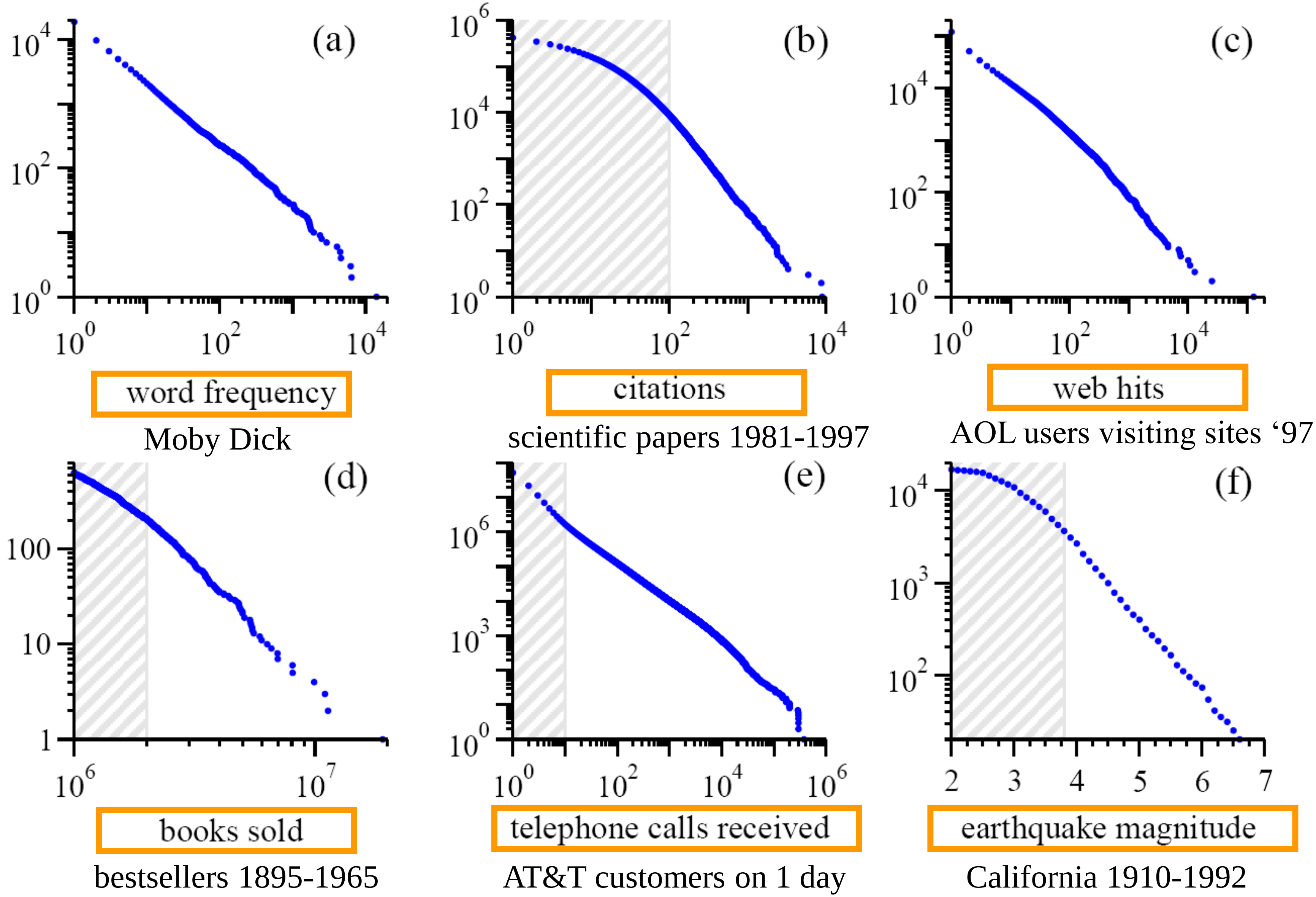

- high skew (asymmetry)
- straight line on a log-log plot

# Power-law vs. Exponential distribution



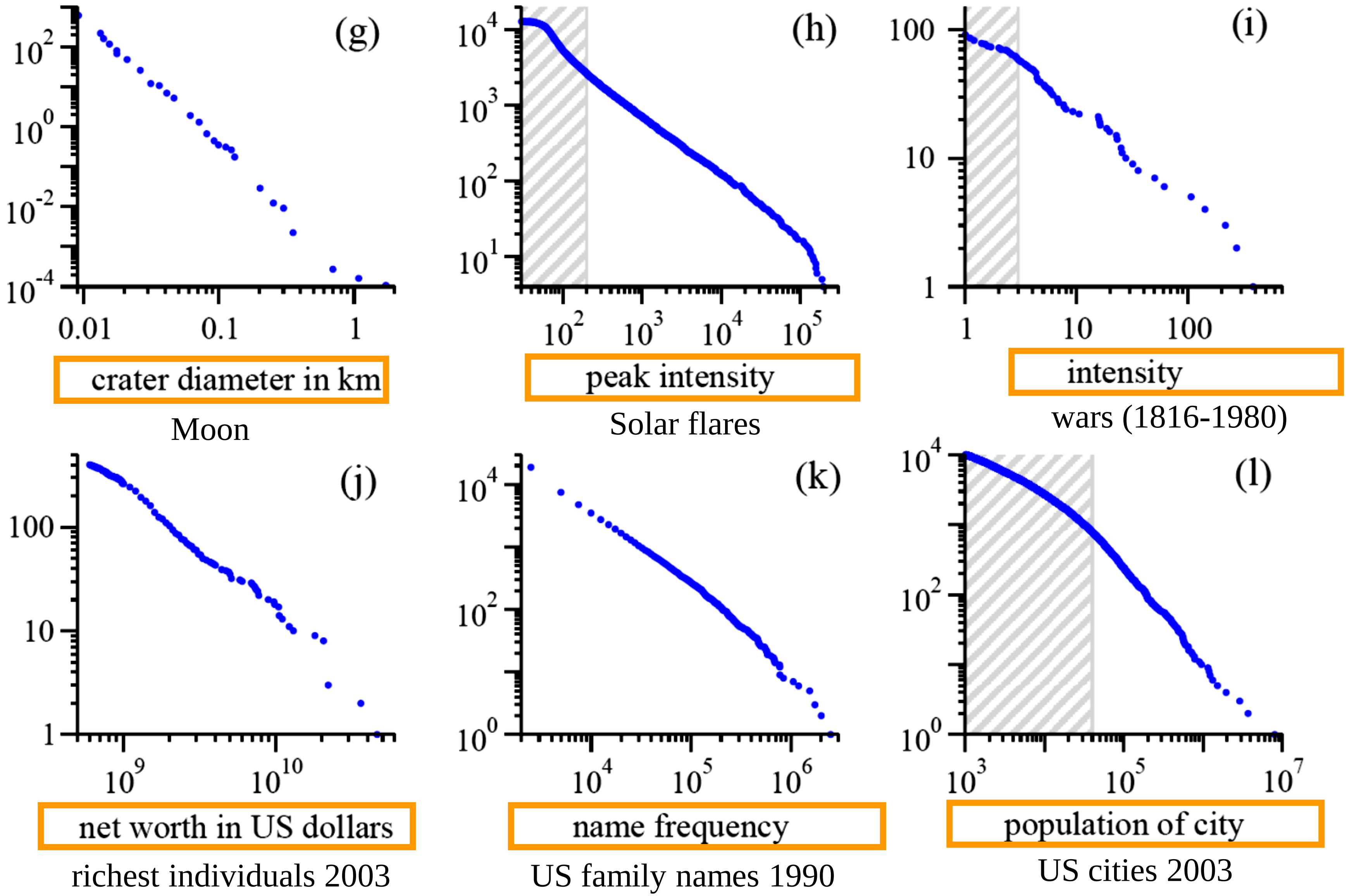
# Power laws are seemingly everywhere

## note: these are cumulative distributions, more about this in a bit...




**Source:** MEJ Newman, 'Power laws, Pareto distributions and Zipf's law', Contemporary Physics 46, 323–351 (2005)

# Yet more power laws




**Source:** MEJ Newman, 'Power laws, Pareto distributions and Zipf's law', Contemporary Physics 46, 323–351 (2005)

# The Ubiquity of the Power Law

- The previous table includes not only technological networks
  - Most real systems and events have a probability distribution that
    - Does not follow the “normal” distribution
    - and obeys to a power law distribution
- Examples, in addition to technological and social networks
  - The distribution of size of files in file systems
  - The distribution of network latency in the Internet
  - The networks of protein interactions (a few protein exists that interact with a large number of other proteins)
  - The power of earthquakes: statistical data tell us that the power of earthquakes follow a power-law distribution
  - The size of rivers: the size of rivers in the world is power law
  - The size of industries, i.e., their overall income
  - The richness of people
  - In these examples, the exponent of the power law distribution is always around 2.5
- The power law distribution is the “normal” distribution for complex systems (i.e., systems of interacting autonomous components)
  - We see later how it can be derived…

