University of Kashan

677

# Proceedings

## of the 51st Annual Iranian Mathematics Conference

## Volume 1: Pure Mathematics

Editors:


Zeinab Saeidian Tarei
(Faculty Member, University of Kashan, I. R. Iran)

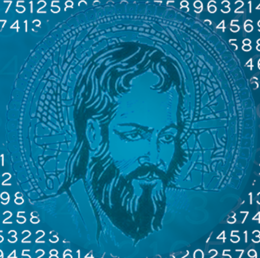
Mardjan Hakimi-Nezhaad

Ali Reza Ashrafi
(Faculty Member, University of Kashan, I. R. Iran)

Ali Ghalavand



February 15-20, 2021
University of Kashan,
I. R. Iran

51st Annual Iranian Mathematics Conference

# $51^{st}$ Annual Iranian Mathematics Conference

15–20 February 2021, University of Kashan

# Proceedings of the $51^{st}$ Annual Iranian Mathematics Conference, Volume 1: Pure Mathematics

Secretariat: Faculty of Mathematical Sciences, University of Kashan,
Kashan 87317-53153, I. R. Iran
Phone: (+9831) 55912918
Fax: (+9831) 55912332
Email: aimc51@kashanu.ac.ir
http://aimc51.kashanu.ac.ir

سرشناسه: کنفرانس ریاضی ایران (پنجاه و یکمین: ۱۳۹۹: کاشان)
Annual Iranian Mathematics Conference (51st: 2021: Kashan)
عنوان و نام پدیدآور: Proceedings of the 51st Annual Iranian Mathematics conference, Volume 1: Pure Mathematics[Book]/ editors Zeinab Saeidian Tarei ...[et al.].
مشخصات نشر: کاشان: دانشگاه کاشان، کاشان: سوره تماشا، ۱۴۰۰=۲۰۲۱م.
مشخصات ظاهری: ۶۳۰ ص.
شابک: ۹۷۸-۶۲۲-۶۵۴۶-۲۲-۵
وضعیت فهرست‌نویسی: فیپا
یادداشت: انگلیسی.
یادداشت: editors Zeinab Saeidian Tarei, Mardjan Hakimi-Nezhaad, Ali Reza Ashrafi, Ali Ghalavand.
موضوع: ریاضیات – کنگره‌ها
موضوع: Mathematics – Congresses
شناسه افزوده: سعیدیان طرئی، زینب، ۱۳۶۳-، ویراستار
شناسه افزوده: Saeidian Tarei, Zeinab، ۱۹۸۴ -
رده‌بندی کنگره: QA۱
رده‌بندی دیویی: ۵۱۰
شماره کتابشناسی ملی: ۷۶۶۲۱۴۲

University of Kashan
677

51st Annual Iranian Mathematics Conference

15-20 February 2021, University of Kashan

**Proceedings of the 51st Annual Iranian Mathematics Conference,**

**Volume 1: Pure Mathematics**

**Publisher:** University of Kashan Press and Sureh Tamasha Publication

**Editors:** Zeinab Saeidian Tarei, Mardjan Hakimi-Nezhaad, Ali Reza Ashrafi, Ali Ghalavand

**Printing and Binding:** Baran

**First Printing:** 2021

**Print Run:** 200

**ISBN:** 978-622-6546-22-5

**Price:** 1500000 IR Rials or 10 Euro

**Editors:**

Zeinab Saeidian Tarei
Faculty Member, University of Kashan, I. R. Iran
saeidian@kashanu.ac.ir

Mardjan Hakimi-Nezhaad
University of Kashan, I. R. Iran
m.hakimi20@gmail.com

Ali Reza Ashrafi
Faculty Member, University of Kashan, I. R. Iran
ashrafi@kashanu.ac.ir

Ali Ghalavand
University of Kashan, I. R. Iran
alighalavand@grad.kashanu.ac.ir

**Library of Congress Control Number:** 7662142

**ISBN:** 978-622-6546-23-5

# Organizers

University of Kashan

# Sponsors

# Message from the Mayor of Kashan

Once again the world's top mathematicians, professors, scholars, and students of mathematics have gathered in a scientific circle in the historical city of Kashan. The Faculty of Mathematics of the University of Kashan has been honored to host the 51st Annual Iranian Mathematics Conference. Undoubtedly, the philosophy of science would be incomplete in the absence of objective examples of phenomena. Mathematics serves as the basic science for understanding the principles of existence and the basis of the order of the universe. As our grasp on mathematic theory tightens, we are humbled by the greatness of this world's creator.

It is not a secret that Kashan has long been a cradle for flourishing men and women like Ghiythal-DnJamshdKashanis who have advanced the boundaries of science.

We were also pleased to have with us an acclaimed mathematician from our city, Dr. Javad Mashreghi; president of the Canadian Mathematical Society.

As the Mayor of Kashan, I wish to welcome all scholars and mathematics enthusiasts to this conference and to thank the esteemed keynote speakers, guests, and participants. I pray that this message finds you in health and ever-increasing prosperity. I wish for a world free of pandemics and a return to normal with physical conferences.

**Mayor of Kashan**
**Saeed Abrishami-Rad**

# Foreword

The 51st Annual Iranian Mathematics Conference was held at University of Kashan in cooperation with the Iranian Mathematical Society from February 15 to February 20, 2021. We were eager to host the presence of the mathematical community of Iran at University of Kashan, and by providing an intimate and academic atmosphere for opportunities for exchange and scientific participation for all in the field of mathematical sciences and their applications. University of Kashan was founded at first as an institution of higher education in 1973. It began its activities in October, 1974 by 200 students of mathematics and physics.

Being in a suitable geographical position, the cultural atmosphere of the region and the long history in science and art have provided the basis for great success for this university and now, for example, University of Kashan has been introduced as the seventh comprehensive university in Iran by ISC National University Ranking.

The Faculty of Mathematical Sciences of University of Kashan is active with nearly forty full-time faculty members in three levels of bachelor's, master's and doctoral degrees and has made a significant contribution to the development and achievements of University of Kashan.

Holding successful conferences, student competitions of the Iranian Mathematical Society and various specialized seminars have been among the activities of this faculty. The editor in chief of the "Bulletin of the Iranian Mathematical Society" and the "Journal of Mathematical Culture and Thought" by the faculty members of this faculty at various times, are some of the effective collaborations with the Iranian Mathematical Society.

Due to the outbreak of the Corona virus, the 51st Iranian Mathematical Conference is being held virtually in University of Kashan for the first time.Besides the limitations created by holding the conference virtually, new opportunities have emerged. We had the great opportunity by using the facilities of cyberspace to invite prominent national and international professors from 22 different countries.

You are all aware that due to various reasons and problems in the educational, economic and social dimensions, the number of mathematics students has decreased significantly in recent years.

The elites of the country, have emphasized on strengthening the basic sciences, especially mathematics, and have introduced them as a treasure for the development of the country. It is up to the Iranian Mathematical Society to use the opportunity and the support the authorities, to plan for the promotion and expansion of mathematics.

As a step towards taking responsibility for this, we added a new section to the conference this year called "Mathematical Promotion". This idea was welcomed by the esteemed officials of the Iranian Mathematical Society and it is hoped that it will be followed as part of the conference in the coming years. In this regard, with the help of the education department of the region, a call was made and so far we have received more than 400 articles, from interested students in different levels of elementary and high school from all over the country.

It was decided to hold the first meeting for the promotion and popularization of mathematics as part of the mathematics conference in the near future and to present the selected works.

I consider it necessary to thank the Ministry of Science, Research and Technology, esteemed officials of University of Kashan, dear colleagues in the Faculty of Mathematical Sciences of the University of Kashan, faculty members of universities and research centers across the country who helped and guided us in particular those who contributed to the accurate judging of the received papers.

I would like to thank all the participants who added value by sending valuable papers and participating in the conference. Holding a conference like Iranian Mathematics Conference virtually was a new experience for us. I hope we have been able to do this great event well and in a desirable and worthy way. Moreover, this will be an experience for the expansion of virtual activities in the future. I apologize in advance for all the shortcomings, which were mainly due to our lack of experience in holding such conferences and virtual activities.

Hoping to see you at the future conferences.

**Conference Chair of AIMC51**
**Hassan Daghigh**

# Welcome to AIMC51

The Annual Iranian Mathematics Conference (AIMC) is the country's most important and oldest mathematical gathering where researchers, students, and professors at home and abroad present their latest scientific findings. The first mathematics conference of the country was held by the University of Shiraz in April 1970, the most important of which was the proposal to establish the Iranian Mathematical Society, which coincided with the second mathematical conference of the country at the Sharif University of Technology in April 1971. Since then, the conference has welcomed a large number of scholars at home and abroad each year.

The Iranian Mathematics Conference has been held for the last fifty years despite all the difficulties. The Faculty of Mathematical Sciences of the University of Kashan is now honored to hold the fifty-first gathering of this important mathematics event of the country from February 15 to February 20, 2021 in the cradle of Iranian civilization and traditional culture, the city of Kashan with seven thousand years history.

We originally planned to hold the conference in person from 7 September to 10 September 2020, but due to the corona pandemic and the laws announced to the universities by the government, we changed the time to February 2021.

AIMC 51 has 31 keynote and 7 invited speakers from 20 different countries, all of whom are among the best and most famous mathematicians in the world in their field. The scope of the conference covered various topics in mathematics, statistics and computer science. The conference was attended by more than 500 researchers from Argentina, Belarus, Brazil, Canada, Check Republic, China, Croatia, India, Iran, Iraq, Italy, Kuwait, Netherland, Nigeria, Oman, Pakistan, Romania, Russia, Saudi Arabia, Serbia, South Africa, South Korea, Thailand, Turkey and USA who held 20, 40 and 60 minutes lectures.

We have fifteen keynote speakers in pure mathematics, seven keynote speakers in applied mathematics, four keynotes in statistics and five keynotes in computer science. There are also seven young invited speakers who are famous mathematicians in their topics.

Our Keynote Speakers in Pure Mathematics are professors: Alireza Abdollahi (University of Isfahan, I. R. Iran), Javad Asadollahi (University of Isfahan, I. R. Iran), Mohammad Bagheri (Historian), Maurizio Brunetti (Universit di Napoli Federico II, Italy), Henri Darmon (McGill University, Canada), Omid Ali Shehni Karamzadeh (Shahid Chamran University of Ahvaz, I. R. Iran), Javad Mashreghi (Laval university, Canada), Mohammad Sal Moslehian (Ferdowsi University of Mashhad, I. R. Iran), Thekiso Seretlo (University of Limpopo, South Africa), Mohammad Shahryari (Sultan Qaboos University, Muscat, Oman), Andrea Solotar (University of Buenos Aires, Argentina), Teerapong Suksumran (Chiang Mai University, Thailand), Mukut Mani Tripathi (Banaras Hindu University, India), Andrei Yu. Vesnin (Russian Academy of Sciences, Russia) and Changchang Xi (Capital Normal University, China).

The AIMC51 Keynote Speakers in Applied Mathematics are professors: Tomislav Došlić (University of Zagreb, Croatia), Roberto Garrappa (University of Bari, Italy), Nezameddin Mahdavi-Amiri (Sharif University of Technology, I. R. Iran), Davoud Mirzaei (University of Isfahan, I. R. Iran), Kees Roos (Delft University of

Technology, Netherland), Majid Soleimani Damaneh (University of Tehran, I. R. Iran) and Zahra Gooya (Shahid Beheshti University, I. R. Iran).

Other main topics of AIMC 51 are Statistics and Computer Science, and the keynote speakers of these topics are professors: Masoud Asgharian (McGill university, Canada), Khalil Shafie (University of Northern Colorado, USA), Ahmad Reza Soltani (Kuwait University, Kuwait), Bijan Zohuri-Zangeneh (Sharif University of Technology, I. R. Iran), Khodakhast Bibak (Miami University, USA), Alain Bretto (University of Caen, France), Luca De Feo (University of Versailles - Saint-Quentin, France), Predrag S. Stanimirovic (University of Nis, Serbia) and Constantine Tsinakis (Vanderbilt University, USA).

Our Invited Speakers are Akbar Ali (University of Ha'il, Saudi Arabia), Mohsen Ghasemi (Urmia University, I. R. Iran), Gülistan Kaya Gök (Hakkari University. Hakkari-Turkey), Mohsen Kian (University of Bojnord, I. R. Iran), Ali Shukur (Belarusian State University, Belarus) and Ebrahim Reyhani (Shahid Rajaee Teacher Training University, I. R. Iran). The annual meeting of the Women's Committee of the Iranian Mathematical Society (WCIMS) will be started by the speech of professor Ashraf Daneshkhah, secretary of WCIMS. This meeting has professor Carolina Araujo as honorary guest. She is the Award Wiener of Ramanujan 2020, Brazil and vice president of the IMU committee for women. Professor Araujo will be presented an invited talk for AIMC 51 participants.

I am very thankful to all of my colleagues in Organizing and Scientific Committee and to all of participants. My special gratitude is going to the Keynote and Invited Speakers. I would also like to thank all the referees for the time they allocated and their help.

**Chair of the Scientific Committee of AIMC51**
**Ali Reza Ashrafi**

**Conference Chair:** Hassan Daghigh

**Chair of Scientific Committee:** Ali Reza Ashrafi

**Chair of Organizing Committee:** Mojtaba Bahramian

**Members of Scientific Committee:**

- Farshid Abdollahi — Shiraz University, I. R. Iran
- Saeid Alikhani — Yazd University, I. R. Iran
- Keyvan Amini — Razi University, I. R. Iran
- Saeid Azam — University of Isfahan, I. R. Iran
- Fariborz Azarpanah — Shahid Chamran University of Ahvaz, I. R. Iran
- Seyed Morteza Babamir — University of Kashan, I. R. Iran
- Mojtaba Bahramian — University of Kashan, I. R. Iran
- Rajab Ali Borzooei — Shahid Beheshti University , I. R. Iran
- Hassan Daghigh — University of Kashan, I. R. Iran
- Kinkar Chandra Das — Sungkyunkwan University, South Korea
- Mostafa Davtalab Olyaie — University of Kashan, I. R. Iran
- Bijan Davvaz — Yazd University, I. R. Iran
- Mohammad Ali Dehghan — Vali-e-Asr University of Rafsanjan, I. R. Iran
- Mahdi Dehghani — University of Kashan, I. R. Iran
- Sohrab Effati — Ferdowsi University Of Mashhad, I. R. Iran
- Ali Eftekhari — University of Kashan, I. R. Iran
- Taraneh Eghlidos — Sharif University of Technology, I. R. Iran
- Hossein Eshraghi — University of Kashan, I. R. Iran
- Gholamhosein Fathtabar — University of Kashan, I. R. Iran
- Majid Gazor — Isfahan University of Technology, I. R. Iran
- Faranak Goodarzi — University of Kashan, I. R. Iran
- Zahra Gooya — Shahid Beheshti University, I. R. Iran
- Massoud Hadian — Iran University of Science and Technology, I. R. Iran
- Masoud Hajarian — Shahid Beheshti University, I. R. Iran
- Ebrahim Hashemi — Shahrood University of Technology, I. R. Iran
- Ali Iranmanesh — Tarbiat Modares University, I. R. Iran
- Reza Jahani-Nezhad — University of Kashan, I. R. Iran
- Reza Kahkeshani — University of Kashan, I. R. Iran
- Vilmos Katona — University of Sopron, Hungary
- Seyed Mohammad Bagher Kashani — Tarbiat Modares University, I. R. Iran
- Rasool Kazemi Najafabadi — University of Kashan, I. R. Iran
- Stefan Kohl — University of St Andrews, Scotland
- Alireza Medghalchi — Kharazmi University, I. R. Iran
- Morteza Moniri — Shahid Beheshti University, I. R. Iran
- Ali Madanshekaf — University of Semnan, I. R. Iran
- Fereshteh Malek — KNT University of Technology, I. R. Iran
- Akbar Mohebi — University of Kashan, I. R. Iran
- Seyfollah Mosazadeh — University of Kashan, I. R. Iran
- Mark Raheb Ghamsary — Loma Linda University, USA
- Farhad Rahmati — Amirkabir University of Technology, I. R. Iran
- Abdolrahman Razani — Imam Khomeini International University, I. R. Iran
- Aliasghar Rezaei — University of Kashan, I. R. Iran
- Abbas Saadatmandi — University of Kashan, I. R. Iran
- Abbas Salemi Parizi — Shahid Bahonar University of Kerman, I. R. Iran
- Mehdi Shams — University of Kashan, I. R. Iran
- Seyyed Mansour Vaezpour — Amirkabir University of Technology, I. R. Iran

**Members of Policy Conucil**

- Ali Reza Ashrafi
- Mojtaba Bahramian
- Behnam Bazigaran
- Hassan Daghigh
- Ali Eftekhari
- Reza Jahani-Nezhad
- Ruhollah Jahanipur
- Akbar Mohebbi
- Amir Hossien Nokhodkar
- Ali Asghar Rezaei

**Members of Organizing Committee**

- Saeed Asaeedi
- Mahdi Asadi
- Jalal Askari Farsangi
- Morteza Bisheh-Niasar
- Mohammad Eghbali
- Ali Eftekhari
- Mardjan Hakimi-Nezhaad
- Elahe Khaldi
- Abolfazl Khedmati
- Mohammad Hassan Malekian
- Seyyed Ali Mohammadiyeh
- Marzieh Pourbabaee
- Mahdi Sabzevari
- Zeinab Saeedian Tarei
- Zeinab Soltani
- Fatemeh Zabihi

**Other people who helped organizing the conference:**

**Ladies:** Maryam Azizi, Narges Barzegran, Leila Goodarzi, Elham Hajirezaei, Shirin Heidari, Marzieh Sadat Hosseini, Zeinab Jafari Tadi, Nazila Jahangir, Sheyda Maddah, Elahe Mahabadian, Nasrin Malek-Mohammadi Faradonbeh, Faezeh Mohammadi, Maryam Nasr-Esfahani, Mohadeseh Nasr-Esfahani, Mahsa Rafiee, Maryam Rezaei Kashi, Mina Shafouri, Maryam Taheri-Sedeh, Ghazal Tavakoli, Armina Zare, Samaneh Zareian

