١) الف)

$$\hat{a}|n\rangle=\sqrt{n}\,|n-1\rangle \qquad \langle m|\hat{a}|n\rangle=\begin{pmatrix}\langle 1|\hat{a}|1\rangle & \langle 1|\hat{a}|2\rangle & \langle 1|\hat{a}|3\rangle\\ \langle 2|\hat{a}|1\rangle & \langle 2|\hat{a}|2\rangle & \langle 2|\hat{a}|3\rangle\\ \langle 3|\hat{a}|1\rangle & \langle 3|\hat{a}|2\rangle & \langle 3|\hat{a}|3\rangle\end{pmatrix}$$

$$=\begin{pmatrix}\sqrt{1}\langle 1|0\rangle & \sqrt{2}\langle 1|1\rangle & \sqrt{3}\langle 1|2\rangle\\ \sqrt{1}\langle 2|0\rangle & \sqrt{2}\langle 2|1\rangle & \sqrt{3}\langle 2|2\rangle\\ \sqrt{1}\langle 3|0\rangle & \sqrt{2}\langle 3|1\rangle & \sqrt{3}\langle 3|2\rangle\end{pmatrix}=\begin{pmatrix}0 & \sqrt{2} & 0\\ 0 & 0 & \sqrt{3}\\ 0 & 0 & 0\end{pmatrix}$$

$$a^{\dagger}|n\rangle=\sqrt{n+1}\,|n+1\rangle=\begin{pmatrix}\langle 1|a^{\dagger}|1\rangle & \langle 1|a^{\dagger}|2\rangle & \langle 1|a^{\dagger}|3\rangle\\ \langle 2|a^{\dagger}|1\rangle & \langle 2|a^{\dagger}|2\rangle & \langle 2|a^{\dagger}|3\rangle\\ \langle 3|a^{\dagger}|1\rangle & \langle 3|a^{\dagger}|2\rangle & \langle 3|a^{\dagger}|3\rangle\end{pmatrix}=\begin{pmatrix}\sqrt{2}\langle 1|2\rangle & \sqrt{3}\langle 1|3\rangle & \sqrt{4}\langle 1|4\rangle\\ \sqrt{2}\langle 2|2\rangle & \sqrt{3}\langle 2|3\rangle & \sqrt{4}\langle 2|4\rangle\\ \sqrt{2}\langle 3|2\rangle & \sqrt{3}\langle 3|3\rangle & \sqrt{4}\langle 3|4\rangle\end{pmatrix}$$

$$=\begin{pmatrix}0 & 0 & 0\\ \sqrt{2} & 0 & 0\\ 0 & \sqrt{3} & 0\end{pmatrix}$$

ب)

$$a=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i\hat{p}}{m\omega}\right),\quad a^{\dagger}=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}-\frac{i\hat{p}}{m\omega}\right)$$

$$\rightarrow \hat{x}=\sqrt{\frac{\hbar}{2m\omega}}(a+a^{\dagger})=\sqrt{\frac{\hbar}{2m\omega}}\left[\begin{pmatrix}0 & \sqrt{2} & 0\\ 0 & 0 & \sqrt{3}\\ 0 & 0 & 0\end{pmatrix}+\begin{pmatrix}0 & 0 & 0\\ \sqrt{2} & 0 & 0\\ 0 & \sqrt{3} & 0\end{pmatrix}\right]=\begin{pmatrix}0 & \sqrt{\frac{\hbar}{m\omega}} & 0\\ \sqrt{\frac{\hbar}{m\omega}} & 0 & \sqrt{\frac{3}{2}\frac{\hbar}{m\omega}}\\ 0 & \sqrt{\frac{3}{2}\frac{\hbar}{m\omega}} & 0\end{pmatrix}$$

$$\hat{x}^2=\begin{pmatrix}0 & \sqrt{\frac{\hbar}{m\omega}} & 0\\ \sqrt{\frac{\hbar}{m\omega}} & 0 & \sqrt{\frac{3}{2}\frac{\hbar}{m\omega}}\\ 0 & \sqrt{\frac{3}{2}\frac{\hbar}{m\omega}} & 0\end{pmatrix}\begin{pmatrix}0 & \sqrt{\frac{\hbar}{m\omega}} & 0\\ \sqrt{\frac{\hbar}{m\omega}} & 0 & \sqrt{\frac{3}{2}\frac{\hbar}{m\omega}}\\ 0 & \sqrt{\frac{3}{2}\frac{\hbar}{m\omega}} & 0\end{pmatrix}=\begin{pmatrix}\frac{\hbar}{m\omega} & 0 & \sqrt{\frac{3}{2}}\frac{\hbar}{m\omega}\\ 0 & \frac{5}{2}\frac{\hbar}{m\omega} & 0\\ \sqrt{\frac{3}{2}}\frac{\hbar}{m\omega} & 0 & \frac{3}{2}\frac{\hbar}{m\omega}\end{pmatrix}$$