# The 20-80 Rule

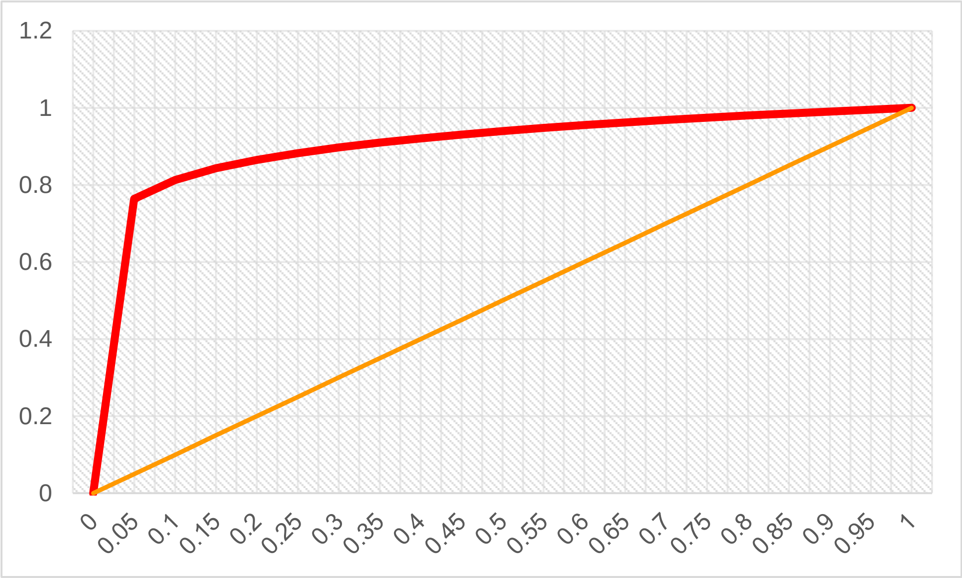
- It's a common "way of saying"
  - But it has scientific foundations
  - For all those systems that follow a power law distribution
- Examples
  - The 20% of the Web sites gests the 80% of the visits (actual data: 15%-85%)
  - The 20% of the Internet routers handles the 80% of the total Internet traffic
  - The 20% of world industries hold the 80% of the world's income
  - The 20% of the world population consumes the 80% of the world's resources
  - The 20% of the Italian population holds the 80% of the lands (that was true before the Mussolini fascist regime, when lands redistribution occurred)
  - The 20% of the earthquakes caused the 80% of the victims
  - The 20% of the rivers in the world carry the 80% of the total sweet water
  - The of the proteins handles the of the most critical metabolic processes
- Does this derive from the power law distribution? YES!

# 80/20 rule

- The fraction W of the wealth in the hands of the richest P of the population is given by

$$W = P^{(\alpha-2)/(\alpha-1)}$$

- Example: US wealth: $\alpha = 2.1$
  - richest 20% of the population holds 86% of the wealth

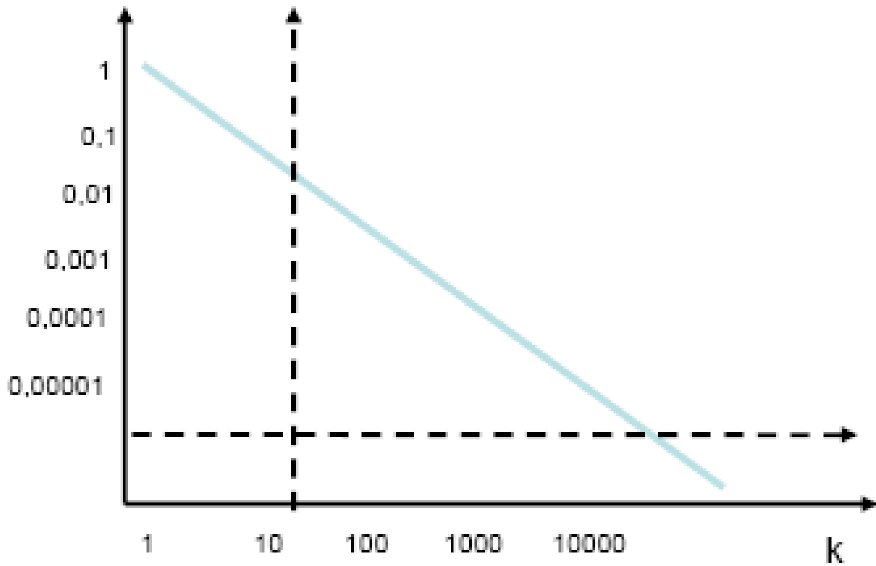


# Hubs and Connectors

- Scale free networks exhibit the presence of nodes that
  - Act as hubs, i.e., as point to which most of the other nodes connects to
  - Act as connectors, i.e., nodes that make a great contributions in getting great portion of the network together
  - "smaller nodes" exists that act as hubs or connectors for local portion of the network
- This may have notable implications, as detailed below