**Gentlemen:** Jalal Abbassi, Mahdi Abedi, Ali Ghalavand, Mohammad Izadi, Bardia Jahangiri, Mostafa Karbalaei Reza, Ali Reza Khalilian, Kourosh Mavaddat-Nezhad, Sajad Raahati, Mohsen Yaghoubi

# Keynote Speakers

| | Name | Family | Affiliation |
|---|---|---|---|
| 1 | Alireza | Abdollahi | University of Isfahan, I. R. Iran |
| 2 | Javad | Asadollahi | University of Isfahan, I. R. Iran |
| 3 | Masoud | Asgharian | McGill Univesity, Canada |
| 4 | Mohammad | Bagheri | Editor in chief of the Journal of the History of Science, I. R. Iran |
| 5 | Khodakhast | Bibak | Miami University, USA |
| 6 | Alain | Bretto | Normandie University, France |
| 7 | Maurizio | Brunetti | Universita Federico II, Italy |
| 8 | Henri | Darmon | McGill University, Canada |
| 9 | Luca | De Feo | University of Versailles, Switzerland |
| 10 | Tomislav | Došlić | University of Zagreb, Croatia |
| 11 | Roberto | Garrappa | Polytechnic University of Bari, Italy |
| 12 | Zahra | Gouya | Shahid Beheshti University, I. R. Iran |
| 13 | Nezam | Mahdavi-Amiri | Sharif University of Technology, I. R. Iran |
| 14 | Javad | Mashreghi | University of Laval, Canada |
| 15 | Davoud | Mirzaei | University of Isfahan, I. R. Iran |
| 16 | Kees | Roos | Technical University Delf, Netherland |
| 17 | Mohammad | Sal Moslehian | Ferdowsi University of Mashhad, I. R. Iran |
| 18 | Thekiso Trevor | Seretlo | University of Limpopo, South Africa |
| 19 | Khalil | Shafie | University of Northern Colorado, USA |
| 20 | Omid Ali | Shehni-Karamzadeh | Shahid Chamran University of Ahvaz, I. R. Iran |
| 21 | Mohammad | Shahryari | Sultan Qaboos University, Muscat, Oman |
| 22 | Majid | Soleimani-Damaneh | University of Tehran, I. R. Iran |
| 23 | Andrea | Solotar | Universidad de Buenos Aires, Argentina |
| 24 | Ahmad Reza | Soltani | Kuwait University, Kuwait |
| 25 | Predrag | Stanimirović | University of Nis, Serbia |
| 26 | Teerapong | Suksumran | Chiang Mai University, Thailand |
| 27 | Mukut Mani | Tripathi | Banaras Hindu University, India |
| 28 | Constantine | Tsinakis | Vanderbilt University, USA |
| 29 | Andrei | Vesnin | Tomsk State University, Russia |
| 30 | Changchang | Xi | Capital Normal University, China |
| 31 | Bijan | Zohuri-Zangeneh | Sharif University of Technology, I. R. Iran |

# Invited Speakers

| | Name | Family | Affiliation |
|---|---|---|---|
| 1 | Akbar | Ali | University of Hail, Saudi Arabia |
| 2 | Mohsen | Ghasemi | Urmia University, I. R. Iran |
| 3 | Gülistan | Kaya Gök | Hakkari University, Turkey |
| 4 | Mohsen | Kian | University of Bojnord, I. R. Iran |
| 5 | Ebrahim | Reihani | Shahid Rajaee Teacher Training University, I. R. Iran |
| 6 | Ali | Shukur | Belarusian State University, Belarus; The Islamic University, Iraq |

# Conference Participants

| | First Name | Last Name | University |
|---|---|---|---|
| 1 | Naser | Abbasi | Lorestan University |
| 2 | Mostafa | Abbaszadeh | Amirkabir University of Technology |
| 3 | Fakhralsadat | Abdenean | Yazd University |
| 4 | Me'raj | Abdi | Bam University |
| 5 | Nasim | Abdi Kourani | Khajeh Nasir Toosi University of Technology |
| 6 | Atefeh | Abdolah Abyaneh | Kharazmi University |
| 7 | Alireza | Abdollahi | University of Isfahan |
| 8 | Farshid | Abdollahi | Shiraz University |
| 9 | Fahimeh | Abdollahi | Khajeh Nasir Toosi University of Technology |
| 10 | Alma | Abedinzadeh | University of Tehran |
| 11 | Mohammed Yahya | Abed | University of Kerbala, Iraq |
| 12 | Mahdi | Abedei | Shahid Bahonar University of Kerman |
| 13 | Marjan | Adib | Payame Noor University |
| 14 | Fatemeh Sadat | Aghaei Maybodi | Yazd University |
| 15 | Fatemeh | Ahangari | Al-Zahra University |
| 16 | Alireza | Ahmadi | Yazd University |
| 17 | Kambiz | Ahmadi | University of Shahrekord |
| 18 | Ghasem | Ahmadi | Payame Noor University |
| 19 | Razieh | Ahmadian | IPM Institute For Research In Fundamental Sciences |
| 20 | Mohammad Ali | Ahmadpoor | University of Guilan |
| 21 | Zohreh | Akbari | University of Mazandaran |
| 22 | Najmeh | Akbari | Isfahan University of Technology |
| 23 | Fahime | Akhavan Ghassabzade | University of Gonabad |
| 24 | Narges | Akhlaghinia | Shahid Beheshti University |
| 25 | Basim | Albuohimad | University of Kerbala, Iraq |
| 26 | Akbar | Ali | University of Hail, Saudi Arabia |
| 27 | Mahdi | Aliakbari | Torbat Heydarieh University |
| 28 | Ghazale | Aliasghari | Shahid Rajaee Teacher Training University |
| 29 | Saeid | Alikhani | Yazd University |
| 30 | Hajar | Alimorad | Jahrom University |
| 31 | Mohammad Reza | Alimoradi | Malayer University |
| 32 | Morteza | Alishahi | Islamic Azad University |
| 33 | Ahmed | Al-Obaidi | University of Kufa, Iraq |
| 34 | Keyvan | Amini | Razi University |
| 35 | Mostafa | Amini | Payame Noor University |
| 36 | Diba | Aminshayan Jahromi | Shiraz University |
| 37 | Letafat | Amiri | Tarbiat Modares University |
| 38 | Sadegh | Amiri | Shahid Sattari Aeronautical University |
| 39 | Hanieh | Amjadian | Amirkabir University of Technology |
| 40 | Mahdi | Anbarloei | Imam Khomeini International University |
| 41 | Hajar | Ansari | Amirkabir University of Technology |
| 42 | Ali | Ansari Ardali | University of Shahrekord |
| 43 | Fereshteh | Arad | Shahid Bahonar University of Kerman |
| 44 | Mahdi | Asadi | University of Kashan |

# Conference Participants

| | First Name | Last Name | Affiliation |
|---|---|---|---|
| 45 | Mohammad Ali | Asadi | Islamic Azad University |
| 46 | Mohammad Bagher | Asadi | University of Tehran |
| 47 | Meysam | Asadipour | Yasouj University |
| 48 | Javad | Asadollahi | University of Isfahan |
| 49 | Saeed | Asaeedi | University of Kashan |
| 50 | Masoud | Asgharian | McGill Univesity, Canada |
| 51 | Ali Reza | Ashrafi | University of Kashan |
| 52 | Jalal | Askari Farsangi | University of Kashan |
| 53 | Hamed | Aslani | University of Guilan |
| 54 | Parvane | Atashpeykar | University of Bonab |
| 55 | Ahmad Reza | Attari Polsangi | Shiraz University |
| 56 | Mehrasa | Ayatollahi | Payame Noor University |
| 57 | Saeid | Azam | University of Isfahan |
| 58 | Mahdieh | Azari | Islamic Azad University |
| 59 | Fariborz | Azarpanah | Shahid Chamran University of Ahvaz |
| 60 | Seyed Morteza | Babamir | University of Kashan |
| 61 | Mohammad | Bagheri | Editor in chief of the Journal of the History of Science |
| 62 | Neda | Bagheri | University of Mazandaran |
| 63 | Karam | Bahari | Razi University |
| 64 | Shima | Baharlouei | Isfahan University of Technology |
| 65 | Erfan | Bahmani | University of Zanjan |
| 66 | Faezeh | Bahmani | University of Kashan |
| 67 | Mojtaba | Bahramian | University of Kashan |
| 68 | Fariba | Bakrani | Shahid Beheshti University |
| 69 | Seddigheh | Banihashemi | University of Mazandaran |
| 70 | Narjes Sadat | Banitaba | Yazd University |
| 71 | Ali | Barani | Lorestan University |
| 72 | Ali | Barati | Razi University |
| 73 | Hasan | Barsam | University of Jiroft |
| 74 | Ali | Barzanouni | Hakim Sabzevari University |
| 75 | Esmaeil | Bashkar | Velayat university |
| 76 | Mostafa | Bayat | Amirkabir University of Technology |
| 77 | Behnam | Bazigaran | University of Kashan |
| 78 | Fatemeh | Bazikar | University of Guilan |
| 79 | Fereshteh | Behboudi | Imam Khomeini International University |
| 80 | Reza | Beyranvand | Lorestan University |
| 81 | Khodakhast | Bibak | Miami University, USA |
| 82 | Morteza | Bisheh-niasar | University of Kashan |
| 83 | Rajab Ali | Borzooei | Shahid Beheshti University |
| 84 | Ali | Bozorgmehr | Iran University of Medical Sciences |
| 85 | Alain | Bretto | Normandie University, France |
| 86 | Maurizio | Brunetti | Universita Federico II, Italy |
| 87 | Kinkar | Chandra Das | Sungkyunkwan University, South Korea |
| 88 | Mehran | Chehlabi | Islamic Azad University |
| 89 | Abbas | Cheraghi | University of Isfahan |
| 90 | Fatemeh | Choopani | Ferdowsi University of Mashhad |
| 91 | Mohammadehsan | Dadkani | University of Sistan and Baluchestan |

# Conference Participants

| | First Name | Last Name | Affiliation |
|---|---|---|---|
| 92 | Hassan | Daghigh | University of Kashan |
| 93 | Mohammadreza | Darafsheh | University of Tehran |
| 94 | Henri | Darmon | McGill University, Canada |
| 95 | Razie | Darvazeban Zade | Payame Noor University |
| 96 | Mahshid | Dashti | Malayer University |
| 97 | Zahra | Davari Shalamzari | Yazd University |
| 98 | Mostafa | Davtalab Olyaie | University of Kashan |
| 99 | Bijan | Davvaz | Yazd University |
| 100 | Luca | De Feo | University of Versailles, Switzerland |
| 101 | Mohammad Ali | Dehghan | Vali-e-Asr University of Rafsanjan |
| 102 | Sakineh | Dehghan | Shahid Beheshti University |
| 103 | Mahdi | Dehghani | University of Kashan |
| 104 | Fatemeh | Dehghani | Yazd University |
| 105 | Najmeh | Dehghani | Persian Gulf University |
| 106 | Zahra | Dehvari | Yazd University |
| 107 | Atefeh | Deris | Arak University |
| 108 | Zahra | Donyari | Shahid Chamran University of Ahvaz |
| 109 | Reza | Doostaki | Shahid Bahonar University of Kerman |
| 110 | Saeed | Doostali | University of Kashan |
| 111 | Fateme | Dorri | Ferdowsi University of Mashhad |
| 112 | Tomislav | Došlić | University of Zagreb, Croatia |
| 113 | Ghodrat | Ebadi | Tabriz University |
| 114 | Javad | Ebadpour Golanbar | Payame Noor University |
| 115 | Neda | Ebrahimi | Shahid Bahonar University of Kerman |
| 116 | Ali | Ebrahimijahan | Amirkabir University of Technology |
| 117 | Asiyeh | Ebrahimzadeh | Farhangian University |
| 118 | Sohrab | Effati | Ferdowsi University of Mashhad |
| 119 | Ali | Eftekhari | University of Kashan |
| 120 | Leila | Eftekhari | Tarbiat Modares University |
| 121 | Mohammad | Eghbali | University of Kashan |
| 122 | Taraneh | Eghlidos | Sharif University of Technology |
| 123 | Hossein | Eshraghi | University of Kashan |
| 124 | Mohammad Reza | Eslahchi | Tarbiat Modares University |
| 125 | Morteza | Essmaili | Kharazmi University |
| 126 | Masoumeh | Etebar | Shahid Chamran University of Ahvaz |
| 127 | Amin | Faghih | Sahand University of Technology |
| 128 | Farhad | Fakhar-Izadi | Amirkabir University of Technology |
| 129 | Farahnaz | Fakhraddin Arani | Nongovermental Collage Nonprofit Refah |
| 130 | Hamid | Faraji | Islamic Azad University |
| 131 | Farzaneh | Farhang Baftani | Islamic Azad University |
| 132 | Mohammad Reza | Farmani | Kharazmi University |
| 133 | Javad | Farokhi-Ostad | Birjand University of Technology |
| 134 | Fariba | Fayazi | University of Qom |
| 135 | Fateme | Fasihi | Bu-Ali Sina University of Hamedan |
| 136 | Gholam Hossien | Fathtabar Firouzjae | University of Kashan |
| 137 | Reza | Fayazi | Ferdowsi University of Mashhad |
| 138 | Mohamad Javad | Fazeli | University of Birjand |
| 139 | Fereshteh | Forouzesh | Bam University |
| 140 | Saba | Fotouhi | Amirkabir University of Technology |
| 141 | Batoul | Ganji Saffar | Al-Zahra University |

# Conference Participants

| | First Name | Last Name | Affiliation |
|---|---|---|---|
| 142 | Roberto | Garrappa | Polytechnic University of Bari, Italy |
| 143 | Majid | Gazor | Isfahan University of Technology |
| 144 | Somayeh | Ghadamyari | University of Sistan and Baluchestan |
| 145 | Mansour | Ghadiri | Yazd University |
| 146 | Ali | Ghafarpanah | Salman Farsi University of Kazerun |
| 147 | Ali | Ghalavand | University of Kashan |
| 148 | Fatemeh | Ghanadian | Damghan University |
| 149 | Fatemeh | Ghandi | University of Kashan |
| 150 | Mohammad Reza | Ghanei | University of Khansar |
| 151 | Hadi | Ghasemi | Hakim Sabzevari University |
| 152 | Mohsen | Ghasemi | Urmia University |
| 153 | Mohammad Hesam | Ghasemi | Shahid Beheshti University |
| 154 | Peyman | Ghiasvand | Payame Noor University |
| 155 | Hamid | Ghorbani | University of Kashan |
| 156 | Ali Reza | Ghorchizadeh | University of Birjand |
| 157 | Azin | Golbaharan | Kharazmi University |
| 158 | Faranak | Goodarzi | University of Kashan |
| 159 | Leila | Goodarzi | University of Kashan |
| 160 | Zahra | Gooya | Shahid Beheshti University |
| 161 | Farzaneh | Gorjizadeh | University of Shahrekord |
| 162 | Punam | Gupta | Dr. Harisingh Gour University, India |
| 163 | Mahnaz | Habibi | Islamic Azad University |
| 164 | Ali | Habibi Moakher | Payame Noor University of Tehran |
| 165 | Ali | Habibirad | Shiraz University of Technology |
| 166 | Masoud | Hadian Dehkordi | Iran University of Science and Technology |
| 167 | Amir Hosein | Hadian Rasanan | Shahid Beheshti University |
| 168 | Armin | Hadjian | University of Bojnord |
| 169 | Somayeh | Hadjirezaei | Vali-e-Asr University of Rafsanjan |
| 170 | Donya | Haghighi | Imam Khomeini International University |
| 171 | Saeid | Haghjoo | Shahid Rajaee Teacher Training University |
| 172 | Narges | Haj Aboutalebi | Islamic Azad University |
| 173 | Masoud | Hajarian | Shahid Beheshti University |
| 174 | Ashraf | Haji Olov Zarnagh | University of Kashan |
| 175 | Elham | Hajirezaei | University of Kashan |
| 176 | Hamid Reza | Hajisharifi | University of Khansar |
| 177 | Nooshin | Hakamipour | Buein Zahra Technical University |
| 178 | Mardjan | Hakimi-Nezhaad | Shahid Rajaee Teacher Training University |
| 179 | Shahad | Hasan | University of Kufa, Iraq |
| 180 | Farzane | Hashemi | University of Kashan |
| 181 | Ebrahim | Hashemi | Shahrood University of Technology |
| 182 | Mehdi | Hassani | University of Zanjan |
| 183 | Mostafa | Hassanlou | Urmia University |
| 184 | Marziyeh | Hatamkhani | University of Arak |
| 185 | Sina | Hedayatian | Shahid Chamran University of Ahvaz |
| 186 | Dariush | Heidari | Mahallat Institute of Higher Education |
| 187 | Mohammad | Heidari | Kharazmi University |