ج)

$$\hat{p}=i\sqrt{\frac{m\omega\hbar}{2}}(a^{\dagger}-a)=i\sqrt{\frac{m\omega\hbar}{2}}\left[\begin{pmatrix}0 & 0 & 0\\ \sqrt{2} & 0 & 0\\ 0 & \sqrt{3} & 0\end{pmatrix}-\begin{pmatrix}0 & \sqrt{2} & 0\\ 0 & 0 & \sqrt{3}\\ 0 & 0 & 0\end{pmatrix}\right]=i\sqrt{\frac{m\omega\hbar}{2}}\begin{pmatrix}0 & -\sqrt{2} & 0\\ \sqrt{2} & 0 & -\sqrt{3}\\ 0 & \sqrt{3} & 0\end{pmatrix}$$

$$\hat{p}^2=-\frac{m\omega\hbar}{2}\begin{pmatrix}0 & -\sqrt{2} & 0\\ \sqrt{2} & 0 & -\sqrt{3}\\ 0 & \sqrt{3} & 0\end{pmatrix}\begin{pmatrix}0 & -\sqrt{2} & 0\\ \sqrt{2} & 0 & -\sqrt{3}\\ 0 & \sqrt{3} & 0\end{pmatrix}=\frac{-m\omega\hbar}{2}\begin{pmatrix}-2 & 0 & \sqrt{6}\\ 0 & -5 & 0\\ \sqrt{6} & 0 & -3\end{pmatrix}=m\omega\hbar\begin{pmatrix}1 & 0 & -\frac{\sqrt{6}}{2}\\ 0 & \frac{5}{2} & 0\\ -\frac{\sqrt{6}}{2} & 0 & \frac{3}{2}\end{pmatrix}$$

(۱

$\hat{H}|n\rangle = \hbar\omega(n+\frac{1}{2})|n\rangle \qquad \langle m|\hat{H}|n\rangle = \begin{pmatrix} \langle 1|\hat{H}|1\rangle & \langle 1|\hat{H}|2\rangle & \langle 1|\hat{H}|3\rangle \\ \langle 2|\hat{H}|1\rangle & \langle 2|\hat{H}|2\rangle & \langle 2|\hat{H}|3\rangle \\ \langle 3|\hat{H}|1\rangle & \langle 3|\hat{H}|2\rangle & \langle 3|\hat{H}|3\rangle \end{pmatrix}$

$= \hbar\omega \begin{pmatrix} \frac{3}{2}\langle 1|1\rangle & \frac{5}{2}\langle 1|2\rangle & \frac{7}{2}\langle 1|3\rangle \\ \frac{3}{2}\langle 2|1\rangle & \frac{5}{2}\langle 2|2\rangle & \frac{7}{2}\langle 2|3\rangle \\ \frac{3}{2}\langle 3|1\rangle & \frac{5}{2}\langle 3|2\rangle & \frac{7}{2}\langle 3|3\rangle \end{pmatrix} = \hbar\omega \begin{pmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{5}{2} & 0 \\ 0 & 0 & \frac{7}{2} \end{pmatrix}$

(۲

در فضای سه بعدی :

$|1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad \langle 1| = (1\ 0\ 0)$

$|2\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \langle 2| = (0\ 1\ 0)$

$|3\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix} \quad \langle 3| = (0\ 0\ 1)$

$\hat{H} = \sum \left(E_0|n\rangle\langle n| + \Delta|n\rangle\langle n+1| + \Delta|n+1\rangle\langle n| \right)$

$= E_0|1\rangle\langle 1| + \Delta|1\rangle\langle 2| + \Delta|2\rangle\langle 1|$

$+ E_0|2\rangle\langle 2| + \Delta|2\rangle\langle 3| + \Delta|3\rangle\langle 2|$

$= E_0\begin{pmatrix}1\\0\\0\end{pmatrix}(1\ 0\ 0) + \Delta\begin{pmatrix}1\\0\\0\end{pmatrix}(0\ 1\ 0) + \Delta\begin{pmatrix}0\\1\\0\end{pmatrix}(1\ 0\ 0)$

$+ E_0\begin{pmatrix}0\\1\\0\end{pmatrix}(0\ 1\ 0) + \Delta\begin{pmatrix}0\\1\\0\end{pmatrix}(0\ 0\ 1) + \Delta\begin{pmatrix}0\\0\\1\end{pmatrix}(0\ 1\ 0)$

$= E_0\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} + \Delta\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix} + \Delta\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}$

$+ E_0\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix} + \Delta\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix} + \Delta\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}$

$= \begin{pmatrix} E_0 & \Delta & 0 \\ \Delta & E_0 & \Delta \\ 0 & \Delta & 0 \end{pmatrix}$