# Why "Scale-Free" Networks

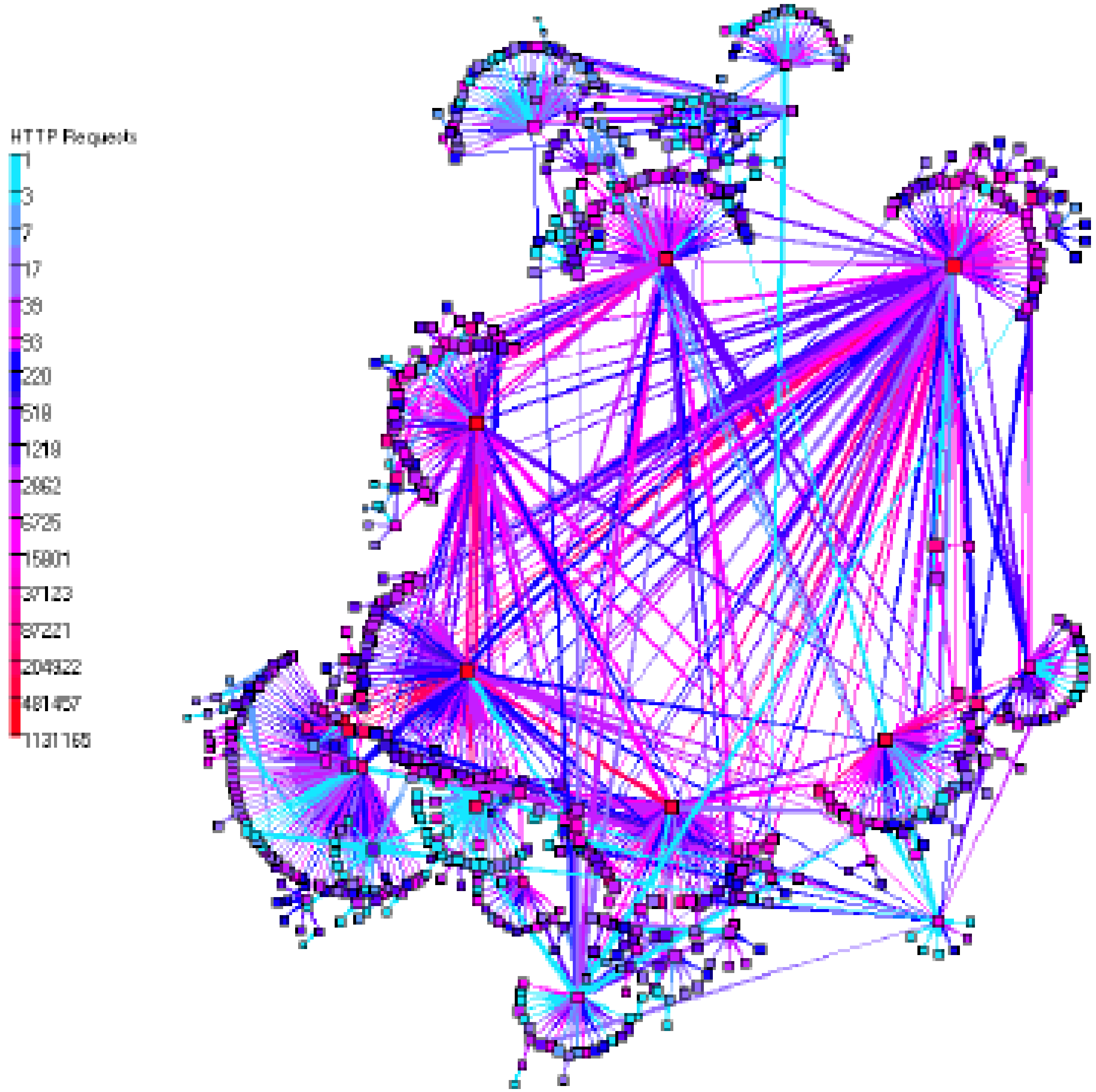
- Why networks following a power law distribution for links are called "**scale free**"?
  - Whatever the scale at which we observe the network
  - The network looks the same, i.e., it looks similar to itself
- The overall properties of the network are preserved independently of the scale




- In particular
  - If we cut off the details of a network – skipping all nodes with a number of links the limited – network will preserve its power-law structure
  - If we consider a sub-portion of any network it have the network, will same overall structure of the whole network

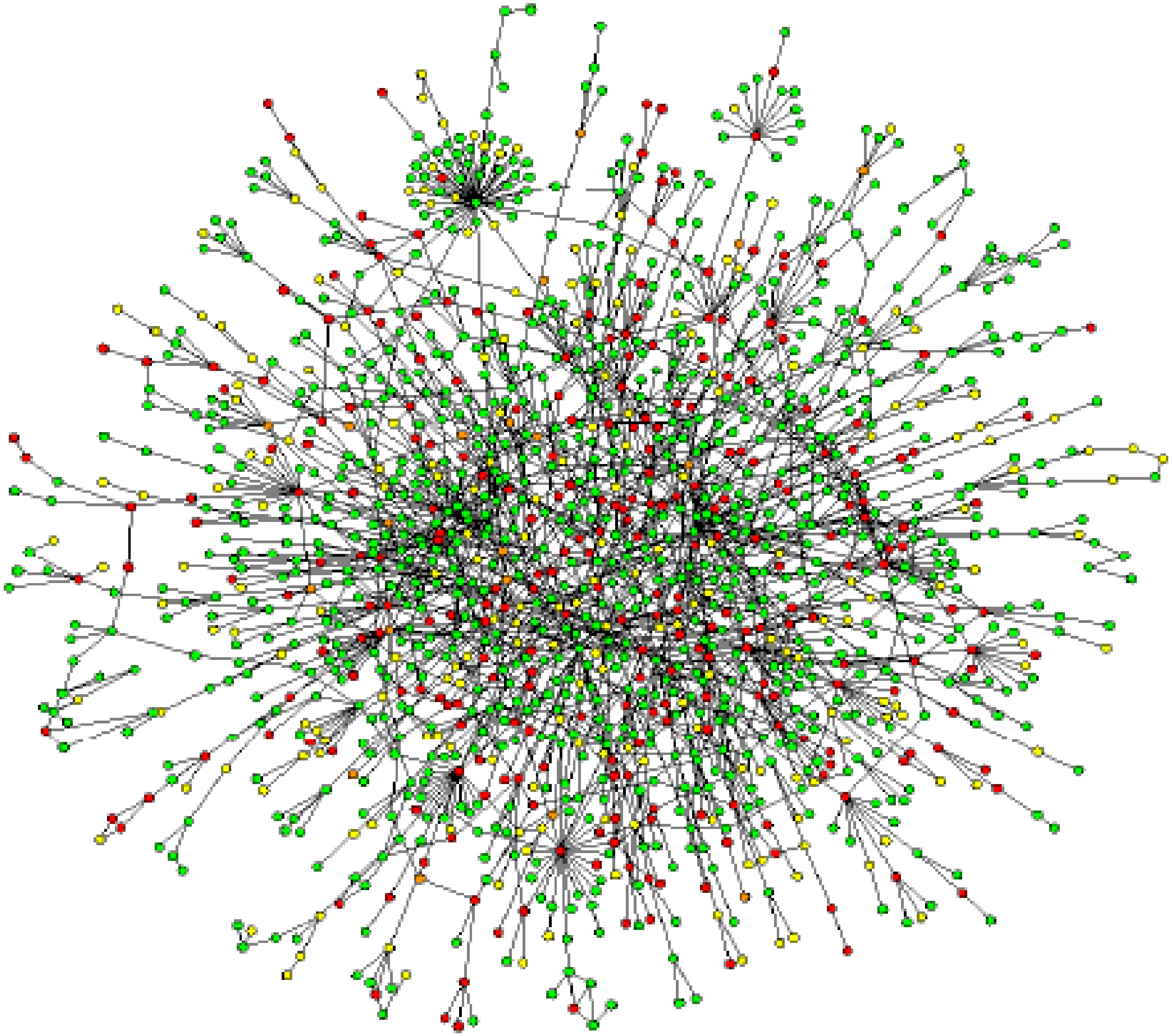
# How do they look like?