# Conference Participants

| | First Name | Last Name | Affiliation |
|---|---|---|---|
| 188 | Samira | Heidari | Imam Khomeini International University |
| 189 | Saghar | Heidari | Shahid Beheshti University |
| 190 | Azam | Hejazi Noghabi | Ferdowsi University of Mashhad |
| 191 | Mohammad | Hemami | Shahid Beheshti University |
| 192 | Esmail | Hesameddini | Shiraz University of Technology |
| 193 | Abdolaziz | Hesari | Shahid Chamran University of Ahvaz |
| 194 | Rasoul | Heydari Dastjerdi | Payame Noor University |
| 195 | Seyedeh Mahya | Hosseini | Payame Noor University |
| 196 | Zahra Sadat | Hosseini | Bu-Ali Sina University of Hamedan |
| 197 | Hasan | Hosseinzadeh | Islamic Azad University |
| 198 | Mohammad | Ilati | Sahand University of Technology |
| 199 | Mohammad Ali | Iranmanesh | Yazd University |
| 200 | Ali | Iranmanesh | Shahid Bahonar University of Kerman |
| 201 | Ali | Iranmanesh | Tarbiat Modares University |
| 202 | Marzieh | Izadi | Shahid Bahonar University of Kerman |
| 203 | Mehdi | Izadi | Shahid Rajaee Teacher Training University |
| 204 | Javad | Izadi | Payame Noor University |
| 205 | Mohammad Mahdi | Izadkhah | Birjand University of Technology |
| 206 | Kiyana | Izadyar | Shahid Chamran University of Ahvaz |
| 207 | Mehsin | Jabel Atteya | University of Al-Mustansiriyah, Iraq |
| 208 | Nasrin | Jafari | Yazd University |
| 209 | Habibollah | Jafari | Islamic Azad University |
| 210 | Mohammad | Jafari | Islamic Azad University |
| 211 | Hosna | Jafarmanesh | Hakim Sabzevari University |
| 212 | Nafisehsadat | Jafarzadeh | Tarbiat Modares University |
| 213 | Mehdi | Jahangiri | University of Maragheh |
| 214 | Reza | Jahani-Nezhad | University of Kashan |
| 215 | Ruhollah | Jahanipur | University of Kashan |
| 216 | Marziye | Jamali | University of Kashan |
| 217 | Sedighe | Jamshidvand | Khajeh Nasir Toosi University of Technology |
| 218 | Mohsen | Jannesari Ladani | Shahreza Higher Education Center |
| 219 | Elham | Javidmanesh | Ferdowsi University of Mashhad |
| 220 | Saeed | Johari | University of Isfahan |
| 221 | Farangis | Johari | Universidade Federal de Minas Gerais, Brazil |
| 222 | Maryam | Joulaei | Islamic Azad University |
| 223 | Alireza | Kabgani | Urmia University of Technology |
| 224 | Akram | Kabiri Samani | Payame Noor University |
| 225 | Azam | Kaheni | University of Birjand |
| 226 | Reza | Kahkeshani | University of Kashan |
| 227 | Zahra | Kamali | Islamic Azad University |
| 228 | Gholamreza | Karamali | Iran University of Science and Technology |
| 229 | Elaheh | Karimi | Islamic Azad University of Bushehr |
| 230 | Elham | Karimi | Al-Zahra University |
| 231 | Sajed | Karimy | Sharif University of Technology |
| 232 | Mohammadreza | Karimzadeh | University of Maragheh |

# Conference Participants

| | First Name | Last Name | Affiliation |
|---|---|---|---|
| 233 | Seyed Mohammad Bagher | Kashani | Tarbiat Modares University |
| 234 | Roghayeh | Katani | Yasouj University |
| 235 | Vilmos | Katona | University of Sopron, Hungary |
| 236 | Gülistan | Kaya Gök | Hakkari University, Turkey |
| 237 | Mohammad Bagher | Kazemi | University of Zanjan |
| 238 | Kianoush | Kazemi | University of Birjand |
| 239 | Ramin | Kazemi | Imam Khomeini International University |
| 240 | Rasool | Kazemi Najafabadi | University of Kashan |
| 241 | Seyed Mehdi | Kazemi Torbaghan | University of Bojnord |
| 242 | Vahid | Keshavarz | Shiraz University of Technology |
| 243 | Niloufar | Keshavarz | Persian Gulf University |
| 244 | Zahra | Keshtkar | Shahid Chamran University of Ahvaz |
| 245 | Maryam | Keyvani Maraghi | University of Maragheh |
| 246 | Mahmood | Khaksar-e Oshagh | Dr. Masaheb Institute of Mathematical Research |
| 247 | Somayeh | Khalashi Ghezelahmad | Islamic Azad University |
| 248 | Elahe | Khaldi | University of Kashan |
| 249 | Ghader | Khaledi | Payame Noor University |
| 250 | Mohsen | Khaleghi Moghadam | Sari Agricultural Sciences and Natural Resources University |
| 251 | Yasser | Khalili | Sari Agricultural Sciences and Natural Resources University |
| 252 | Alireza | Khalili Asboei | Farhangian University |
| 253 | Amir | Khamseh | Kharazmi University |
| 254 | Seyed Mohammad Amin | Khatami | Birjand University of Technology |
| 255 | Abolfazl | Khedmati | University of Kashan |
| 256 | Ekhtiar | Khodadadi | Islamic Azad University |
| 257 | Hamid | Khodaei | Malayer University |
| 258 | Davod | Khojasteh Salkuyeh | University of Guilan |
| 259 | Hassan | Khosravi | Gonbad Kavous University |
| 260 | Eisa | Khosravi Dehdezi | Persian Gulf University |
| 261 | Mohsen | Kian | University of Bojnord |
| 262 | Stefan | Kohl | University of St Andrews, Scotland |
| 263 | Masoumeh | Koohestani | Khajeh Nasir Toosi University of Technology |
| 264 | Majid | Kowkabi | University of Gonabad |
| 265 | Zeinab | Kowsari | Kharazmi University |
| 266 | Behnaz | Lajmiri | Amirkabir University of Technology |
| 267 | Sanaz | Lamei | University of Guilan |
| 268 | Seyed Jalal | Langari | Farhangian University |
| 269 | Samira | Latifi | University of Mohaghegh Ardabili |
| 270 | Ehsan | Lotfali Ghasab | Shahid Chamran University of Ahvaz |
| 271 | Maryam | Lotfipour | Fasa University |

# Conference Participants

| | First Name | Last Name | Affiliation |
|---|---|---|---|
| 272 | Abbas | Maarefparvar | IPM Institute For Research In Fundamental Sciences |
| 273 | Ali | Madanshekaf | University of Semnan |
| 274 | Nezam | Mahdavi-Amiri | Sharif University of Technology |
| 275 | Soheila | Mahdavi Zafarghandi | University of Kashan |
| 276 | Ali | Mahdipoor | University of Kashan |
| 277 | Zahra | Mahmoodi | Islamic Azad University |
| 278 | Mojgan | Mahmoudi | Shahid Beheshti University |
| 279 | Roya | Makrooni | University of Sistan and Baluchestan |
| 280 | Fereshteh | Malek | Khajeh Nasir Toosi University of Technology |
| 281 | Hassan | Maleki | Malayer University |
| 282 | Mohammad Hassan | Malekian | University of Kashan |
| 283 | Somayeh | Malekinejad | Payame Noor University |
| 284 | Sepideh | Maleki-Roudposhti | University of Guilan |
| 285 | Nasrin | Malek-Mohammadi | University of Kashan |
| 286 | Maryam | Malekpour | Al-Zahra University |
| 287 | Mahdiyeh | Manavi | Khajeh Nasir Toosi University of Technology |
| 288 | Mukut | Mani Tripathi | Banaras Hindu University, India |
| 289 | Hossien | Mashayekhi | University of Kashan |
| 290 | Javad | Mashreghi | University of Laval, Canada |
| 291 | Maryam | Masoudi Arani | Technical and Vocational University |
| 292 | Iman | Masoumi | Tafresh University |
| 293 | Kurosh | Mavaddat Nezhaad | University of Kashan |
| 294 | Majid | Mazrooei | University of Kashan |
| 295 | Alireza | Medghalchi | Kharazmi University |
| 296 | Hussain | Mehdi | University of Kufa, Iraq |
| 297 | Elahe | Mehraban | University of Guilan |
| 298 | Samira | Mehrangiz | Shiraz University |
| 299 | Hamid | Mehravaran | Islamic Azad University |
| 300 | Sadegh | Merati | Shiraz University |
| 301 | Ali | Mesforush | Shahrood University of Technology |
| 302 | Azar | Mirzaei | Razi University |
| 303 | Davoud | Mirzaei | University of Isfahan |
| 304 | Fatemeh | Mirzaei | Payame Noor University |
| 305 | Fatemeh | Mirzaei Gaskarei | Islamic Azad University |
| 306 | Mahsa | Mirzargar | Mahallat Institute of Higher Education |
| 307 | Mohammad Mahdi | Moayeri | Shahid Beheshti University |
| 308 | Alireza | Mofidi | Amirkabir University of Technology |
| 309 | Amir Abbas | Mofidian Naeini | Isfahan University of Technology |
| 310 | Hoda | Mohammadi | Payame Noor University |
| 311 | Maryam | Mohammadi | Isfahan University of Technology |
| 312 | Shahnaz | Mohammadi | Tabriz University |
| 313 | Reza | Mohammadiarani | Amirkabir University of Technology |
| 314 | Seyyed Ali | Mohammadiyeh | University of Kashan |
| 315 | Zahra | Mohammadzadeh | University of Birjand |
| 316 | Mahdi | Mohammadzadeh Karizaki | Torbat Heydarieh University |

# Conference Participants

| | First Name | Last Name | Affiliation |
|---|---|---|---|
| 317 | Akbar | Mohebbi | University of Kashan |
| 318 | Mina | Moini | University of Malayer |
| 319 | Zahra | Mohtasham | Shahid Beheshti University |
| 320 | Reza | Mokhtari | Isfahan University of Technology |
| 321 | Tahereh | Molaee | Al-Zahra University |
| 322 | Mahdieh | Molaeiderakhtenjani | University of Birjand |
| 323 | Ehsan | Momtahan | Yasouj University |
| 324 | Morteza | Moniri | Shahid Beheshti University |
| 325 | Mansooreh | Moosapoor | Farhangian University, Bentolhoda Sadr |
| 326 | Rasoul | Moradi | Persian Gulf University |
| 327 | Sirous | Moradi | Lorestan University |
| 328 | Ali | Moradzadeh-Dehkordi | Shahreza Higher Education Center |
| 329 | Seyed Adel | Moravveji | Laval University, Canada |
| 330 | Maysam | Mosadeq | Islamic Azad University |
| 331 | Seyfollah | Mosazadeh | University of Kashan |
| 332 | Zohreh | Mostaghim | Iran University of Science and Technology |
| 333 | Marziyeh | Motahari | Tarbiat Modares University |
| 334 | Hamid | Mousavi | University of Tabriz |
| 335 | Fatemeh Sadat | Mousavinejad | Yazd University |
| 336 | Ehsan | Movahednia | Behbahan Khatam Alanbia University of Technology |
| 337 | Kamran | Musazadeh | Islamic Azad University |
| 338 | Mohammad Javad | Nadjafi-Arani | Mahallat Institute of Higher Education |
| 339 | Razieh | Naghibi | Yazd University |
| 340 | Mohammadali | Naghipoor | Jahrom University |
| 341 | Reza | Naghipour | Kharazmi University |
| 342 | Mehran | Naghizadeh Qomi | University of Mazandaran |
| 343 | Alireza | Najafizadeh | Payame Noor University |
| 344 | Maryam | Najafvand Derikvandi | University of Kashan |
| 345 | Mehran | Namjoo | Vali-e-Asr University of Rafsanjan |
| 346 | Seyed Mojtaba | Naser Sheykholislami | Semnan University |
| 347 | Nasim | Nasrabadi | University of Birjand |
| 348 | Zohreh | Nazari | Vali-e-Asr University of Rafsanjan |
| 349 | Ali Mohammad | Nazari | University of Arak |
| 350 | Tahere | Nazari | Payame Noor University |
| 351 | Ali | Naziri-Kordkandi | Payame Noor University |
| 352 | Behzad | Nemati Saray | Institute for Advanced Studies in Basic Sciences |
| 353 | Mohsen | Niazi | University of Birjand |
| 354 | Sogol | Niazian | Islamic Azad University |
| 355 | Zohre | Nikooravesh | Birjand University of Technology |
| 356 | Ashkan | Nikseresht | Shiraz University |
| 357 | Fateme | Nikzad | Payame Noor University |
| 358 | Amir Hossein | Nokhodkar | University of Kashan |
| 359 | Monireh | Nosrati | University of Bonab |
| 360 | Bahareh | Nouri | Kharazmi University |

# Conference Participants

| | First Name | Last Name | Affiliation |
|---|---|---|---|
| 361 | Mohammad Reza | Oboudi | Shiraz University |
| 362 | Jafar | Ojbag | Azarbaijan Shahid Madani University |
| 363 | Fateme | Olia | Khajeh Nasir Toosi University of Technology |
| 364 | Reza | Orfi | Kharazmi University |
| 365 | Fatemeh | Parishani | University of Isfahan |
| 366 | Mehdi | Parsinia | Shahid Chamran University of Ahvaz |
| 367 | Rohollah | Parvinianzadeh | Yasouj University |
| 368 | Leila | Pedram | Imam Khomeini International University |
| 369 | Sajjad | Piradl | Payame Noor University |
| 370 | Vahid | Pirhadi | University of Kashan |
| 371 | Shima | Pirmohammadi | University of Isfahan |
| 372 | Mina | Pirzadeh | University of Guilan |
| 373 | Marzieh | Pourbabaee | University of Kashan |
| 374 | Hossein | Pourbashash | University of Garmsar |
| 375 | Alireza | Pourmoslemi | Payame Noor University |
| 376 | Maryam | Rabiee Farahani | Ferdowsi University of Mashhad |
| 377 | Farzad | Radmehr | Western Norway University of Applied Sciences, Norway |
| 378 | Majed | Raeesi | University of Sistan and Baluchestan |
| 379 | Marzieh | Raei | Malek Ashtar University of Technology |
| 380 | Ahmadreza | Raeisi Dehkordi | University of Isfahan |
| 381 | Mark | Raheb Ghamsary | Loma Linda University, USA |
| 382 | Saeid | Rahimi | Persian Gulf University |
| 383 | Morteza | Rahimi Khorzughi | University of Tehran |
| 384 | Parisa | Rahimkhani | Al-Zahra University |
| 385 | Gholamreza | Rahimlou | Technical and Vocational University |
| 386 | Mohammad Hossein | Rahmani Doust | University of Neyshabur |
| 387 | Hormoz | Rahmatan | Payame Noor University |
| 388 | Farhad | Rahmati | Amirkabir University of Technology |
| 389 | Marzieh | Rahmati | Payame Noor University |
| 390 | Ali | Rajaei | Tarbiat Modares University |
| 391 | Sayyed Mehrab | Ramezani | Yasouj University |
| 392 | Pardis | Ramezani | Payame Noor University |
| 393 | Jalil | Rashidinia | Iran University of Science and Technology |
| 394 | Abdolrahman | Razani | Imam Khomeini International University |
| 395 | Ebrahim | Reihani | Shahid Rajaee Teacher Training University |
| 396 | Parisa | Rezaei | University of Sistan and Baluchestan |
| 397 | Ali Asghar | Rezaei | University of Kashan |
| 398 | Akbar | Rezaei | Payame Noor University |
| 399 | Maryam | Rezaei Kashi | University of Kashan |
| 400 | Reza | Rezavand | University of Tehran |
| 401 | Monireh | Riahi | Damghan University |
| 402 | Mehdi | Riazi-Kermani | Wichita State University, USA |
| 403 | Mohammad Saber | Roohi | Shahid Beheshti University |

# Conference Participants

| | First Name | Last Name | Affiliation |
|---|---|---|---|
| 404 | Kees | Roos | Technical University Delft, Netherland |
| 405 | Nazanin | Roshandel Tavana | Amirkabir University of Technology |
| 406 | Mehdi | Rostami | Amirkabir University of Technology |
| 407 | Salimeh | Rostami | Yazd University |
| 408 | Esmaeil | Rostami | Shahid Bahonar University of Kerman |
| 409 | Mohsen | Rostamian Delavar | University of Bojnord |
| 410 | Arta | Rouhi | Semnan University |
| 411 | Reza | Saadati | Iran University of Science and Technology |
| 412 | Maryam | Saadati | Imam Khomeini International University |
| 413 | Abbas | Saadatmandi | University of Kashan |
| 414 | Rahele | Sabagh | University of Sistan and Baluchestan |
| 415 | Samaneh | Saberali | Urmia University |
| 416 | Sedigheh | Sabermahani | Al-Zahra University |
| 417 | Mehdi | Sabzevari | University of Kashan |
| 418 | Somayeh | Sadeghi | University of Isfahan |
| 419 | Behruz | Sadeqi | Payame Noor University |
| 420 | Nasrin | Sadri | IPM Institute For Research In Fundamental Sciences |
| 421 | Farhad | Saeediyoun | Amirkabir University of Technology |
| 422 | Hojatollah | Saeidi | University of Shahrekord |
| 423 | Zeinab | Saeidian | University of Kashan |
| 424 | Jamshid | Saeidian | Kharazmi University |
| 425 | Ali | Safaie | University of Maragheh |
| 426 | Farzaneh | Safari | Imam Khomeini International University |
| 427 | Akram | Safari-Hafshejani | Payame Noor University |
| 428 | Marziyeh | Saffarian | University of Kashan |
| 429 | Maryam | Saghalorzadeh | Jundi-Shapur University of Technology |
| 430 | Zahra | Sajjadnia | Shiraz University |
| 431 | Saman | Saki | Iran University of Science and Technology |
| 432 | Mohammad | Sal Moslehian | Ferdowsi University of Mashhad |
| 433 | Worod | Salah | University of Kufa, Iraq |
| 434 | Mohammad Ali | Salahshour | University of Kashan |
| 435 | Alireza | Salehi | Petroleum University of Technology |
| 436 | Abbas | Salemi Parizi | Shahid Bahonar University of Kerman |
| 437 | Hamid | Salimi | University of Tehran |
| 438 | Mahdi | Salmanpour | University of Kashan |
| 439 | Nasrin | Samadyar | Al-Zahra University |
| 440 | Mohammad Esmael | Samei | Bu-Ali Sina University |
| 441 | Karim | Samei | Bu-Ali Sina University |
| 442 | Amir Hossein | Sanatpour | Kharazmi University |
| 443 | Samaneh | Saneifar | Yazd University |
| 444 | Behnaz | Savizi | Islamic Azad University |
| 445 | Yamin | Sayyari | Sirjan University of Technology |
| 446 | Khadijeh | Sayyari | Kharazmi University |
| 447 | Salameh | Sedaghat | Buein Zahra Technical University |
| 448 | Monireh | Sedghi | University of Tabriz |