Web Cache Network



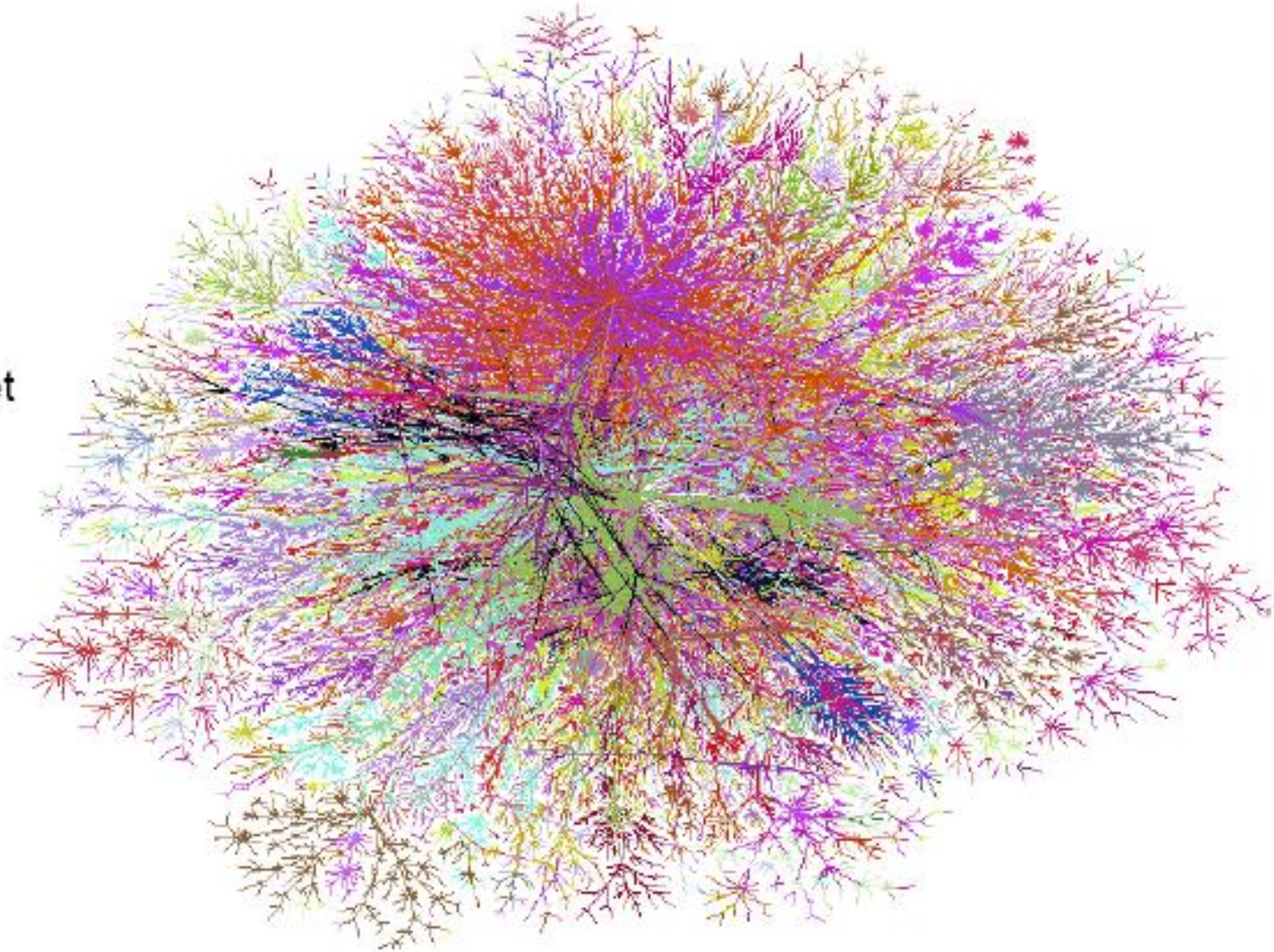
# How do they look like?

Protein
Network



# How do they look like?

The Internet
Routers



# Fractals and Scale Free Networks

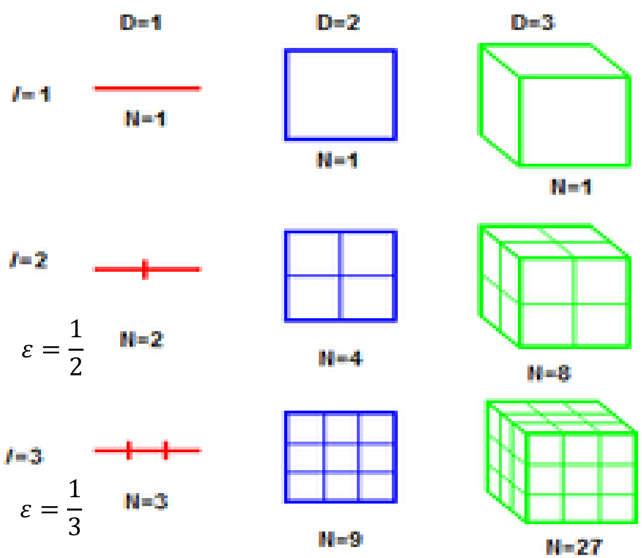
- The nature is made up of mostly “fractal objects”
  - The fractal term derives from the fact that they have a non-integer dimension
  - 2-d objects have a “size” (i.e., a surface) that scales with the square of the linear size A=kL2
  - 3-d objects have a “size” (i.e., a volume) that scales with the cube of the linear size V=kL3
  - Fractal objects have a “size” that scales with some fractions of the linear size S=kLa/b
- Fractal objects have the property of being “self-similar” or “scale-free”
  - Their “appearance” is independent from the scale of observation
  - They are similar to itself independently of whether you look at the from near and from far
  - That is, they are scale-free

# Examples of Fractals

- The Koch snowflake
  - Coastal Regions & River systems
  - Lymphatic systems
  - The distribution of masses in the universe

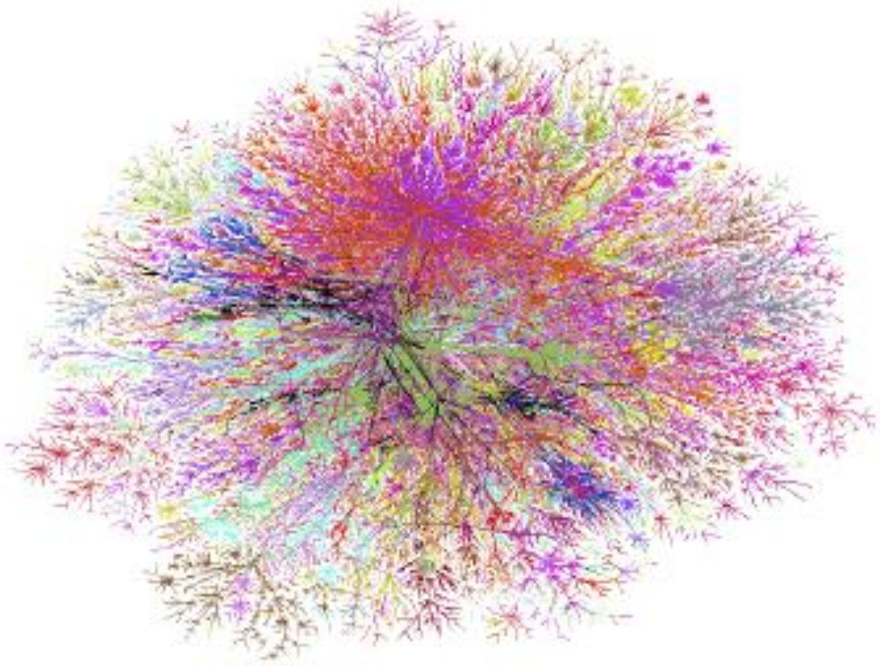
$$\log_{\varepsilon} N = -D = \frac{\log N}{\log \varepsilon}$$

$$\varepsilon = \frac{1}{3}, N = 4 \Rightarrow D = 1.2619$$



# Scale Free Networks are Fractals?

- Yes, in fact:
  - They are the same at whatever dimension we observe them
  - Also, the fact that they grow according to a power law can be considered as a sort of fractal dimension of the network…
- Having a look at the figures clarifies the analogy



# Power law distribution

- Straight line on a log-log plot

$$\ln(p(x)) = c - \alpha \ln(x)$$

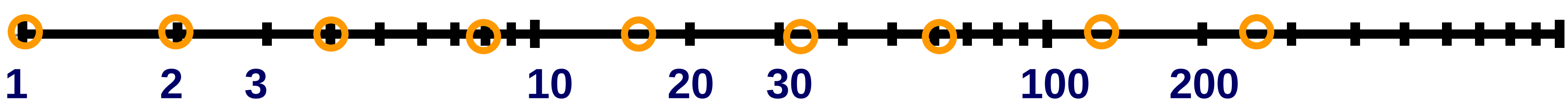
- Exponentiate both sides to get that *p(x)*, the probability of observing an item of size ‘x’ is given by

$$p(x) = Cx^{-\alpha}$$

power law exponent $\alpha$

Normalization constant (probabilities over all *x* must sum to 1)

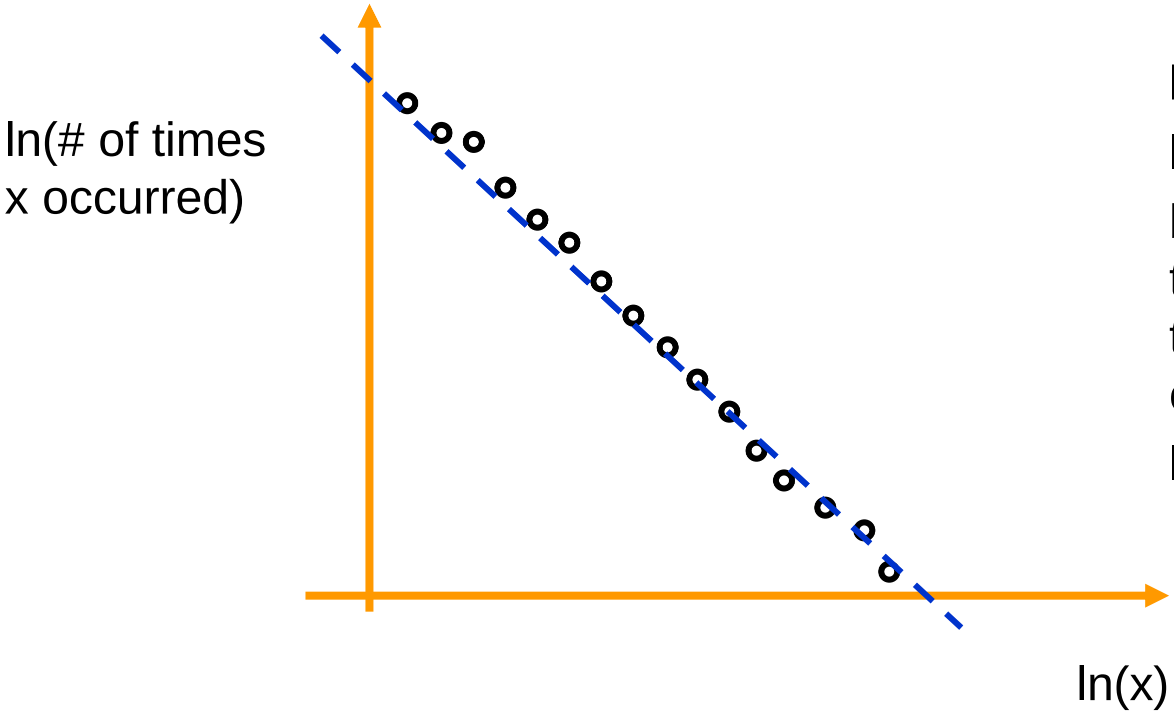
- powers of a number will be uniformly spaced (Logarithmic axes)




- $2^0=1$, $2^1=2$, $2^2=4$, $2^3=8$, $2^4=16$, $2^5=32$, $2^6=64$,….

# Fitting power-law distributions

- Most common and not very accurate method:
  - Bin the different values of x and create a frequency histogram




ln(*x*) is the natural logarithm of x,

but any other base of the logarithm will give the same exponent of a because

$\log_{10}(x) = \ln(x)/\ln(10)$

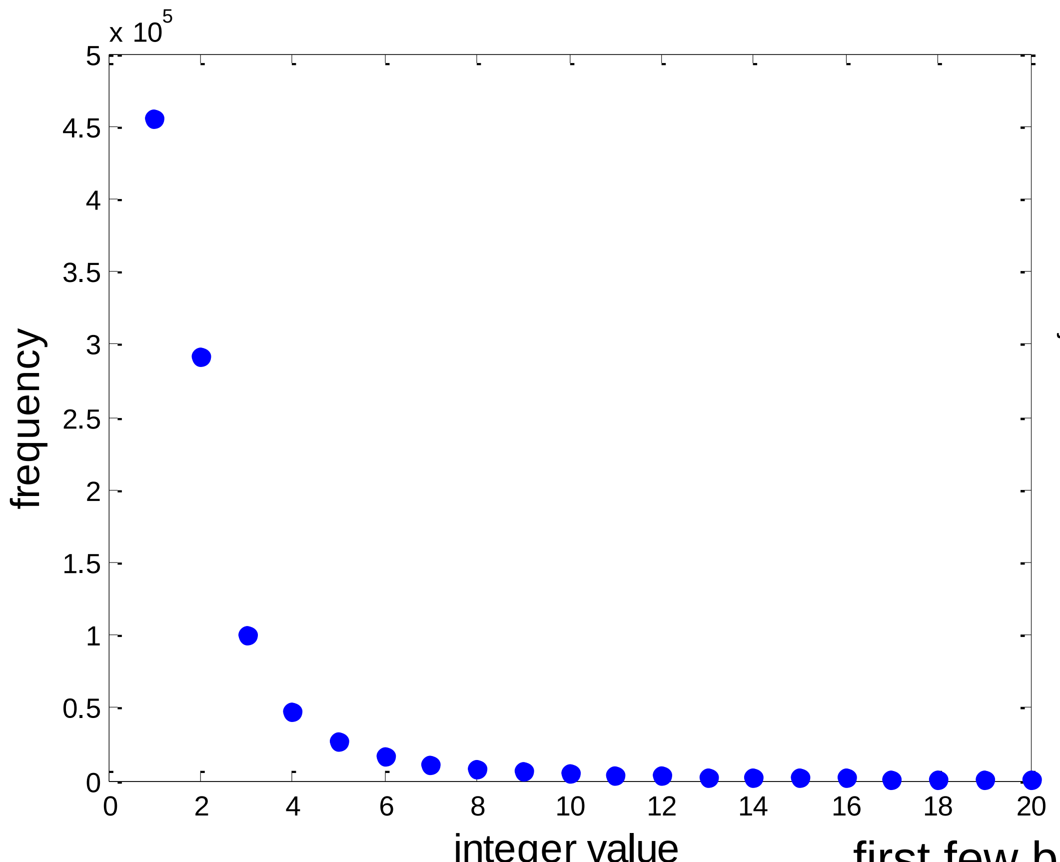
x can represent various quantities, the indegree of a node, the magnitude of an earthquake, the frequency of a word in text