# Conference Participants

| | First Name | Last Name | Affiliation |
|---|---|---|---|
| 449 | Somayeh | Seifollahzadeh | University of Tabriz |
| 450 | Thekiso | Seretlo | University of Limpopo, South Africa |
| 451 | Zahra | Seyedi Lahrodi | Tarbiat Modares University |
| 452 | Saeed | Shaabanian | Tarbiat Modares University |
| 453 | Mohsin | Shaalan Abdulhussein Alakaashi | University of Kufa, Iraq |
| 454 | Zahra | Shabani Siahkalde | University of Sistan and Baluchestan |
| 455 | Mehrnoosh | Shadravan | Shahid Beheshti University |
| 456 | Khalil | Shafie | University of Northern Colorado, USA |
| 457 | Negur | Shehani Karamzadeh | Shahid Beheshti University |
| 458 | Amin | Shahkarami | Lorestan University |
| 459 | Hakimeh | Shahriaripour | Kerman Graduate University of Technology |
| 460 | Mohammad | Shahryari | Sultan Qaboos University, Muscat, Oman |
| 461 | Maryam | Shahsiah | University of Isfahan |
| 462 | Mina | Shamgani | University of Kashan |
| 463 | Mehdi | Shams | University of Kashan |
| 464 | Afsaneh | Shamsaki | Damghan University |
| 465 | Alireza | Shamsian | Institute for Advanced Studies in Basic Sciences |
| 466 | Reza | Sharafdini | Persian Gulf University |
| 467 | Javad | Sharafi | University of Kashan |
| 468 | Seyedeh Fatemeh | Shariati | Amirkabir University of Technology |
| 469 | Kamran | Sharifi | Shahroud University of Technology |
| 470 | Farzad | Shaveisi | Razi University |
| 471 | Omid Ali | Shehni-Karamzadeh | Shahid Chamran University of Ahvaz |
| 472 | Marjan | Sheibani Abdolyousefi | Semnan University |
| 473 | Hayder Baqer Ameen | Shelash | University of Kufa, Iraq |
| 474 | Efat | Shikhi | Payam Noor University |
| 475 | Maryam | Shirali | Shahid Chamran University of Ahvaz |
| 476 | Nasrin | Shirali | Shahid Chamran University of Ahvaz |
| 477 | Parisa | Shiri | Sahand University of Technology |
| 478 | Farrokh | Shirjian | Tarbiat Modares University |
| 479 | Hasan | Shlaka | University of Kufa, Iraq |
| 480 | Shirin | Shoaee | Shahid Beheshti University |
| 481 | Raheleh | Shokrpour | University of Tabriz |
| 482 | Ali | Shukur | Belarusian State University, Belarus; The Islamic University, Iraq |
| 483 | Amirhossein | Sobhani | Semnan University |
| 484 | Mahsa | Soheil Shamaee | University of Kashan |
| 485 | Mahdi | Sohrabi-Haghighat | University of Arak |
| 486 | Maryam | Soleimani | IPM Institute For Research In Fundamental Sciences |
| 487 | Majid | Soleimani-Damaneh | University of Tehran |
| 488 | Rasoul | Soleimani | Payame Noor University |
| 489 | Fazlollah | Soleymani | Institute for Advanced Studies in Basic Sciences |
| 490 | Mahdieh | Soleymani Baghsha | Sharif University of Technology |

# Conference Participants

| | First Name | Last Name | Affiliation |
|---|---|---|---|
| 491 | Farnaz | Soliemany | Urmia University |
| 492 | Andrea | Solotar | Universidad de Buenos Aires, Argentina |
| 493 | Vali | Soltani Masih | Payame Noor University |
| 494 | Zeinab | Soltani | University of Kashan |
| 495 | Ahmad Reza | Soltani | Kuwait University, Kuwait |
| 496 | Sima | Soltani Renani | Isfahan University of Technology |
| 497 | Gholamreza | Soltaniabri | University of Tehran |
| 498 | Somayeh | Soltanpour | Petroleum University of Technology |
| 499 | Ghiyam | Soudan | Bu-Ali Sina University |
| 500 | Predrag | Stanimirović | University of Nis, Serbia |
| 501 | Teerapong | Suksumran | Chiang Mai University, Thailand |
| 502 | Hamid Reza | Tabrizidooz | University of Kashan |
| 503 | Mojgan | Taghavi | Shahid Beheshti University |
| 504 | Meysam | Taheri-Dehkordi | University of Kashan |
| 505 | Maryam | Taheri Sedeh | University of Kashan |
| 506 | Maryam | Tahmasbi | Shahid Beheshti University |
| 507 | Haleh | Tajadodi | University of Sistan and Baluchestan |
| 508 | Farkhondeh | Takhteh | Persian Gulf University |
| 509 | Ebrahim | Tamimi | Semnan University |
| 510 | Somayyeh | Tari | Azarbaijan Shahid Madani University |
| 511 | Mostafa | Tavakoli | Ferdowsi University of Mashhad |
| 512 | Mohammadreza | Tavakkoli Moghaddam | Shahid Beheshti University |
| 513 | Reza | Tayebi Khorami | Islamic Azad University |
| 514 | Atieh | Teymourzadeh | University of Mazandaran |
| 515 | Faezeh | Tiba | University of Sistan and Baluchestan |
| 516 | Abdolsaleh | Toghdori | Yazd University |
| 517 | Fateme | Torabi | Damghan University |
| 518 | Soraya | Torkaman | Yazd University |
| 519 | Vali | Torkashvand | Islamic Azad University |
| 520 | Thekiso | Trevor Seretlo | University of Limpopo, South Africa |
| 521 | Constantine | Tsinakis | Vanderbilt University, USA |
| 522 | Seyed Mansour | Vaezpour | Amirkabir University of Technology |
| 523 | Farzaneh | Vahdanipour | University of Mohaghegh Ardabili |
| 524 | Seryas | Vakili | University of Tabriz |
| 525 | Amir | Veisi | Yasouj University |
| 526 | Andrei | Vesnin | Tomsk State University, Russia |
| 527 | Changchang | Xi | Capital Normal University, China |
| 528 | Marjan | Yaghmaei | Kharazmi University |
| 529 | Mohamad | Yar Ahmadi | Shahid Chamran University of Ahvaz |
| 530 | Zahra | Yarahmadi | Islamic Azad University |
| 531 | Mohammad Reza | Yasamian | Payame Noor University |
| 532 | Azam | Yazdani | Amirkabir University of Technology |
| 533 | Mohsen | Yousefnezhad | Shiraz University |
| 534 | Omid | Zabeti | University of Sistan and Baluchestan |
| 535 | Sayyed Mahmood | Zabetzadeh | Payame Noor University |
| 536 | Fatemeh | Zabihi | University of Kashan |
| 537 | Amirhesam | Zaeim | Payame Noor University |

# Conference Participants

| | First Name | Last Name | Affiliation |
|---|---|---|---|
| 538 | Hassan | Zaherifar | Yazd University |
| 539 | Elham | Zangiabadi | Vali-e-Asr University of Rafsanjan |
| 540 | Hossein | Zare | Tarbiat Modares University |
| 541 | Rohollah | Zarei | Yasouj University |
| 542 | Behnam | Zarpak | Shahed University |
| 543 | Maryam | Zeinali | Shahid Chamran University of Ahvaz |
| 544 | Ali | Zeydi Abdian | Lorestan University |
| 545 | Elham | Zeynal | Islamic Azad University |
| 546 | Mosayeb | Zohrehvand | Malayer University |
| 547 | Bijan | Zohuri-Zangeneh | Sharif University of Technology |

# List of Presented Papers

# Keynote and Invited Talks

# Higher Dimensional Ideal Approximation Theory

Javad Asadollahi*
Department of Pure Mathematics, University of Isfahan, Isfahan, Iran

---

Abstract. Ideal approximation theory is a gentle generalization of the classical approximation theory and deals with morphisms and ideals instead of objects and subcategories. Our aim in this presentation is to study ideal approximation theory over $n$-exact categories. In particular, the higher version of the notions such as ideal cotorsion pairs, phantom ideals, Salce's Lemma and Wakamatsu's Lemma for ideals will be introduced and studied. The main source of $n$-exact categories are $n$-cluster tilting subcategories of exact categories.

**Keywords:** $n$-Exact categories, $n$-Cluster tilting subcategories, Phantom morphisms.
**AMS Mathematical Subject Classification [2010]:** 18E05, 18G25, 18G15.

---

## 1. Introduction

The starting point of approximation theory is the discovery of the existence of injective envelopes by Baer in 1940. Approximation theory, that is approximation of complicated objects of a category by simpler objects in a specific subcategory, is essentially based on the notions of preenvelopes and precovers. Recall that a class $\mathscr{F}$ of $R$-modules is precovering if for every $R$-module $M$, there exists a morphism $\varphi : F \to M$ with $F \in \mathscr{F}$ such that the induced morphism $\mathrm{Hom}_R(F', F) \to \mathrm{Hom}_R(F', M)$ is surjective, for all $F' \in \mathscr{F}$. Dually the notion of preenveloping classes is defined. An important problem in this context is to investigate whether a class of modules is (pre) enveloping or/and (pre)covering.

Approximation theory also plays a central role in the representation theory of algebras under the name of left approximations (preenvelopings) and right approximations (precoverings). For a good account on approximation theory see the monograph [5].

A nice generalization of the classical approximation theory, known as ideal approximation theory is studied systematically in [4] and [6], that gives morphisms and ideals of categories equal importance as objects and subcategories. In this theory, the role of the objects and subcategories in classical approximation theory is replaced by morphisms and ideals of the category. An ideal of a category is an additive subfunctor of the Hom functor, which is closed under compositions by morphisms from left and right. For instance, the phantom ideal and phantom cover in module category are studied extensively.

On the other hand, in a successful attempt to build up a higher version of Auslander's correspondence and also generalizing the classical theory of almost

*Speaker

split sequences of Auslander-Reiten, Iyama [7, 8] introduced the notion of $n$-cluster tilting subcategories, where $n$ is an integer greater or equal than 1. Soon it is realized that these subcategories play a crucial role in the theory and so cluster tilting subcategories became the subject of several researches.

In particular, study of the structure of such subcategories leads Jasso [9] to a higher version of the classical homological algebra and as a consequence new notions such as $n$-abelian and $n$-exact categories were born. These notions provide appropriate higher versions of the classical abelian and exact categories, in the sense that 1-abelian and 1-exact categories are the usual abelian and exact categories. Instead of the usual kernels and cokernels, resp. inflations and deflations, in these categories we have the notions of $n$-kernels and $n$-cokernels and the role of short exact sequences, resp. conflations, are played by exact complexes with $n+2$ terms.

Following these ideas, the general goal of this presentation is to introduce ideal approximation theory into the higher homological algebra. Our results show that the correct context in which to carry these arguments out is that of an $n$-cluster tilting subcategory of an exact category. By [9, §4] we know that these subcategories are $n$-exact, i.e. with 'admissible' sequences with $n+2$ terms as conflations. Using this structure, a 'higher ideal approximation theory' is developed. We state and prove some foundational results in this subject to motivate the theory.

## 2. Main Results

Let us begin with some basic facts and backgrounds we need throughout. We are mainly work in an exact category $(\mathscr{A}, \mathscr{E})$, where $\mathscr{A}$ is an additive category and $\mathscr{E}$ is the class of conflations, see [2].

Let $n \geq 1$ be a fixed integer. The notion of $n$-exact categories is defined by Jasso in [9, §4] as a natural generalization of exact categories. Let $\mathscr{C}$ be an additive category. Let $f^0 : X^0 \longrightarrow X^1$ be a morphism in $\mathscr{C}$. An $n$-cokernel of $f^0$ is a sequence

$$X^1 \xrightarrow{f^1} X^2 \longrightarrow \cdots \longrightarrow X^n \xrightarrow{f^n} X^{n+1},$$

of morphisms in $\mathscr{C}$ such that for every $X \in \mathscr{C}$ the induced sequence

$$0 \longrightarrow \mathscr{C}(X^{n+1}, X) \xrightarrow{f^n_*} \cdots \xrightarrow{f^1_*} \mathscr{C}(X^1, X) \xrightarrow{f^0_*} \mathscr{C}(X^0, X),$$

of abelian groups is exact. Here and throughout we write $\mathscr{C}(-,-)$ instead of $\mathrm{Hom}_{\mathscr{C}}(-,-)$. We denote the $n$-cokernel of $f^0$ by $(f^1, f^2, \ldots, f^n)$. The notion of $n$-kernel of a morphism $f^n : X^n \longrightarrow X^{n+1}$ is defined similarly, or rather dually.

A sequence $X^0 \xrightarrow{f^0} X^1 \longrightarrow \cdots \longrightarrow X^n \xrightarrow{f^n} X^{n+1}$ of objects and morphisms in $\mathscr{C}$, is called $n$-exact [9, Definitions 2.2, 2.4] if $(f^0, f^1, \ldots, f^{n-1})$ is an $n$-kernel of $f^n$ and $(f^1, f^2, \ldots, f^n)$ is an $n$-cokernel of $f^0$. An $n$-exact sequence like the above one, usually will be denoted by

$$X^0 \overset{f^0}{\rightarrowtail} X^1 \xrightarrow{f^1} X^2 \longrightarrow \cdots \longrightarrow X^n \overset{f^n}{\twoheadrightarrow} X^{n+1}.$$

An $n$-exact structure on $\mathscr{C}$ is a class $\mathscr{X}$ of $n$-exact sequences, called $\mathscr{X}$-admissible $n$-exact sequences, that satisfies axioms of [9, Definition 4.2]. An $n$-exact category is a pair $(\mathscr{C}, \mathscr{X})$, where $\mathscr{C}$ is an additive category and $\mathscr{X}$ is an $n$-exact structure on $\mathscr{C}$.

Typical examples of $n$-exact categories are $n$-cluster tilting subcategories of exact categories, see [9, Theorem 4.14].

Definition 2.1. [9, Definition 4.13] Let $(\mathscr{A}, \mathscr{E})$ be a small exact category. A subcategory $\mathscr{C}$ of $\mathscr{A}$ is called an $n$-cluster tilting subcategory if it satisfies the following conditions.

i) For every object $A \in \mathscr{A}$, there exists an admissible monomorphism $A \rightarrowtail C$, which is also a left $\mathscr{C}$-approximation of $A$.
ii) For every object $A \in \mathscr{A}$, there exists an admissible epimorphism $C' \twoheadrightarrow A$, which is also a right $\mathscr{C}$-approximation of $A$.
iii) There exists equalities $\mathscr{C}^{\perp_n} = \mathscr{C} = {}^{\perp_n}\mathscr{C}$, where

$$\mathscr{C}^{\perp_n} = \{A \in \mathscr{A} : \mathrm{Ext}^i_{\mathscr{E}}(C, A) = 0 \text{ for all } C \in \mathscr{C} \text{ and all } 1 \leq i \leq n-1\},$$
$${}^{\perp_n}\mathscr{C} = \{A \in \mathscr{A} : \mathrm{Ext}^i_{\mathscr{E}}(A, C) = 0 \text{ for all } C \in \mathscr{C} \text{ and all } 1 \leq i \leq n-1\}.$$

For a detailed explanation of the notion of Ext in exact categories see [3, Subsection 6.2].

Definition 2.2. Let $\mathscr{A}$ be an additive category. A two sided ideal $\mathscr{I}$ of $\mathscr{A}$ is a subfunctor

$$\mathscr{I}(-,-) : \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \longrightarrow \mathscr{A}b,$$

of the bifunctor $\mathscr{A}(-,-)$ that associates to every pair $A$ and $A'$ of objects in $\mathscr{A}$ a subgroup $\mathscr{I}(A, A') \subseteq \mathscr{A}(A, A')$ such that

i) If $f \in \mathscr{I}(A, A')$ and $g \in \mathscr{A}(A', C)$, then $gf \in \mathscr{I}(A, C)$.
ii) If $f \in \mathscr{I}(A, A')$ and $g \in \mathscr{A}(D, A)$, then $fg \in \mathscr{I}(D, A')$.

Let $\mathscr{I}$ be an ideal of $\mathscr{A}$ and $A \in \mathscr{A}$ be an object of $\mathscr{A}$. An $\mathscr{I}$-precover of $A$ is a morphism $C \xrightarrow{\varphi} A$ in $\mathscr{I}$ such that any other morphism $C' \xrightarrow{\varphi'} A$ in $\mathscr{I}$ factors through $\varphi$, i.e. there exists a morphism $\psi : C' \longrightarrow C$ such that $\varphi\psi = \varphi'$. $\mathscr{I}$ is called a precovering ideal if every object $A \in \mathscr{A}$ admits an $\mathscr{I}$-precover. The notions of $\mathscr{I}$-preenvelope and preenveloping ideals are defined dually. See [4] for definitions and details.

Let $\mathscr{F}$ be a sub-bifunctor of $\mathrm{Ext}^1(-,-)$. By [4, page 759], a morphism $f : X \longrightarrow A$ in $\mathscr{C}$ is called $\mathscr{F}$-projective if for every object $B$ in $\mathscr{C}$, $\mathscr{F}(f, B) = 0$. In other words, $f : X \to A$ in $\mathscr{C}$ is $\mathscr{F}$-projective if the $n$-pullback of any $\mathscr{F}$-admissible $n$-exact sequence along $f$ is contractible. An object $A$ in $\mathscr{C}$ is called $\mathscr{F}$-projective if the identity morphism is an $\mathscr{F}$-projective morphism. The ideal of $\mathscr{F}$-projective morphisms is denoted by $\mathscr{F}$-proj. The notions of $\mathscr{F}$-injective morphisms and $\mathscr{F}$-injective objects are defined dually. The ideal of $\mathscr{F}$-injective morphisms is denoted by $\mathscr{F}$-inj.

These notions form the basics of ideal approximation theory. Another important notion in this context, is the notion of phantom ideals and phantom cover that are studied extensively by Herzog in [6].

We study these notions in an $n$-cluster tilting subcategory of an $n$-exact category. For instance higher phantom morphisms are defined as follows.

Definition 2.3. Let $\mathscr{C}$ be an $n$-cluster tilting subcategory of an exact category $(\mathscr{A}, \mathscr{E})$ with $n$-exact structure $\mathscr{X}$. Let $\mathscr{F}$ be a sub-bifunctor of $\mathrm{Ext}^n_{\mathscr{X}}(-,-)$. A morphism $\varphi$ in $\mathscr{C}$ is called an $n$-$\mathscr{F}$-phantom morphism if the $n$-pullback of every

$\mathscr{X}$-admissible $n$-exact sequence along $\varphi$ is an $\mathscr{F}$-admissible $n$-exact sequence. In other words, $\varphi : X \longrightarrow A$ in $\mathscr{C}$ is an $n$-$\mathscr{F}$-phantom morphism if for every object $A'$ in $\mathscr{C}$, the morphism

$$\mathrm{Ext}^n(\varphi, A') : \mathrm{Ext}^n(A, A') \longrightarrow \mathrm{Ext}^n(X, A'),$$

of abelian groups takes values in the subgroup $\mathscr{F}(X, A')$. We denote the collection of all $n$-$\mathscr{F}$-phantom morphisms by $\Phi(\mathscr{F})$. Note that it is easy to see that $\Phi(\mathscr{F})$ forms an ideal of $\mathscr{C}$.

Based on such definitions, we study higher cotorsion ideals, higher Salce's Lemma and Wakamatsu's Lemma, all are pillars of classical approximation theory. For example, a higher version of Wakamatsu's Lemma can be stated as follows.

THEOREM 2.4. *Let $(\mathscr{C}, \mathscr{X})$ be an $n$-cluster tilting subcategory of an exact category $(\mathscr{A}, \mathscr{E})$ with enough $\mathscr{X}$-injective objects. Let $\mathscr{I}$ be an ideal of $\mathscr{C}$ which is left closed under $n$-extensions by objects in $\mathscr{I}$. Let $A$ be an object of $\mathscr{C}$ and $i : I \longrightarrow A$ be the $\mathscr{I}$-cover of $A$. Then for every $X \in \mathscr{I}$, there exists the exact sequence*

$$0 \longrightarrow \mathrm{Ext}^n(X, K_n) \longrightarrow \mathrm{Ext}^n(X, K_{n-1}) \longrightarrow \cdots \longrightarrow \mathrm{Ext}^n(X, K_1) \longrightarrow 0,$$

*of abelian groups, where $K_n \longrightarrow K_{n-1} \longrightarrow \cdots \longrightarrow K_1$ is an $n$-kernel of $i$.*

## References


1. J. Asadollahi and S. Sadeghi, *Higher ideal approximation theory,* (2020). arXiv:2010.03423v1
2. Th. Bühler, *Exact categories,* Expo. Math. **28** (1) (2010) 1–69.
3. L. Frerick and D. Sieg, *Exact categories in Functional Analysis,* preprint 2010. (On-line ready at www.math.uni-trier.de/abteilung/analysis/HomAlg.pdf)
4. X. H. Fu, P. A. Guil Asensio, I. Herzog and B. Torrecillas, *Ideal approximation theory,* Adv. Math. **244** (2013) 750–790.
5. R. Göbel and J. Trlifaj, *Approximations and Endomorphism Algebras of Modules,* de Gruyter Expositions in Mathematics, Vol. 41, Walter de Gruyter, Berlin, 2006.
6. I. Herzog, *The phantom cover of a module,* Adv. Math. **215** (2007) 220–249.
7. O. Iyama, *Higher-dimensional Auslander-Reiten theory on maximal orthogonal subcategories,* Adv. Math. **210** (2007) 22–50.
8. O. Iyama, *Cluster tilting for higher Auslander algebras,* Adv. Math. **226** (2011) 1–61.
9. G. Jasso, *n-Abelian and n-exact categories,* Math. Z. **283** (2016) 703–759.



E-mail: asadollahi@sci.ui.ac.ir
E-mail: so.sadeghi@sci.ui.ac.ir

# On the Structure of Profinite Polyadic Groups

Mohammad Shahryari*
Department of Mathematics, College of Science, Sultan Qaboos University, Muscat, Oman

Abstract. We introduce profinite polyadic groups as the $n$-ary generalizations of a the ordinary profinite groups. The structure of such profinite systems will be investigated and we will show that a topological polyadic group $(G, f)$ is profinite, if and only if, it is compact, Hausdorff, totally disconnected. It is also shown that a topological polyadic group $\mathrm{der}_{\theta,b}(G, \bullet)$ is profinite, if and only if, the corresponding retract group $(G, \bullet)$ is profinite and the automorphism $\theta$ is continuous. Also, for a variety (formation) $\mathfrak{X}$ of finite polyadic groups, we show that a polyadic group $(G, f)$ is pro-$\mathfrak{X}$, if and only if it is compact, Hausdorff, totally disconnected and for every open congruence $R$, the finite polyadic group $(G/R, f_R)$ belongs to $\mathfrak{X}$.
**Keywords:** Polyadic groups, $n$-Ary groups, Profinite groups and polyadic groups, Post's cover and retract of a polyadic group.
**AMS Mathematical Subject Classification [2010]:** 20N15.

## 1. Introduction

In this talk, we introduce the class of the profinite polyadic groups: polyadic groups which are the inverse limit of a system of finite polyadic groups. A polyadic group is a natural generalization of the concept of group to the case where the binary operation of group replaced with an $n$-ary associative operation, one variable linear equations in which have unique solutions. So, *polyadic group* means an $n$-ary group for a fixed natural number $n \geq 2$. These interesting algebraic objects are introduced by Kasner and Dörnte ([1, 2]) and studied extensively by Emil Post during the first decades of the last century, [3]. During decades, many articles are published on the structure of polyadic groups.

It is easy to define topological polyadic groups, and so, one can ask which topological polyadic groups are profinite. In this talk, we discuss this problem and as the main result, we show that a topological polyadic group $(G, f)$ is profinite, if and only if, it is compact, Hausdorff, totally disconnected.

**1.1. Polyadic Groups.** A polyadic group is a pair $(G, f)$ where $G$ is a non-empty set and $f : G^n \to G$ is an $n$-ary operation, such that

i) the operation is associative, i.e.

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}),$$

for any $1 \leq i < j \leq n$ and for all $x_1, \ldots, x_{2n-1} \in G$, and

*Speaker

ii) for all $a_1, \dots, a_n, b \in G$ and $1 \leq i \leq n$, there exists a unique element $x \in G$ such that
$$f(a_1^{i-1}, x, a_{i+1}^n) = b.$$

Note that, here we use the compact notation $x_i^j$ for every sequence
$$x_i, x_{i+1}, \dots, x_j,$$
of elements in $G$, and in the special case when all terms of this sequence are equal to a fixed $x$, we denote it by $\overset{(t)}{x}$, where $t$ is the number of terms.

Suppose $(G, f)$ is a polyadic group and $a \in G$ is a fixed element. Define a binary operation
$$x \bullet y = f(x, \overset{(n-2)}{a}, y).$$
Then $(G, \bullet)$ is an ordinary group, called the *retract* of $(G, f)$ over $a$. Such a retract will be denoted by $ret_a(G, f)$. All retracts of a polyadic group are isomorphic.

One of the most fundamental theorems of polyadic group is the following, now known as *Hosszú -Gloskin's theorem.* We will use it frequently to determine the connections between the polyadic and ordinary profinite groups. According to this theorem, for any polyadic group $(G, f)$, there exists an ordinary group $(G, \bullet)$, an automorphism $\theta$ of $(G, \bullet)$ and an element $b \in G$ such that

1. $\theta(b) = b$,
2. $\theta^{n-1}(x) = bxb^{-1}$, for every $x \in G$,
3. $f(x_1^n) = x_1\theta(x_2)\theta^2(x_3)\cdots\theta^{n-1}(x_n)b$, for all $x_1, \dots, x_n \in G$.

Because of this, we use the notation $\mathrm{der}_{\theta,b}(G, \bullet)$ for $(G, f)$ and we say that $(G, f)$ is $(\theta, b)$-derived from the group $(G, \bullet)$.

## 2. Main Results

A profinite polyadic group is the inverse limit of an inverse system of finite polyadic groups. More precisely, let $(I, \leq)$ be a directed set and suppose $\{(G_i, f_i), \varphi_{ij}, I\}$ is an inverse system of finite polyadic groups. This means that for every pair $(i, j)$ of elements of $I$ with $j \leq i$, we are given a polyadic homomorphism
$$\varphi_{ij} : (G_i, f_i) \to (G_j, f_j)$$
such that the equality $\varphi_{jk}\varphi_{ij} = \varphi_{ik}$ holds for all $k \leq j \leq i$. Now, assume that
$$(G, f) = \varprojlim_i (G_i, f_i).$$

Then $(G, f)$ is called a profinite polyadic group. Note that as each $G_i$ is finite, being a closed subspace of the direct product of a family of finite sets, $(G, f)$ is compact, Hausdorff, and totally disconnected topological polyadic group.

Recall that, according to Hosszú -Gloskin's theorem, we have $(G_i, f_i) = \mathrm{der}_{\theta_i,b_i}(G_i, \bullet_i)$, for some ordinary group $(G_i, \bullet_i)$, an element $b_i \in G_i$, and an automorphism $\theta_i$. We will prove
$$(G, \bullet) = \varprojlim_i (G_i, \bullet_i).$$

This shows that the group $(G, \bullet)$ is profinite. Our first result is a characterization of the profinite polyadic groups.

THEOREM 2.1. *Let $(G, f)$ be a polyadic group. Then $(G, f)$ is profinite, if and only if, it is compact, Hausdorff, totally disconnected.*

The next result concerns the connection between a profinite polyadic group and its retract:

THEOREM 2.2. *A topological polyadic group $(G, f) = \mathrm{der}_{\theta,b}(G, \bullet)$ is profinite if and only if, its retract $(G, \bullet)$ is profinite and the automorphism $\theta$ is continuous.*

To explain the last result, let $\mathfrak{X}$ be a class of groups. We define $\mathrm{Pol}_n(\mathfrak{X})$ to be the class of all $n$-ary groups, the corresponding retract in which belongs to $\mathfrak{X}$. It is shown that if $\mathfrak{X}$ is a variety or a formation (of finite groups), then $\mathrm{Pol}_n(\mathfrak{X})$ has also the same property. For this reason, we show the second class by the same notation $\mathfrak{X}$.

THEOREM 2.3. *For a variety (formation) $\mathfrak{X}$ of finite polyadic groups, a polyadic group $(G, f)$ is pro-$\mathfrak{X}$, if and only if, it is compact, Hausdorff, totally disconnected, and for every open congruence $R$, the finite polyadic group $(G/R, f_R)$ belongs to $\mathfrak{X}$.*

## References

1. W. Dörnte, *Unterschungen über einen verallgemeinerten Gruppenbegriff*, Math. Z. **29** (1929) 1–19.
2. E. Kasner, *An extension of the group concept*, Bull. Amer. Math. Soc. **10** (1904) 290–291.
3. E. Post, *Polyadic groups*, Trans. Amer. Math. Soc. **48** (1940) 208–350.

E-mail: m.ghalehlar@squ.edu.om

# On the Behavior of Birkhoff Sums Generated by Irrational Rotation

Ali Shukur*
The Islamic University, Iraq
Belarusian State University, Belarus
Anatolij Borisovich Antonevich
Belarusian State University, Minsk, Belarus
Moscow State University, Moscow, Russian
and Andri Vasilyevich Kochergin
Belarusian State University,Minsk, Belarus
Moscow State University, Moscow, Russian

ABSTRACT. In this talk, we will consider the Birkhoff sums $f(n, x, h)$, where $f$ is a continuous function with zero average on the unit circle, generated by irrational rotation. We show that the unique boundary condition of growth rate of sequence $\max f(n, x, h)$ for $n \to \infty$, if the average of the Birkhoff sums, i.e. $\frac{1}{n} f(n, x, h)$ is approaching to zero.
**Keywords:** Birkhoff sums, Irrational rotation, Resolvent, Weighted shift operator.
**AMS Mathematical Subject Classification [2010]:** 47B37, 34C29.

## 1. Introduction

Let $X$ be a compact topological space and $\alpha : X \to X$ be a continuous invertable map. This kind of maps generate a dynamical systems (cascades)$suchas : \alpha^k(x) = \alpha(\alpha^{k-1}(x)),\ k \in \mathbb{Z}$. For $f : X \to \mathbb{C}$ and $n \in \mathbb{Z}$, the Birkhoff sums $f(n, x)$ is represented by

$$f(n,x) = f(n,x,\alpha) = \begin{cases} \sum_{k=0}^{n-1} f(\alpha^k(x)), & \text{for } n > 0, \\ 0, & \text{for } n = 0, \\ -\sum_{k=n}^{-1} f(\alpha^k(x)) = -f(-n;\alpha^n(x)), & \text{for } n < 0. \end{cases}$$

Particularly, the behavior of the Birkhoff sums is related to ergodic theorem, this fact is shown in the next discussion.

Let $PM_\alpha(X)$ be a set of probability Borel measures in $X$, which invariant relatively to $\alpha$. The Birkhoff's ergodic theorem says, if $\mu \in PM_\alpha(X)$ and $f \in L_1(X, \mu)$, then the limit of the Birkhoff average exist, $\mu$-almost everywhere (See [6]). In case of continuous functions, the following result presented:

*Speaker

Theorem 1.1. [5] *If $X$ be a compact topological space, $\alpha : X \to X$ be a continuous map and $f \in C(X)$, then*

$$\lim_{n\to\infty} \max_X \frac{1}{n} f(n,x,\alpha) = \max\{\int_X f(x)d\mu : \mu \in PM_\alpha(X)\},$$

$$\lim_{n\to\infty} \min_X \frac{1}{n} f(n,x,\alpha) = \min\{\int_X f(x)d\mu : \mu \in PM_\alpha(X)\}.$$

Moreover, the map $\alpha$ is called *strictly ergodic,* if there exist only one invariant probability measure $\mu$. From Theorem 1.1, follows that the following convergent, where $f \in C(X)$ holds:

$$\lim_{n\to\infty} \max_X \frac{1}{n} f(n;x) = \int_X f(x)d\mu, \tag{1}$$

$$\lim_{n\to\infty} \min_X \frac{1}{n} f(n;x) = \int_X f(x)d\mu. \tag{2}$$

In the present work, we will provide a detailed description about the convergence of (1) and (2). The estimates of powers of operators generated by irrational are given.

## 2. Main Results

Let $T = \mathbb{R}/\mathbb{Z}$ be the unit circle and the map $x \to x+h$ generates the rotation such that $\alpha_h : T \to T$ with angle $2\pi h$, where $h$ is irrational number. For a function $f \in C(T)$ the Birkhoff sums $f(n,x,h)$ is represented by

$$f(n,x;h) = \begin{cases} f(x) + f(x+h) + \cdots + f(x+(n-1)h), & \text{for } n > 0, \\ 0, & \text{for } n = 0, \\ -[f(x-h) + f(x-2h) + \cdots + f(x-nh)], & \text{for } n < 0. \end{cases}$$

Theorem 2.1. [3] *Let $h$ be irrational number. For any sequence of numbers $\sigma_n$, which monotonic converge to zero, there exist a continuous function $\varphi$ with zero average such that Birkhoff sums $f(n,h,\varphi)$ is growing such as faster than*

$$f(n,h,\varphi) \geq n\sigma_n.$$

Theorem 2.2. [3] *Let $\varphi$ be a continuous function with zero average, which is not trigonometrical polynomial. For any monotonic converge to zero $\sigma_n$, there exist an irrational number $h$, such that $f(n_k,h,\varphi)$ is growing such as faster than*

$$f(n_k,h,\varphi) \geq n_k\sigma_{n_k}.$$

*If $\varphi$ is smooth, then $f(q_k,h,\varphi)$ is bounded.*

The proof was based on some facts of number theory and ergodic theory in [3].

### 3. Estimate of Powers of Weighted Shift Operator

An operator $T_\gamma$ acting on $C(\mathbb{S}^1)$ by formula

$$T_\gamma u(x) = u(\gamma(x)),$$

is called *a rotation operator*. For any $a \in C(\mathbb{S}^1)$, the operator acting by formula

$$(aT_h u)(x) = a(x)u(x+h), \tag{3}$$

is called *a weighted shift operator generated by rotation* and it is norm of the powers is given by

$$\|[aT_h]^n\| = \max_x \prod_{j=0}^{n-1} |a(x+jh)|.$$

In the following, we denote by $\sigma(T)$ the spectrum of a bounded operator $T : F \to F$ on a Banach space $F$ and by $r(T)$ the spectral radius. From Gelfand's formula follows that the spectral radius can be calculated by norm of the powers of operator $T$, such that

$$r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}.$$

Howeover, the behavior of the resolvent $(T - \lambda I)^{-1}$ depends on the growth rate of the powers of operator. On the relation between $\|T^n\|$ and $\|(T - \lambda I)^{-1}\|$ we refer to [**4**, **7**].