# Example on an artificially generated data set

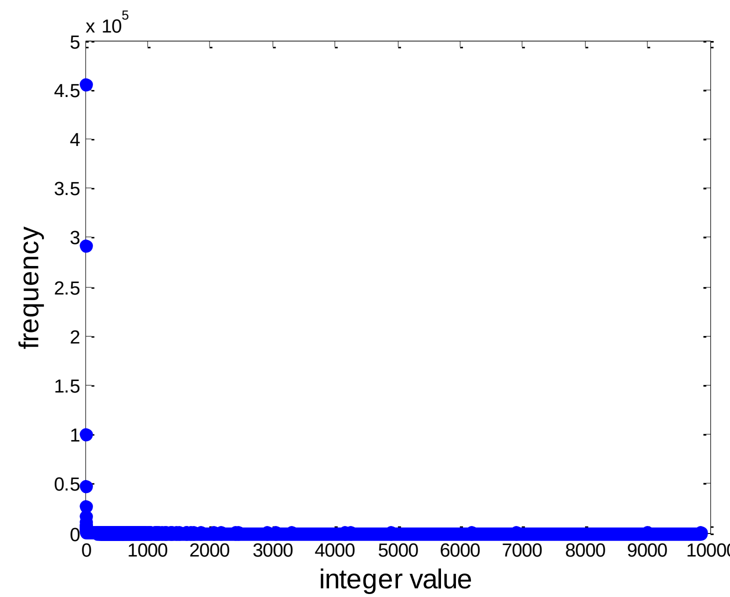
- Take 1 million random numbers from a distribution with $\alpha = 2.5$
- Can be generated using the so-called ‘transformation method’
  - Generate random numbers r on the unit interval $0 \le r < 1$
  - then $x = (1-r)^{-1/(\alpha-1)}$ is a random power law distributed real number in the range $1 \le x < \infty$

# Linear scale plot of straight bin of the data

- How many times did the number 1 or 3843 or 99723 occur
- Power-law relationship not as apparent
- Only makes sense to look at smallest bins




first few bins

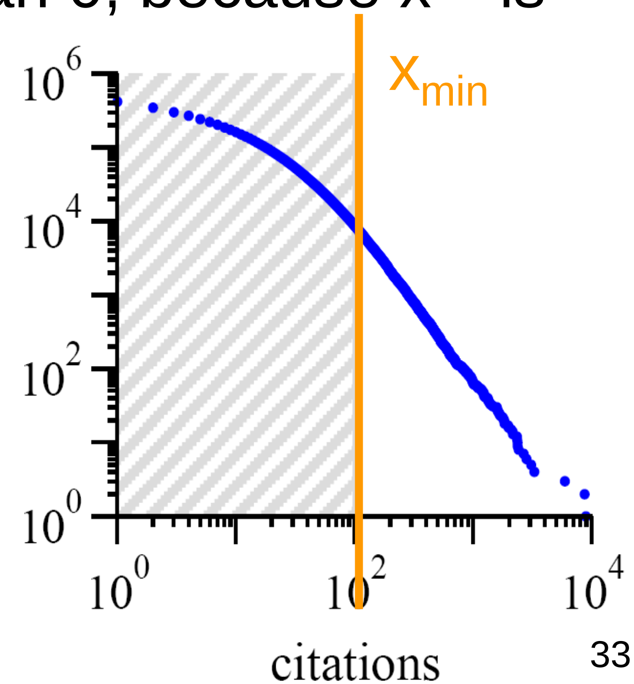



whole range

# Where to start fitting?

- some data exhibit a power law only in the tail
- after binning or taking the cumulative distribution you can fit to the tail
- so need to select an $x_{min}$ the value of x where you think the power-law starts
- certainly $x_{min}$ needs to be greater than 0, because $x^{-\alpha}$ is infinite at x = 0

**Example:** Distribution of citations to papers where power law is evident only in the tail ($x_{min}$ > 100 citations)




**Source:** MEJ Newman, 'Power laws, Pareto distributions and Zipf's law', Contemporary Physics **46**, 323–351 (2005)

# Some exponents for real world data

| | $x_{min}$ | exponent $\alpha$ |
|---|---|---|
| frequency of use of words | 1 | 2.20 |
| number of citations to papers | 100 | 3.04 |
| number of hits on web sites | 1 | 2.40 |
| copies of books sold in the US | 2 000 000 | 3.51 |
| telephone calls received | 10 | 2.22 |
| magnitude of earthquakes | 3.8 | 3.04 |
| diameter of moon craters | 0.01 | 3.14 |
| intensity of solar flares | 200 | 1.83 |
| intensity of wars | 3 | 1.80 |
| net worth of Americans | $600m | 2.09 |
| frequency of family names | 10 000 | 1.94 |
| population of US cities | 40 000 | 2.30 |

# Many real world networks are power law

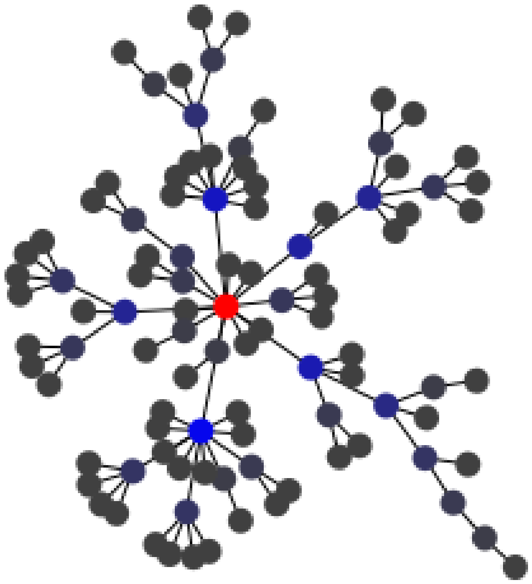
| | exponent $\alpha$ (in/out degree) |
|---|---|
| film actors | 2.3 |
| telephone call graph | 2.1 |
| email networks | 1.5/2.0 |
| sexual contacts | 3.2 |
| WWW | 2.3/2.7 |
| internet | 2.5 |
| peer-to-peer | 2.1 |
| metabolic network | 2.2 |
| protein interactions | 2.4 |

# Preferential Attachment in Networks

- First considered by [Price 65] as a model for citation networks
  - each new paper is generated with m citations (mean)
  - new papers cite previous papers with probability proportional to their indegree (citations)
  - what about papers without any citations?
    - each paper is considered to have a “default” citation
- Power law with exponent

# generating power-law networks

- Nodes appear over time (growth)
- Nodes prefer to attach to nodes with many connections (preferential attachment, cumulative advantage)

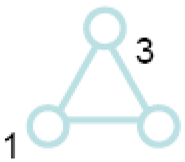


# Barabási-Albert model (BA model)

- Undirected model: each node connects to other nodes with probability proportional to their degree
  - the process starts with some initial subgraph ($m_0$ all-all connected node)
  - each node comes with m edges
  - the probability of tipping the new nodes to the old ones is proportional to the degrees of old nodes is a kind of preferential attachment algorithm
  - After t time steps, the network will have $n=t+m_0$ nodes and $M=m_0+mt$ edges
- It can be shown that this leads to a power law network!

# Basic BA-model

- Very simple algorithm to implement
  - start with an initial set of $m_0$ fully connected nodes
    - e.g. $m_0 = 3$

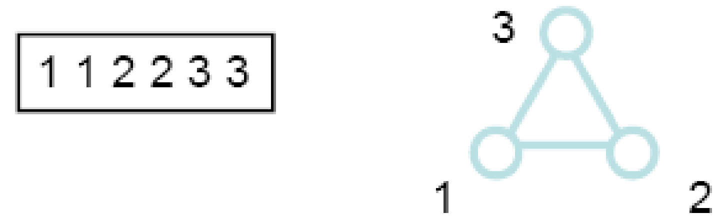



1 1 2 2 2 3 3 4 5 6 6 7 8 ....