Theorem 3.1. [**1**] *Let $aT_h$ be a weighted shift operator generated by:*

1) *If $h$ is a rational number, i.e. $h =: \frac{m}{N}, N \neq 0$, – some fractions, then*

$$\sigma(aT_h) = \{\lambda : \exists x \in X, \lambda^N = \prod_{j=0}^{N-1} a(x + \frac{jm}{N})\}.$$

*As well as*

$$R(aT_h) = [\max_x \prod_{j=0}^{N-1} |a(x + \frac{jm}{N})|]^{\frac{1}{N}}.$$

2) *If $h$ is irrational number and $a(x) \neq 0$ for all $x$, then $\sigma(aT_h) = \{\lambda : |\lambda| = \Phi(a)\}$, where $\Phi(a)$ is the geometric average of $a$, i.e.*

$$\Phi(a) = \exp[\int_0^1 \ln |a(x)|dx]. \tag{4}$$

*In particular, $R(aT_h) = \Phi(a)$.*

Moreover, we assume that the spectral radius in (4) is equal to 1, so, if $\varphi(x) = \ln |a(x)|$, then

$$\int_0^1 \varphi(x) = 0, \tag{5}$$

$$\frac{1}{n} \ln \|[aT_h]^n\| \to 0 \text{ and } \ln \|[aT_h]^n\| = \max_x \sum_{j=0}^{n-1} \varphi(x + jh).$$

THEOREM 3.2. [2] *Let $\varphi(x)$ be not trigonometrical polynomial and it satisfies condition (5). For any sequence $\omega_n$ such that $\frac{\omega_n}{n} \to 0$, there exists irrational number $h$, such that for some subsequence $n_j$ holds*

$$\|[aT_h]^{n_j}\| \geq e^{\omega_{n_j}}.$$

In what follows, we consider a special kind of irrational numbers defined by:

$$A_\sigma = \{h \in \mathbb{R} : \exists\, C, M, \text{ such that } \quad |h - \frac{m}{N}| \geq \frac{C}{N^{2+\sigma}}\ \forall\ m \in \mathbb{Z}, N > M\}.$$

THEOREM 3.3. *Let $h \in A_\sigma$, where $\sigma > 0$ and let the operator (3) satisfies (5) and $|a| \in C^m(\mathbb{S}^1)$. If $m > \sigma + 3$, then the sequence of power operator (3) $aT_h$ is bounded.*

**Proof.** For $h \in A_\sigma$ satisfies

$$|h - \frac{p}{k}| \geq \frac{M_1}{k^{2+\sigma}},$$

which equal to

$$|kh - p| \geq \frac{M_1}{k^{1+\sigma}}.$$

Thus,

$$\frac{1}{|1 - e^{i2\pi kh}|} \leq M_2|k|^{1+\sigma}.$$

Due to the condition $|a(x)| > 0$, we have $\varphi(x) = \ln|a(x)|$ and $|a(x)|$ belongs to $C^m(\mathbb{S}^1)$. Therefore the Fourier Coefficient of $|(a(x))|$ hold

$$|C_k| \leq \frac{M_3}{|k|^{m-1}}.$$

Thus, for Fourier Coefficient of function $\varphi_n(x)$ we have

$$|C_k \frac{1 - e^{i2\pi knh}}{1 - e^{i2\pi kh}}| \leq |C_k \frac{2}{1 - e^{i2\pi kh}}| \leq M_2 M_3 \frac{1}{|k|^{m-2-\sigma}},$$

which does not depend on $n$.

If $m - 2 - \sigma > 1$, then

$$\sum_{k \neq 0} \frac{1}{|k|^{m-2-\sigma}},$$

convergent.

Therefore

$$\max_x |\varphi_n(x)| \leq M_2 M_3 \sum_{k \neq 0} \frac{1}{|k|^{m-2-\sigma}}.$$

## References

1. A. B. Antonevich, *Linear Functional Eequation. Operator Approach*, Universitetskoe, Minsk, 1988 (in Russian), Birkhaüser Basel, 1996.
2. A. B. Antonevich and A. Shukur, *On the powers of operator generated by rotation*, J. Anal. Appl. **16** (2018) 57–67.
3. A. B. Antonevich, A. Kochergin and A. Shukur, *On the behavior of Birkhoff sums generated by rotation*, Sbornik Math. to appear.
4. T. Bermudez, M. Gonzalez and M. Mbekhta, *Operators with an ergodic power*, Studia Math. **141** (2000) 201–208.
5. I. P. Cornfeld, S. V. Fomin and Y. G. Sinai, *Ergodic Theory*, Springer-Verlag, New York, 1980.
6. A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems,* Cambridge University Press, Cambridge, 2012.
7. Y. Katznelson and L. Tzafriri, *On power bounded operators,* J. Functional Anal. **68** (1986) 313–328.

E-mail: shukur.math@gmail.com

E-mail: antonevich@bsu.by

E-mail: a.kochergin@gmail.com; avk@econ.msu.ru

# Gyrogroups: Generalization of Groups

Teerapong Suksumran*
Department of Mathematics, Faculty of Science, Chiang Mai University, Thailand

Abstract. A gyrogroup is a non-associative algebraic structure, which is a natural generalization of a group, arising from the study of the parametrization of the Lorentz transformation group by Abraham A. Ungar. Gyrogroups share many properties with groups and, in fact, every group may be viewed as a gyrogroup with trivial gyroautomorphisms. In this talk, we indicate strong connections between gyrogroups and classical structures such as groups, linear spaces, topological spaces, and metric spaces from the algebraic point of view.
**Keywords:** Gyrogroup, Gyrogroup action, Representation of gyrogroup, Topological gyrogroup, Gyronorm.
**AMS Mathematical Subject Classification [2010]:** 20N05.

## 1. Introduction

Roughly speaking, a gyrogroup (also called a Bol loop with the $A_\ell$-property) is a non-associative group-like structure that shares many properties with groups. One of the most important examples of gyrogroups is the *complex Möbius gyrogroup*, which consists of the complex open unit disk $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ and *Möbius addition* $\oplus_M$ defined by

$$a \oplus_M b = \frac{a+b}{1+\bar{a}b}, \qquad \text{for all } a, b \in \mathbb{D}.$$

It is not difficult to check that Möbius addition is not associative so that $(\mathbb{D}, \oplus_M)$ does not form a group. However, it has several properties like groups, which eventually motivate the notion of a gyrogroup. In the following definition, we present an abstract version of the axioms of being a gyrogroup.

Denote by $\mathrm{Aut}(G)$ the group of automorphisms of $(G, \oplus)$, where $G$ is a non-empty set and $\oplus$ is a binary operation on $G$.

Definition 1.1 (Gyrogroups). A non-empty set $G$, together with a binary operation $\oplus$ on $G$, is called a *gyrogroup* if it satisfies the following properties.

(G1) There exists an element $e \in G$ such that $e \oplus a = a$ for all $a \in G$.(identity)

(G2) For each $a \in G$, there exists an element $b \in G$ such that $b \oplus a = e$. (inverse)

(G3) For all $a$, $b \in G$, there is an automorphism $\mathrm{gyr}[a, b] \in \mathrm{Aut}(G)$ such that

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c,$$

for all $c \in G$. (left gyroassociative law)

(G4) For all $a$, $b \in G$, $\mathrm{gyr}[a \oplus b, b] = \mathrm{gyr}[a, b]$. (left loop property)

*Speaker

It can be proved that every gyrogroup has a unique two-sided identity, denoted by $e$ and that any element $a$ of a gyrogroup has a unique two-sided inverse, denoted by $\ominus a$. The automorphism $\mathrm{gyr}[a, b]$ is called the *gyroautomorphism* generated by $a$ and $b$. The gyroautomorphisms play a fundamental role in gyrogroup theory, as they come to remedy the absence of associativity in gyrogroups and lead to the *gyroassociative law*, a weak form of the associative law. In fact, any group can be made into a gyrogroup by defining the gyroautomorphisms to be the identity automorphism and, conversely, any gyrogroup with *trivial* gyroautomorphisms is a group. From this point of view, the notion of gyrogroups suitably generalizes that of groups. A gyrogroup that satisfies a commutative-like law,

$$a \oplus b = \mathrm{gyr}[a, b](b \oplus a), \qquad \text{for all elements } a, b,$$

is called a *gyrocommutative* gyrogroup, in order to emphasize similarity of an abelian group.

## 2. Gyrogroups and Related Structures

**2.1. Groups and Gyrogroups.** Gyrogroups and groups are related in various ways. For instance, if $G$ is a gyrogroup, then the symmetric group of $G$, denoted by $\mathrm{Sym}(G)$, admits the gyrogroup structure and $G$ can be embedded as a twisted subgroup of $\mathrm{Sym}(G)$ via the embedding $a \mapsto L_a, a \in G$, where $L_a$ is the *left gyrotranslation* defined by $L_a(x) = a \oplus x$ for all $x \in G$. One of the most important equations that connects group and gyrogroup operations is the following composition law,

$$L_a \circ L_b = L_{a \oplus b} \circ \mathrm{gyr}[a, b],$$

which is an abstract version of the composition law of Lorentz boosts as well as Möbius translations.

Another strong connection between groups and gyrogroups, which provides the machinery for studying gyrogroups via group theory, is shown in the next theorem. Recall that a subset $B$ of a group $\Gamma$ is a *twisted subgroup* of $\Gamma$ if the following properties hold: (i) $1 \in B$, 1 being the identity of $\Gamma$; (ii) if $b \in B$, then $b^{-1} \in B$; and (iii) if $a, b \in B$, then $aba \in B$ [3]. Recall also that a subset $B$ of a group $\Gamma$ is a (left) *transversal* to a subgroup $\Xi$ of $\Gamma$ if each element $g$ of $\Gamma$ can be written uniquely as $g = bh$ for some $b \in B$ and $h \in \Xi$ [4]. Let $B$ be a transversal to a subgroup $\Xi$ in a group $\Gamma$. Given two elements $a$ and $b$ of $B$, define $a \odot b$ to be the unique element of $B$ arising from the product $ab$ in $\Gamma$. Therefore, any transversal $B$ to $\Xi$ gives rise to a binary operation $\odot$ on $B$, called the *transversal operation.*

Definition 2.1. [6, Gyrotriples] Let $\Gamma$ be a group, let $B$ be a subset of $\Gamma$, and let $\Xi$ be a subgroup of $\Gamma$. A triple $(\Gamma, B, \Xi)$ is called a *gyrotriple* if the following properties hold:

i) $B$ is a transversal to $\Xi$ in $\Gamma$;
ii) $B$ is a twisted subgroup of $\Gamma$;
iii) $\Xi$ normalizes $B$, that is, $hBh^{-1} \subseteq B$ for all $h \in \Xi$.

Theorem 2.2. [6, Section 2.1] *If $G$ is a gyrogroup, then there exists a group $\Sigma$ containing an isomorphic copy $\hat{G}$ of $G$ such that $(\Sigma, \hat{G}, \mathrm{Aut}(G))$ is a gyrotriple.*

*Conversely, if* $(\Gamma, B, \Xi)$ *is a gyrotriple, then* $B$ *equipped with the transversal operation is a gyrogroup.*

**2.2. Gyrogroup Actions and Gyrogroup Representations.** Viewing a group action as a homomorphism, we can extend the notion of group actions to the case of gyrogroups in a natural way. Let $G$ be a gyrogroup and let $X$ be a non-empty set. A function from $G \times X$ to $X$, written $(a, x) \mapsto a \cdot x$, is a *gyrogroup action* of $G$ on $X$ if the following properties hold:

i) $e \cdot x = x$ for all $x \in X$;
ii) $a \cdot (b \cdot x) = (a \oplus b) \cdot x$ for all $a, b \in G, x \in X$.

As proved in [5], every gyrogroup action of $G$ on $X$ induces a gyrogroup homomorphism from $G$ to $\mathrm{Sym}(X)$ and vice versa. This leads to the notion of permutation representations of a gyrogroup. Several results in the theory of group actions remain true in the case of gyrogroups, including the orbit-stabilizer theorem [5, Theorem 3.9], the orbit decomposition theorem [5, Theorem 3.10], and the Burnside lemma—also known as the Cauchy–Frobenius lemma [5, Theorem 3.11].

Imposing the linear structure on the set $X$ acted by a gyrogroup enables us to study linear representations of $G$ on the linear space $X$ in the same way as one studies linear representations of groups. This method allows us to examine the structure of a gyrogroup, using tools from linear algebra. Let $G$ be a gyrogroup and let $V$ be a linear space. A gyrogroup action of $G$ on $V$ is said to be *linear* if in addition for each $a \in G$, the map defined by $v \mapsto a \cdot v, v \in V$, is a linear transformation on $V$. As proved in [8], every linear action of a gyrogroup $G$ on a linear space $V$ induces a gyrogroup homomorphism from $G$ to $\mathrm{GL}(V)$ and vice versa, where $\mathrm{GL}(V)$ is the general linear group of $V$. Several classical theorems are extended to the case of gyrogroups, including Schur's lemma [8, Theorem 3.13] and Maschke's theorem [8, Theorem 3.2].

**2.3. Topological Gyrogroups.** In 2017, W. Atiponrat introduced the notion of topological gyrogroups, which is motivated by well-known concrete gyrogroups such as Euclidean Einstein gyrogroups and Möbius gyrogroups [1]. A gyrogroup $G$ equipped with a topology is called a *topological gyrogroup* if (i) the gyroaddition map $\oplus\colon (x, y) \mapsto x \oplus y$ is jointly continuous and (ii) the inversion map $\ominus\colon x \mapsto \ominus x$ is continuous, where $G \times G$ carries the product topology. Let $(G, \tau)$ be a topological gyrogroup and let $\mathrm{H}(G)$ be the group of homeomorphisms of $G$. In the case when $\tau$ possesses a nice property and $\mathrm{H}(G)$ is endowed with a suitable topology, we obtain a topological version of Cayley's theorem, as shown in the following theorem:

THEOREM 2.3. [9, Theorem 3.4] *Let* $G$ *be a locally compact Hausdorff topological gyrogroup and suppose that* $\mathrm{H}(G)$ *carries the g-topology. Then* $\mathrm{H}(G)$ *is a completely regular topological group and* $G$ *is embedded into* $\mathrm{H}(G)$ *as a twisted subgroup via the topological embedding* $a \mapsto L_a, a \in G$.

Here, the g-topology on $\mathrm{H}(G)$ is the topology generated by the subbase

$$\{[C, O]\colon C \text{ is closed in } G,\ O \text{ is open in } G, \text{ and } C \text{ or } X \setminus O \text{ is compact}\},$$

where $[A, B] = \{f \in \mathrm{H}(G)\colon f(A) \subseteq B\}$.

A topological gyrogroup $G$ is said to be *strong* if there exists an open base $\mathcal{U}$ at the identity $e$ of $G$ such that $\mathrm{gyr}[a, b](U) = U$ for all $a, b \in G, U \in \mathcal{U}$ [2]. Several results that are true for topological groups can be extended to the case of strongly topological gyrogroups. Among other things, we obtain the following theorem:

Theorem 2.4. [10, Proposition 5] *Every strongly topological gyrogroup $G$ can be embedded as a closed subgyrogroup of a path-connected and locally path-connected topological gyrogroup $G^{\bullet}$. Furthermore, gyrocommutativity, first countability, and metrizability are shared by $G$ and $G^{\bullet}$.*

**2.4. Normed Gyrogroups.** Recall that the most standard metric on groups is the word metric (with respect to some generating set), which allows us to study a (finitely generated) group as a geometric object. Groups with word metric fall in the category of normed groups. This inspires us to define a normed gyrogroup.

Definition 2.5. [7, Gyronorms] Let $G$ be a gyrogroup. A function $\|\cdot\|\colon G \to \mathbb{R}$ is called a *gyronorm* on $G$ if the following properties hold:

i) $\|x\| \geq 0$ for all $x \in G$ and $\|x\| = 0$ if and only if $x = e$; (positivity)
ii) $\|\ominus x\| = \|x\|$ for all $x \in G$; (invariant under taking inverses)
iii) $\|x \oplus y\| \leq \|x\| + \|y\|$ for all $x, y \in G$; (subadditivity)
iv) $\|\mathrm{gyr}[a, b]x\| = \|x\|$ for all $a, b, x \in G$. (invariant under gyrations)

Any gyrogroup with a specific gyronorm is called a *normed gyrogroup*.

Let $(G, \|\cdot\|)$ be a normed gyrogroup. Then the function $d\colon G \times G \to \mathbb{R}$ defined by

$$d(x, y) = \|\ominus x \oplus y\|, \qquad \text{for all } x, y \in G,$$

is a metric on $G$, called a *gyronorm metric*, and so $(G, d)$ becomes a metric space. We emphasize that a normed gyrogroup need not be a topological gyrogroup. Therefore, sufficient conditions for a normed gyrogroup to be a topological gyrogroup are worth finding. We present a few conditions below.

Theorem 2.6. [7, Theorem 11] *Let $G$ be a normed gyrogroup with the corresponding metric $d$. If one of the following conditions holds, then $G$ is a topological gyrogroup with respect to the topology induced by $d$:*

1) *Right-gyrotranslation inequality: $d(x \oplus a, y \oplus a) \leq d(x, y)$ for all $a, x, y \in G$;*
2) *Klee's condition: $d(x \oplus y, a \oplus b) \leq d(x, a) + d(y, b)$ for all $a, b, x, y \in G$.*

Theorem 2.7. [10, Theorem 15] *Let $G$ be a normed gyrogroup with the corresponding metric $d$. If every right gyrotranslation $R_a\colon x \mapsto x \oplus a, x \in G$, and the inversion function $\ominus$ are continuous, then $G$ is a topological gyrogroup with respect to the topology induced by $d$.*

**Acknowledgement**

I would like to thank Professor Alireza Ashrafi for his kind invitation to participate in the 51th Annual Iranian Mathematics Conference.

## References

[1] W. Atiponrat, *Topological gyrogroups: Generalization of topological groups*, Topology Appl. **224** (2017) 73–82.

[2] M. Bao and F. Lin, *Feathered gyrogroups and gyrogroups with countable pseudocharacter*, Filomat **33** (16) (2019) 5113–5124.