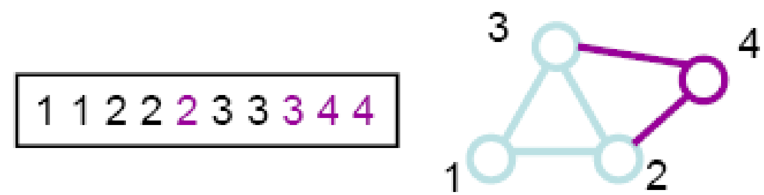
  - now add new vertices one by one, each one with exactly m edges
  - each new edge connects to an existing vertex in proportion to the number of edges that vertex already has → **preferential attachment**
  - easiest if you keep track of edge endpoints in one large array and select an element from this array at random
    - the probability of selecting any one vertex will be proportional to the number of times it appears in the array – which corresponds to its degree

# Generating BA graphs

- To start, each vertex has an equal number of edges (2)
  - the probability of choosing any vertex is 1/3

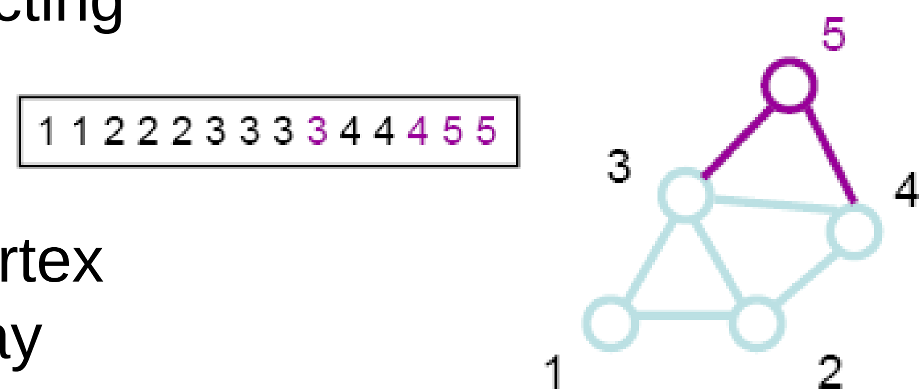



- We add a new vertex, and it will have m edges, here take m=2
  - draw 2 random elements from the array – suppose they are 2 and 3




- Now the probabilities of selecting 1,2,3,or 4 are

  1/5, 3/10, 3/10, 1/5




- Add a new vertex, draw a vertex for it to connect from the array
- etc.

# Proof of the scale-freeness

- Assume for simplicity that the degree $k_i$ for any node i is a continuous variable
- The probability of the tipping a node to node i is

$$\Pi(k_i) = \frac{k_i}{\sum_j k_j}$$

- Because of the assumptions, $k_i$ is expected to grow proportionally to $\Pi(k_i)$, that is to its probability of having a new edge
- Consequently, and because m edges are attached at each time, $k_i$ should obey the differential equation aside

$$\frac{\partial k_i}{\partial t} = m\Pi(k_i) = m\frac{k_i}{\sum_{j=1}^{n-1} k_j}$$

# Proof of the scale-freeness

- The sum $\sum_{j=1}^{n-1} k_j$
- Goes over all nodes except the new ones
- This it results in

$$\sum_{j=1}^{n-1} k_j = 2mt - m$$

- Remember that the total number of edges is almost mt and that here is edge is twice
- Substituting in the differential equation

$$\frac{\partial k_i}{\partial t} = m\frac{k_i}{\sum_{j=1}^{n-1} k_j} = m\frac{k_i}{2mt - m} \approx \frac{k_i}{2t}$$

# Proof of the scale-freeness

- We have now to solve this equation $\frac{\partial k_i}{\partial t} = \frac{k_i}{2t}$
  - That is, we have find a $k_i(t)$ function such as its derivative is equal to itself, divided by 2t
- We now show this is:

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{\beta}; \quad \text{with } \beta = \frac{1}{2}$$

- In fact:

$$\frac{\partial}{\partial t}\left(m\left(\frac{t}{t_i}\right)^{\beta}\right) = \frac{1}{2}\frac{m}{t_i^{\beta}}\frac{1}{t^{\beta}} = \frac{1}{2}\frac{m}{t_i^{\beta}}\frac{1}{t^{\beta}}\frac{t^{\beta}}{t^{\beta}} = \frac{m}{2}\frac{t^{\beta}}{t_i^{\beta}}\frac{1}{t^{2\beta}} = \frac{k_i(t)}{2t}$$

- where we also consider the initial condition $k_i(t_i)$=m, where ti is the time at which node i has arrived

# Proof of the scale-freeness

- The $k_i(t)$ function that we have not calculated shows that the degree of each node grown with a power law with time
- Now, let's calculate the probability that a node has a degree $k_i(t)$ smaller than k
- We have

$$P[k_i(t) < k] = P\left[m \frac{t^{\beta}}{t_i^{\beta}} < k\right] = P\left[m^{\frac{1}{\beta}} \frac{t^{\beta\frac{1}{\beta}}}{t_i^{\beta\frac{1}{\beta}}} < k^{\frac{1}{\beta}}\right] =$$

$$= P\left[m^{\frac{1}{\beta}} \frac{t}{t_i} < k^{\frac{1}{\beta}}\right] = P\left[t_i > \frac{m^{\frac{1}{\beta}} t}{k^{\frac{1}{\beta}}}\right]$$

# Proof of the scale-freeness

- Now let's remember that we add nodes at each time interval
- Therefore, the probability $t_j$ for a node, that is the probability for a node to have arrived at time $t_i$ is a constant and is:

$$P(t_i) = \frac{1}{t + m_0}$$

- Substituting this into the previous probability distribution

$$P[k_i(t) < k] = P\left[t_i > \frac{m^{\frac{1}{\beta}} t}{k^{\frac{1}{\beta}}}\right] = 1 - P\left[t_i \leq \frac{m^{\frac{1}{\beta}} t}{k^{\frac{1}{\beta}}}\right] = 1 - \frac{m^{\frac{1}{\beta}} t}{k^{\frac{1}{\beta}} (t + m_0)}$$

# Proof of the scale-freeness

- Now given the probability distribution

$$P[k_i(t) < k]$$

- Which represents the probability that a node i has less than k link

$$P(k) = \frac{\partial P[k_i(t) < k]}{\partial k}$$

- The probability that a node has exactly k link can be derived by the derivative of the probability distribution

$$P(k) = \frac{\partial P[k_i(t) < k]}{\partial k} = \frac{\partial}{\partial k}\left(1 - \frac{m^{\frac{1}{\beta}} t}{k^{\frac{1}{\beta}}(t + m_0)}\right) = \frac{2m^{\frac{1}{\beta}} t}{m_0 + t} \frac{1}{k^{\frac{1}{\beta}+1}}$$

# Conclusion of the Proof

- Given P(k):

$$P(k) = \frac{2m^{\frac{1}{\beta}} t}{m_0 + t} \frac{1}{k^{\frac{1}{\beta}+1}}$$