[3] T. Foguel, M. K. Kinyon and J. Phillips, *On twisted subgroups and Bol loops of odd order*, Rocky Mountain J. Math. **36** (2006) 183–212.

[4] T. Foguel and A. A. Ungar, *Involutory decomposition of groups into twisted subgroups and subgroups*, J. Group Theory **3** (2000) 27–46.

[5] T. Suksumran, *Gyrogroup actions: A generalization of group actions*, J. Algebra **454** (2016) 70–91.

[6] T. Suksumran, *Involutive groups, unique* 2*-divisibility, and related gyrogroup structures*, J. Algebra Appl. **16** (6) (2017) 1750114.

[7] T. Suksumran, *On metric structures of normed gyrogroups*, Mathematical Analysis and Applications (Th. M. Rassias and P. M. Pardalos, eds.), Springer Optimization and Its Applications, Vol. 154, Springer, Cham, 2019, pp. 529–542.

[8] T. Suksumran and K. Wiboonton, *Möbius's functional equation and Schur's lemma with applications to the complex unit disk*, Aequat. Math. **91** (3) (2017) 491–503.

[9] J. Wattanapan, W. Atiponrat and T. Suksumran, *Embedding of locally compact Hausdorff topological gyrogroups in topological groups*, Topology Appl. **273** (2020) 107102.

[10] J. Wattanapan, W. Atiponrat and T. Suksumran, *Embedding of strongly topological gyrogroups in path-connected and locally path-connected gyrogroups*, Symmetry **12** (11) (2020) 1817.

E-mail: teerapong.suksumran@cmu.ac.th

# The Wiener Index and Hyperbolic Geometry of Fullerene Graphs

Andrei Vesnin*
Tomsk State University, Tomsk, Russia
Sobolev Institute of Mathematics, Novosibirsk, Russia

Abstract. We observe that fullerene graphs are one-skeletons of polytopes which can be realized in a hyperbolic 3-dimensional space with all dihedral angles equal to $\pi/2$. We are referring volume of such polytope as a hyperbolic volume of a fullerene. We demonstrate that hyperbolic volumes of fullerenes correlate with few important topological indices and can serve as a chemical descriptor for fullerenes.

**Keywords:** Fullerene, Wiener index, Hyperbolic space, Right-angled polyhedron.
**AMS Mathematical Subject Classification [2010]:** 05C12, 51M10, 92E10.

## 1. Introduction

A fullerene is a spherically shaped molecule consisting of carbon atoms in which every carbon ring is a pentagon or a hexagon. Every atom of a fullerene has bounds with exactly three neighboring atoms. The molecule may be a hollow sphere, ellipsoid, tube, or many other shapes and sizes. Fullerenes are the subjects of intensive research in chemistry, and they have found promising technological applications, especially in nanotechnology and material sciences.

Molecular graphs of fullerenes are called *fullerene graphs*. A fullerene graph is a 3-connected planar graph in which every vertex has degree 3, and every face is pentagonal or hexagonal. By Euler's polygonal formula, the number of pentagonal faces is always 12, and the total number $f$ of faces in fullerene graph with $n$ vertices is equal to $n/2 + 2$. It is known that fullerene graphs having $n$ vertices exist for $n = 20$ and for all even $n \geq 24$. The number of all non-isomorphic fullerene graphs $C_n$ for many values of $n$ can be found in [2]. Fullerenes without adjacent pentagons, i.e., each pentagon is surrounded only by hexagons, satisfy the isolated pentagon rule (IPR), and are called *IPR fullerene graphs*.

Mathematical studies of fullerenes include applications of topological and graph theory methods, information theory approached, design of combinatorial and computational algorithms, etc.

In the present talk we will discuss a new point of views on fullerenes based on non-Euclidean geometry of corresponding polytopes. The talk is based on papers [4, 5, 6].

*Speaker

## 2. Fullerenes as Hyperbolic Polytopes

Let $\mathbb{H}^3$ be a 3-dimensional hyperbolic space, i.e, 3-dimensional connected and simply connected Riemann manifold with constant sectional curvature equals to $-1$, see [7]. Its conformal Poincare ball model $\mathbb{B}^3$ is given by the unit ball $\mathbb{B}^3 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : ||x|| < 1\}$, where $||x||^2 = x_1^2 + x_2^2 + x_3^2$, with the metric

$$ds^2 = 4\frac{dx_1^2 + dx_2^2 + dx_3^2}{(1 - ||x||^2)^2}.$$

Geodesics in $\mathbb{B}^3$ are either line segments through the origin or arcs of circles orthogonal to its boundary $\partial\mathbb{B}^3$. The totally geodesic subspaces of $\mathbb{B}^3$ are the intersections with $\mathbb{B}^3$ of generalized spheres (spheres or hyperplanes) orthogonal to $\partial\mathbb{B}^3$.

A polytope is called *acute-angled* if all its dihedral angles are at most $\pi/2$. The following rigidity holds in a 3-dimensional hyperbolic space $\mathbb{H}^3$.

Theorem 2.1. [7] *A bounded acute-angled polytope in $\mathbb{H}^3$ is uniquely (up to isometry) determined by its combinatorial type and dihedral angles.*

We say that polyhedron is *right-angled* if all its dihedral angles equal to $\pi/2$. A connected graph is said to be *cyclically $k$-connected* if at least $k$ edges have to be removed to split it into two connected components both having a cycle.

Theorem 2.2. (Pogorelov, Andreev) *A polyhedral graph is 1-skeleton of a bounded right-angled hyperbolic polytope if and only if the graph is 3-regular and cyclically 5-connected.*

The combinatorially smallest example of right-angled hyperbolic polytope is a dodecahedron. The class of right-angled hyperbolic polytopes has many interesting properties and can be used to construct hyperbolic 3-manifolds by four-coloring of faces [8, 9]. Topological properties of corresponding 3-manifolds are discussed in [10]. Observe, that any fullerene graph satisfies Theorem 2.2 and can be realized as 1-skeleton of a right-angled hyperbolic polytope, see Figure 1 for two isomers of 48-vertex fullerene in $\mathbb{H}^3$. By Theorem 2.1 any geometric invariant of its right-angled realization in $\mathbb{H}^3$, for example a volume, can be taken as a fullerene invariant. The fullerene, presented on the right-hand side in Figure 1, has volume 17.034558, that is minimal among all $C_{48}$ fullerenes, and the fullerene, presented in the right-hand side in Figure 1, has volume 18.61.7604, that is maximal among all $C_{48}$ fullerenes.

Volumes of bounded right-angled hyperbolic polytopes can be estimated by number of vertices.

Theorem 2.3. [6] *If $P$ is a bounded right-angled hyperbolic polytope with $n \geq 24$ vertices, then*

$$(n-8) \cdot \frac{v_8}{32} \leqslant \mathrm{vol}(P) < (n-14) \cdot \frac{5v_3}{8},$$

*where $v_8$ is the volume of a regular ideal hyperbolic octahedron and $v_3$ is the volume of a regular ideal hyperbolic tetrahedron.*

Constants $v_8$ and $v_3$ have expressions in terms of the Lobachevsky function

$$\Lambda(x) = -\int_0^x \log|2\sin t| dt.$$

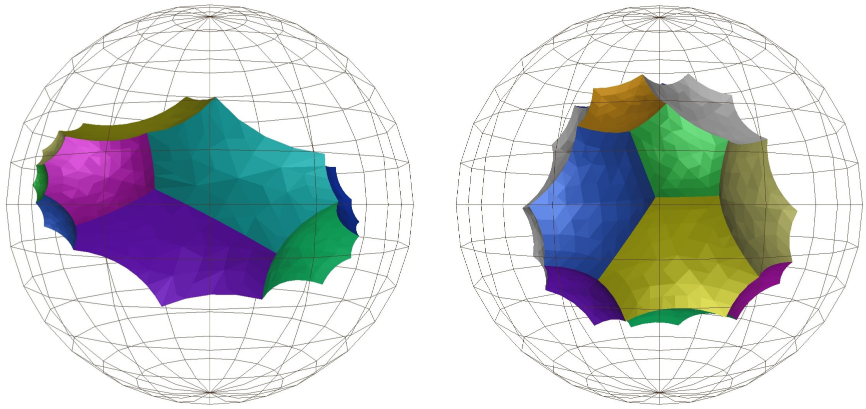


FIGURE 1. Two isomers of fullerene $C_{48}$ as right-angled polytopes in a hyperbolic space.

Namely, $v_8 = 8\Lambda(\pi/4)$ and $v_3 = 2\Lambda(\pi/6)$. To six decimal places $v_8$ is 3.663862, and $v_3$ is 1.014941.

### 3. Wiener Complexity of Fullerene Graphs

The vertex set of a graph $G$ is denoted by $V(G)$. The distance $d(u,v)$ between vertices $u, v \in V(G)$ is the number of edges in a shortest path connecting $u$ and $v$ in $G$. By *transmission* of $v \in V(G)$, we means the sum of distances from vertex $v$ to all other vertices of $G$,

$$\mathrm{tr}(v) = \sum_{u \in V(G)} d(u,v).$$

Transmissions of vertices are used to design of many distance-based topological indices. Usually, a topological index is a graph invariant that maps a family of graphs to a set of numbers such that values of the invariant coincide for isomorphic graphs. The *Wiener index* is a topological index defined as follows

$$W(G) = \sum_{\{u,v\} \subset V(G)} d(u,v) = \frac{1}{2} \sum_{v \in V(G)} \mathrm{tr}(v).$$

It was introduced as a structural descriptor for tree-like organic molecules by Harold Wiener in 1947. The Wiener index that has found important applications in chemistry. Various aspects of the theory and practice of the Wiener index of fullerene graphs are discussed in many works [1]. For other topological indices which are useful to study fullerenes, see e.g. [3].

The number of different vertex transmissions in a graph $G$ is known as the *Wiener complexity* [4] (or the *Wiener dimension*), $C_W(G)$. This graph invariant can be regarded as a measure of transmission variety. A graph is called *transmission irregular* if all vertices of the graph have pairwise different transmissions, i.e., it has the largest possible Wiener complexity. It is obvious that a transmission irregular graph has the trivial automorphism group.

The computer search of transmission irregular graphs was realized in [4] for hundreds of millons of graphs.

Theorem 3.1. [4] *There do not exist transmission irregular fullerene graphs with $n \leq 232$ vertices and IPR fullerene graphs with $n \leq 270$ vertices.*

Since the almost all fullerene graphs have no symmetries, we conject that transmission irregular graphs exist for a large number of vertices (when the interval of possible values of transmissions will be sufficiently large with respect to the number of vertices).

Question 3.2. Does there exist a transmission irregular fullerene graph (IPR fullerene graph)? If yes, then what is the order of such graphs?

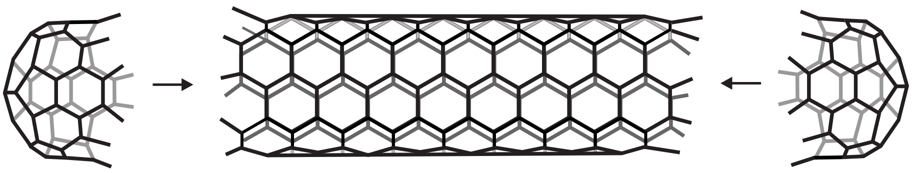


Figure 2. Construction of a nanotubical fullerene with two caps.

Next we consider fullerene graphs with the maximal Wiener index. A class of fullerene graphs of tubular shapes is called *nanotubical fullerene graphs.* They have cylindrical shape with the two ends capped by subgraphs containing six pentagons and possible some hexagons called caps (See an illustration in Figure 2).

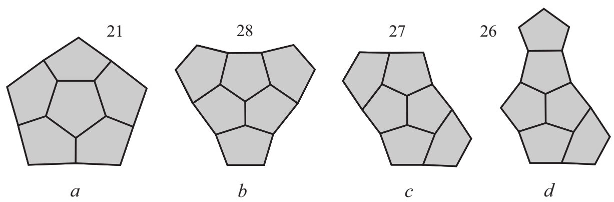



Figure 3. Pentagonal parts of caps for nanotubical fullerene graphs with the maximal Wiener index.

It was observed in [4] that if $n = 32$ or $36 \leq n \leq 232$, then maximal Wiener index fullerene with $n$ vertices looks as a nanotube with one of four types of caps presented in Figure 3. Type (a) appears 21 times, type (b) appears 28 times, type (c) appears 27 times, and type (d) appears 28 times.

## 4. Hyperbolic Volume, Topological Indices and Stability of Fullerenes

It is known that topological indices can serve as descriptors for some properties of chemical compounds. It was shown in [5] that hyperbolic volumes of fullerenes, i.e., volumes of right-angled hyperbolic polytopes with fullerenes as 1-skeletons, correlate with some properties of fullerenes and can be considered as descriptors too. It can be seen from Figure 4 that there are two isomers of $C_{60}$ with the largest volume coincide with two having the smallest relative energy, and also three isomers of $C_{60}$ with the smallest volume coincide with three having the largest relative energy.

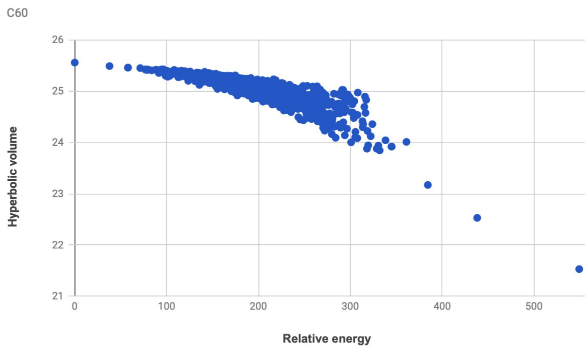



FIGURE 4. Scatter chart of volume and relative energy.

Moreover, the observed correlation between hyperbolic volumes and Weiner indices suggest few conjectures about minimal volume polytopes for various classes of fullerenes. Here we formulate one of them.

**Conjecture 4.1.** *If fullerene with $n = 10k$, $k \geq 2$, carbon atoms has the minimal hyperbolic volume in the class $C_n$, then it is a nanotubical fullerene with caps of type $(a)$.*

Numerical computations confirm the conjecture for $n \leq 64$.

### Acknowledgement

The author was supported by the Russian Foundation for Basic Research (grant 19-01-00569).

### References

1. A. R. Ashrafi and V. M. Diudea, *Distance, Symmetry, and Topology in Carbon Nanomaterials*, Carbon Materials: Chemistry and Physics, Vol. 9, Springer: Cham, Switzerland, 2016.
2. G. Brinkmann and A. W. M. Dress, *A constructive enumeration of fullerenes.* J. Algorithms **309** (1997) 345358.
3. A. Dobrynin, O. Ori, M. Putz and A. Vesnin, *Generalized topological efficiency case study with $C_{84}$ fullerene*, Fuller. Nanotub. Car. N. **28** (2020) 545–550.
4. A. Dobrynin and A. Vesnin, *On the Wiener complexity and the Wiener index of fullerene graphs*, Mathematics **7** (11) (2019) 1071.
5. A. Egorov and A. Vesnin, *On correlation of hyperbolic volumes of fullerenes with their properties*, Comput. Math. Biophys. **8** (2020) 150–167.
6. A. Egorov and A. Vesnin, *Volume estimates for right-angled hyperbolic polyhedra*, Rend. Istit. Mat. Univ. Trieste **52** (2020) 1–12. (On-line ready at `https://rendiconti.dmi.units.it/volumi/fa/B10.pdf`)
7. J. Ratcliffe, *Foundations of Hyperbolic Manifolds*, Springer-Verlag, New York, 1994.

8. A. Vesnin, *Three-dimensional hyperbolic manifolds of Löbell type*, Sib Math. J. **28** (1987) 731–734.
9. A. Vesnin, *Right-angled polyhedra and hyperbolic* 3*-manifolds*, Russian Math. Surveys **72** (2017) 335–374.
10. A. Vesnin and A. Mednykh, *Three-dimensional hyperelliptic manifolds and Hamiltonian graphs,* Sib. Math. J. **40** (1999) 628–643.

E-mail: vesnin@math.nsc.ru

# Exact Contexts and Recollements of Derived Module Categories

Changchang Xi*
School of Mathematical Sciences, Capital Normal University, Beijing, China

Abstract. This talk surveys some of our recent joint works on constructions of recollements of derived categories of algebras. Generalising Milnor squares and usual tensor products over commutative rings, we introduce exact contexts and their noncommutative tensor products, respectively. We then construct universal localizations and characterize when they are homological in terms of the data of exact contexts. In this way we establish new methods to get recollements of derived categories of rings.
**Keywords:** Derived module category, Exact context, Noncommutative tensor product, Recollement, Rigid morphism.
**AMS Mathematical Subject Classification [2010]:** 16E35, 18G35, 13B30.

## 1. Introduction

Recollements of triangulated categories were introduced by Beilinson, Bernstein and Deligne in 1982 to investigate derived categories of perverse sheaves over singular spaces. Now, they are widely applied in algebraic geometry and topology, and recently in representation theory, particularly in the contexts of homological dimensions and tilting theory (See [**1**, **3**, **4**, **7**]).

Many methods are available for constructing recollements of triangulated categories, such as taking Verdier quotients of triangulated categories, or forming stable categories of Frobenius categories. But little is known about constructing recollements of derived module categories of rings. The importance of such recollements is that they reduce the study of derived, homological or numerical invariants (for instance, finitistic dimension and algebraic $K$-theory) of one algebra to those of the other two 'smaller' algebras involved in a recollement (See, for example, [**6**, **7**, **8**]).

In the talk, we will present a general method to construct recollements of derived module categories of rings by homological exact contexts. In such a construction all of the three rings or algebras involved can be described explicitly by the data of given exact contexts. The method can consequently be applied to a large variety of circumstances: localizations, ring extensions, Milnor squares, and ring epimorphisms (e.g., [**5**]).