- After a while, that is for t→∞

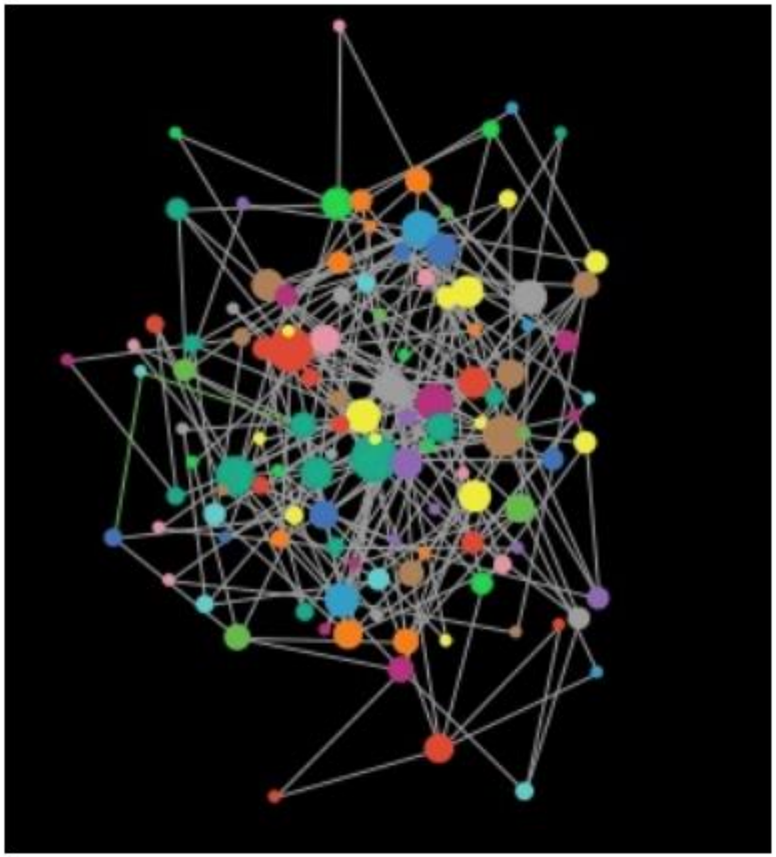
$$P(k) \approx 2m^{\frac{1}{\beta}} k^{-\frac{1}{\beta}-1} = 2m^{\frac{1}{\beta}} k^{-\gamma} \quad \text{where } \gamma = \frac{1}{\beta} + 1 = 3$$

- **we have obtained a power law probability density,** with an exponent which is independent of any parameter (being the only initial parameter m)

# Generality of the BA Model

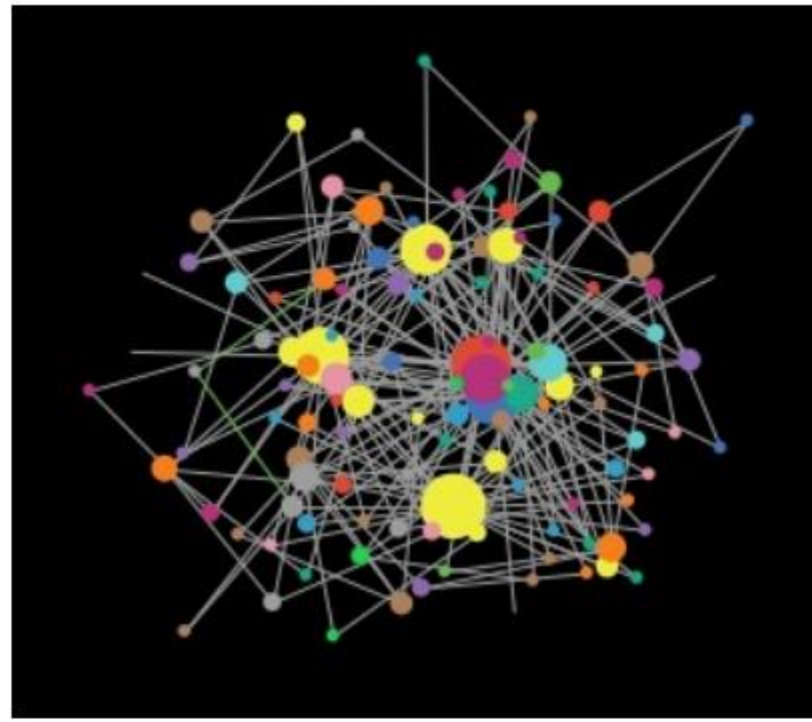
- In its simplicity, the BA model captures the essential characteristics of a number of phenomena
  - In which events determining “size” of the individuals in a network
  - Are not independent from each other
  - Leading to a power law distribution
- So, it can somewhat explain why the power law distribution is as ubiquitous as the normal Gaussian distribution
- Examples
  - **Gnutella (the first decentralized P2P network)**: a peer which has been there for a long time, has already collected a strong list of acquaintances, so that any new node has higher probability of getting aware of it
  - **Rivers**: the eldest and biggest a river, the more it has probability to break the path of a new river and get its water, thus becoming even bigger
  - **Industries**: the biggest an industry, the more its capability to attract clients and thus become even bigger
- **Richness**: the rich I am, the more I can exploit my money to make new money $\rightarrow$ “RICH GET RICHER”

# random non-preferential and preferential growth



random

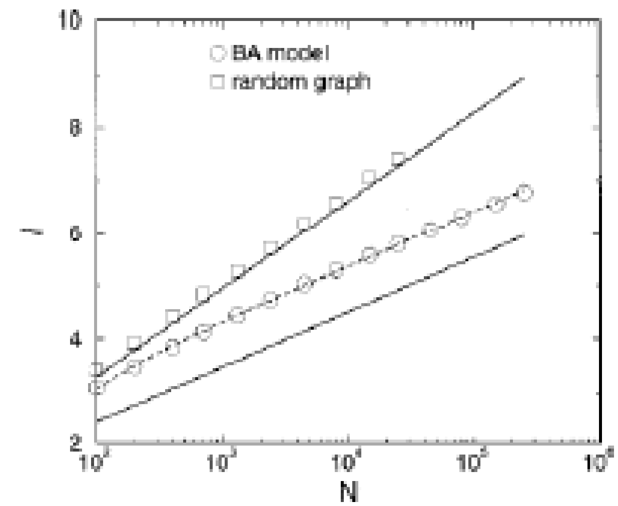
m = 2

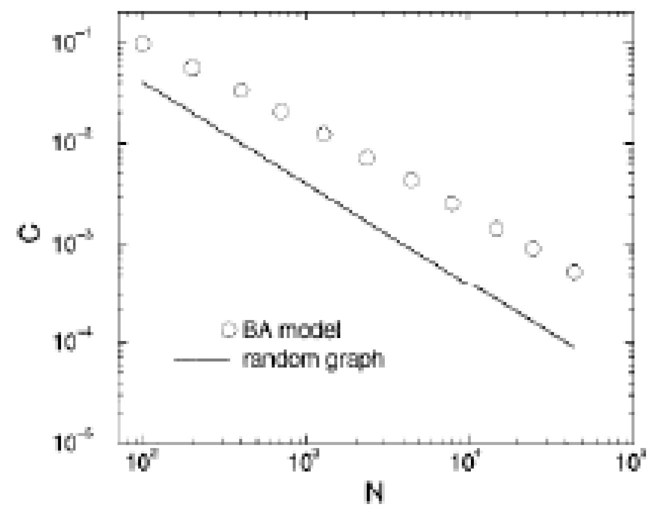


preferential

# Additional Properties of the BA Model

- Characteristic Path Length
  - It can be shown (but it is difficult) that the BA model has a length proportional to log(n)/log(log(n))
  - Which is even shorter than in random networks
  - And which is often in accord with – but sometimes underestimates – experimental data
- Clustering
  - There are no analytical results available
  - Simulations shows that in scale-free networks the clustering decreases with the increases of the network order
  - As in random graph, although a bit less
  - This is not in accord with experimental data!



## Problems of the BA Model

- The BA model is a nice one, but is not fully satisfactory!
- The BA model does not give satisfactory answers with regard to clustering
  - While the small world model of Watts and Strogatz does!
  - So, there must be something wrong with the model..
- The BA model predicts a fixed exponent of 3 for the power law
  - However, real networks shows exponents between 1 and 3
  - So, there most be something wrong with the model

# Reading

- M. E. J. Newman, **Power laws, Pareto distributions and Zipf's law**, Contemporary Physics 46, 323-351 (2005)
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