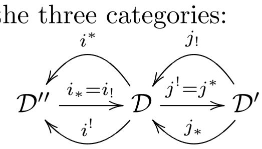
Let $R$ be a ring with identity. We denote by $R$-Mod the category of all unitary left $R$-modules. If $f : M \to N$ and $g : N \to L$ are homomorphisms of $R$-modules, the composite of $f$ and $g$ is denoted by $fg : M \to L$ and the image of $x \in M$ under $f$ is denoted by $(x)f$ instead of $f(x)$. By End$(_R(M)$ we means the endomorphism ring of the $R$-module $M$.

*Speaker

We denote by $\mathscr{C}(R\text{-Mod})$ the category of (cochain) complexes of left $R$-modules, and by $\mathscr{K}(R\text{-Mod})$ and $\mathscr{D}(R\text{-Mod})$ the homotopy and derived categories of $\mathscr{C}(R\text{-Mod})$, respectively. For brevity of the notation, we write $\mathscr{C}(R)$, $\mathscr{K}(R)$ and $\mathscr{D}(R)$ for $\mathscr{C}(R\text{-Mod})$, $\mathscr{K}(R\text{-Mod})$ and $\mathscr{D}(R\text{-Mod})$, respectively, and identify $R$-Mod with the subcategory of $\mathscr{D}(R\text{-Mod})$ consisting of all stalk complexes concentrated in degree zero. Further, we denote by $\mathscr{D}^b(R)$ the full subcategory of $\mathscr{D}(R)$ consisting of all complexes which are isomorphic in $\mathscr{D}^b(R)$ to bounded complexes of $R$-modules. The categories $\mathscr{D}(R)$ and $\mathscr{D}^b(R)$ are often called derived module categories.

Now, we recall the definition of recollements of triangulated categories in [2].

Definition 1.1. Let $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$ be triangulated categories. We say that $\mathcal{D}$ is a *recollement* of $\mathcal{D}'$ and $\mathcal{D}''$ (or there is a recollement $(\mathcal{D}'', \mathcal{D}, \mathcal{D}')$) if there are six triangle functors among the three categories:




such that

(1) $(i^*, i_*), (i_!, i^!), (j_!, j^!)$ and $(j^*, j_*)$ are adjoint pairs,

(2) $i_*, j_*$ and $j_!$ are fully faithful functors,

(3) $i^! j_* = 0$ (and thus also $j^! i_! = 0$ and $i^* j_! = 0$), and

(4) for each object $X \in \mathcal{D}$, there are triangles $i_! i^!(X) \to X \to j_* j^*(X) \to i_! i^!(X)[1]$ and $j_! j^!(X) \to X \to i_* i^*(X) \to j_! j^!(X)[1]$ in $\mathcal{D}$.

Examples of recollements of triangulated categories can be gained by homological ring epimorphisms. Let $\lambda : R \to S$ be a ring epimorphism. It is said to be homological if the functor $\mathscr{D}(\lambda_*) : \mathscr{D}(S) \to \mathscr{D}(R)$ induced by the restriction functor $\lambda_* : S\text{-Mod} \to R\text{-Mod}$ is fully faithful. For a homological ring epimorphism $\lambda : R \to S$, there exists a recollement $\big(\mathscr{D}(S), \mathscr{D}(R), \mathrm{Tria}({}_RQ^\bullet)\big)$ of triangulated categories, where ${}_RQ^\bullet$ is the mapping cone of $\lambda$ as a complex of $R$-modules, and where $\mathrm{Tria}({}_RQ^\bullet)$ stands for the smallest full triangulated subcategory of $\mathscr{D}(R)$ containing ${}_RQ^\bullet$ and being closed under coproducts. In general, $\mathrm{Tria}({}_RQ^\bullet)$ does not have to be equivalent to a derived module category.

Next, we recall the notion of exact contexts introduced in [4].

Definition 1.2. An exact context is a quadruple $(\lambda, \mu, M, m)$ consisting of ring homomorphisms $\lambda : R \to S$, $\mu : R \to T$, an $R$-$S$-bimodule $M$ and an element $m \in M$, such that the sequence of abelian groups is exact:

$$0 \longrightarrow R \xrightarrow{(\lambda,\mu)} S \oplus T \xrightarrow{\binom{\cdot m}{-m\cdot}} M \longrightarrow 0,$$

where $\cdot m$ and $m\cdot$ stand for the right and left multiplication maps by $m$, respectively. The exact context $(\lambda, \mu, M, m)$ is said to be homological if $\mathrm{Tor}_i^R(T, S) = 0$ for all $i \geq 1$.

Clearly, exact contexts generalize Milnor squares which are pullbacks of two ring homomorphisms with one of the homomorphisms being surjective.

For an exact context $(\lambda, \mu, M, m)$, there is constructed in [4] an associative ring $T \boxtimes_R S$ with identity, called the noncommutative tensor product of $(\lambda, \mu, M, m)$.

Such a tensor product captures the notion of both coproducts and usual tensor products in ring theory, and is of crucial significance in describing the left-hand side ring in a recollement of derived module categories.

The contexts of this talk are mainly based on the joint works with Professor Hongxing Chen (e.g., [4, 5]).

## 2. Main Results

Exact contexts can be constructed from rigid morphisms in any additive categories. Recall that a morphism $f: Y \to X$ in an additive $\mathcal{C}$ is said to be rigid if $\mathrm{Hom}_{\mathcal{C}}(Y, X) = \mathrm{End}_{\mathcal{C}}(Y)f + f\mathrm{End}_{\mathcal{C}}(X)$ (e.g., [4]). This is equivalent to saying that $\mathrm{Hom}_{\mathscr{K}(\mathcal{C})}(f, f[1]) = 0$, where $[1]$ denotes the shift functor of the homotopy category $\mathscr{K}(\mathcal{C})$. Thus we have an exact sequence of abelian groups:

$$0 \longrightarrow R \xrightarrow{(\lambda,\mu)} \mathrm{End}_{\mathcal{C}}(Y) \oplus \mathrm{End}_{\mathcal{C}}(X) \xrightarrow{\begin{pmatrix} \cdot f \\ -f\cdot \end{pmatrix}} \mathrm{Hom}_{\mathcal{C}}(Y, X) \longrightarrow 0,$$

where $(\lambda, \mu)$ is the kernel of the map $\begin{pmatrix} \cdot f \\ -f\cdot \end{pmatrix}$. This provides clearly an exact context.

Lemma 2.1. [4] *Each rigid morphism in an additive category $\mathcal{C}$ gives rise to an exact context. Conversely, each exact context is obtained from a rigid morphism in an additive category.*

Examples of rigid morphisms include source maps and sink maps of almost split sequences.

Theorem 2.2. [4, 5] *Let $(\lambda : R \to S, \mu : R \to T, M, m)$ be an exact context and $\Lambda := \begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$ be the triangular matrix ring defined by $R$, $S$ and $M$.*

1) *There exist a ring $T \boxtimes_R S$, two ring homomorphisms $\rho : S \to T \boxtimes_R S$ and $\phi : T \to T \boxtimes_R S$, and a homomorphism $\beta : M \to T \boxtimes_R S$ of $S$-$T$-bimodules, such that the ring homomorphism*

$$\theta := \begin{pmatrix} \rho & \beta \\ 0 & \phi \end{pmatrix} : \quad \Lambda = \begin{pmatrix} S & M \\ 0 & T \end{pmatrix} \longrightarrow \begin{pmatrix} T \boxtimes_R S & T \boxtimes_R S \\ T \boxtimes_R S & T \boxtimes_R S \end{pmatrix}$$

   *is a universal localization in the sense of Cohen-Schofield. This ring is uniquely determined by $(\lambda, \mu, M, m)$ and called the noncommutative tensor product of $(\lambda, \mu, M, m)$.*
2) *The homomorphism $\theta$ in (1) is homological if and only if the exact context is homological.*
3) *If the exact context is homological, then there exists a recollement of the derived module categories of rings:*

$$\mathscr{D}(T \boxtimes_R S) \longrightarrow \mathscr{D}(\Lambda) \longrightarrow \mathscr{D}(R).$$

   *If the projective dimensions of ${}_RS$ and $T_R$ are additionally finite, then this recollement can be restricted to a recollement of bounded derived module categories:*

$$\mathscr{D}^b(T \boxtimes_R S) \longrightarrow \mathscr{D}^b(\Lambda) \longrightarrow \mathscr{D}^b(R).$$

For the definition of universal localizations of noncommutative rings, we refer the reader to the book [9] or [4]. Note that the notion of universal localizations is a generalization of usual localizations in commutative algebra.

As a consequence, we get recollements of derived categories of rings from localizations of commutative rings.

Corollary 2.3. [5, Corollary 1.2] *Let $R$ be a commutative ring, $\Phi$ a multiplicative subset of $R$, and $\lambda : R \to S$ the localization of $R$ at $\Phi$. If $\lambda$ is injective (for example, if $R$ is an integral domain), then there exists a recollement of derived module categories*

$$\xymatrix{\mathscr{D}(\Psi^{-1}S') \ar[r] & \mathscr{D}\big(\mathrm{End}_R(S \oplus S/R)\big) \ar@/_1pc/[l] \ar@/^1pc/[l] \ar[r] & \mathscr{D}(R) \ar@/_1pc/[l] \ar@/^1pc/[l]}$$

*where $S' := \mathrm{End}_R(S/R)$ is commutative, $\Psi$ is the image of $\Phi$ under the induced map $R \to S'$ by right multiplication and $\Psi^{-1}S'$ is the localization of $S'$ at $\Psi$.*

Theorem 2.2 can also be applied to trivially twisted extensions of finite-dimensional algebras for free since all requirements in Theorem 2.2 are automatically satisfied. We refer the reader to [10] for the definition of twisted extensions and to [5] for the construction of recollements.

Thus the inputs of our construction for recollements of derived module categories are exact contexts $(\lambda, \mu, M, m)$, under the requirement of the condition $\mathrm{Tor}_i^R(T, S) = 0$ for all $i \geq 1$, the outputs are recollements of derived categories of rings which are built from the given data of exact contexts.

**Acknowledgement**

The author is grateful to Professor Hossein Eshraghi at the University of Kashan for his invitation to give a talk at the 51th Annual Iranian Mathematics Conference held in Kashan, Iran, February 16-17, 2021. Also, the author thanks the NNSFC (grant 12031014) and BNSF (grant (1192004)) for partial support.

## References

1. S. Bazzoni, *Equivalences induced by infinitely generated tilting modules*, Proc. Amer. Math. Soc. **138** (2010) 533–544.
2. A. A. Beilinson, J. Bernstein and P. Deligne, *Faisceaux pervers*, Asterisque **100** (1982) 5–171.
3. H. X. Chen and C. C. Xi, *Good tilting modules and recollements of derived module categories, II*, J. Math. Soc. Japan **71** (2019) 515–554.
4. H. X. Chen and C. C. Xi, *Exact contexts, noncommutative tensor products, and unversal localizations*, Trans. Amer. Math. Soc. **371** (2019) 3647–3672.
5. H. X. Chen and C. C. Xi, *Recollements of derived categories I: Construction from exact contexts*, J. Pure Appl. Algebra **225** (2021) 106648.
6. H. X. Chen and C. C. Xi, *Recollements of derived categories III: Finitistic dimensions*, J. Lond. Math. Soc. **95** (2017) 633–658.
7. D. Happel, *Reduction techniques for homological conjectures*, Tsukuba J. Math. **17** (1993) 115–130.
8. Y. Y. Qin and Y. Han, *Reducing homological conjectures by n-recollements*, Algebr. Represent. Theory **19** (2016) 377–395.

9. A. H. Schofield, *Representation of Rings over Skew Fields*, London Mathematical Society Lecture Note Series, Vol. 92, Cambridge, England, 1985.
10. C. C. Xi, *Characteristic tilting modules and Ringel duals,* Sci. China Ser. A **43** (2000) 1121–1130.

E-mail: xicc@@cnu.edu.cn

# Contributed Talks

## Algebra

# On the List Distinguishing Number of Graphs

Saeid Alikhani*
Department of Mathematics, Yazd University, 89195-741, Yazd, Iran
and Samaneh Soltani
Department of Mathematics, Yazd University, 89195-741, Yazd, Iran

ABSTRACT. A graph $G$ is said to be $k$-distinguishable if every vertex of the graph can be colored from a set of $k$ colors such that no non-trivial automorphism fixes every color class. The distinguishing number $D(G)$ is the least integer $k$ for which $G$ is $k$-distinguishable. A list assignment to $G$ is an assignment $L = \{L(v)\}_{v\in V(G)}$ of lists of labels to the vertices of $G$. A distinguishing $L$-labeling of $G$ is a distinguishing labeling of $G$ where the label of each vertex $v$ comes from $L(v)$. The list distinguishing number of $G$, $D_l(G)$ is the minimum $k$ such that every list assignment to $G$ in which $|L(v)| = k$ for all $v \in V(G)$ yields a distinguishing $L$-labeling of $G$. In this paper, we study and compute the list-distinguishing number of some families of graphs. We also study graphs with the distinguishing number equal the list distinguishing number.

**Keywords:** Distinguishing number, List-distinguishing labeling, List distinguishing chromatic number.
**AMS Mathematical Subject Classification [2010]:** 05C15, 05E18.

## 1. Introduction

Let $G = (V, E)$ be a simple graph. The set of all *automorphisms* of $G$, with the operation of composition of permutations, is a permutation group on $V$ and is denoted by $\mathrm{Aut}(G)$. A coloring of $G$, $\phi : V \to \{1, 2, \dots, r\}$, is *$r$-distinguishing*, if no non-trivial automorphism of $G$ preserves all of the vertex colors. In other words, $\phi$ is $r$-distinguishing if for every non-trivial $\sigma \in \mathrm{Aut}(G)$, there exists $x$ in $V$ such that $\phi(x) \neq \phi(\sigma(x))$. The *distinguishing number* of a graph $G$ is the minimum number $r$ such that $G$ has a coloring that is $r$-distinguishing; this was defined in [1]. The introduction of the distinguishing number was a great success; by now about one hundred papers have been written motivated by this seminal paper. The core of the research has been done on the invariant itself, either on finite [3, 8, 9].

Ferrara et al. [6] extended the notion of a distinguishing labeling to a list distinguishing labeling. A *list assignment* to $G$ is an assignment $L = \{L(v)\}_{v\in V(G)}$ of lists of labels to the vertices of $G$. A *distinguishing $L$-labeling* of $G$ is a distinguishing labeling of $G$ where the label of each vertex $v$ comes from $L(v)$. The *list distinguishing number* of $G$, $D_l(G)$ is the minimum $k$ such that every list assignment to $G$ in which $|L(v)| = k$ for all $v \in V(G)$ yields a distinguishing $L$-labeling of $G$. Since all of the lists can be identical, we observe that $D(G) \leq D_l(G)$. In some

*Speaker

cases, it is easy to show that the list-distinguishing number can equal the distinguishing number. For example, it is not difficult to see that $D(K_n) = n = D_l(K_n)$, $D(K_{n,n}) = n+1 = D_l(K_{n,n})$ and $D_l(C_n) = D(C_n) = 2$ [6]. In particular, Ferrara et al. [7] extended an enumerative technique of Cheng [5], to show that for any tree $T$, $D_l(T) = D(T)$. Ferrara et al. [6] asked the following question at the end of their paper.

QUESTION 1.1. Does there exist a graph $G$ such that $D(G) \neq D_l(G)$?

Amusingly, Ferrara feels that no such graph $G$ exists, while Gethner believes this question can be answered in the affirmative.

In this paper we first study and compute the list-distinguishing number for some families of graphs, such as power of hypercubes, friendship and book graphs. We also state a necessary and sufficient condition for graph $G$ satisfying $D_l(G) = D(G)$.

## 2. Main Results

The Cartesian product of graphs $G$ and $H$ is a graph $G\Box H$ with vertex set $V(G)\times V(H)$. Two vertices $(u,v)$ and $(u_0,v_0)$ are adjacent in $G\times H$ if and only if $u = u_0$ and $vv_0 \in E(H)$ or $uu_0 \in E(G)$ and $v = v_0$. The $r$th Cartesian power of a graph $G$, denoted by $G^r$, is the Cartesian product of $G$ with itself taken $r$ times. That is $G^r = G\Box G\Box \dots \Box G$, $r$-times. The graphs $G$ and $H$ are called factors of the product $G\Box H$. A graph $G$ is prime with respect to the Cartesian product if it is nontrivial and cannot be represented as the product of two nontrivial graphs. Recently, Chandran, Padinhatteeri, and Ravi Shankar in [4] proved the following results:

THEOREM 2.1. *Let $G$ be a connected prime graph, then*

i) *If $|G| \neq 2$, then $D_l(G^r) = 2$ for $r \geq 3$.*
ii) *If $|G| = 2$ then $D_l(G^r) = 2$ for $r \geq 4$ and $D_l(G^r) = 3$ when $r \in \{2,3\}$.*

COROLLARY 2.2. *If a connected graph $G$ is prime with respect to the Cartesian product, then $D_l(G^r) = D(G^r)$ for $r \geq 3$, where $G^r$ is the Cartesian product of the graph $G$ taken $r$ times.*

The $p$th power of a graph $G$ is the graph whose vertex set is $V(G)$ and in which two vertices are adjacent when they have distance less than or equal to $p$. They also determined the list distinguishing number of $p$th power of hypercube.

Here, we consider the friendship graphs and the book graphs and compute their list-distinguishing number. We begin with the friendship graph. The friendship graph $F_n$ $(n \geq 2)$ can be constructed by intersecting $n$ copies of $C_3$ at a common vertex.

THEOREM 2.3. *For every $n \geq 2$, $D_l(F_n) = D(F_n) = \left\lceil \dfrac{1+\sqrt{8n+1}}{2} \right\rceil$.*

The $n$-book graph $(n \geqslant 2)$ is defined as the Cartesian product of $K_{1,n}$ and $P_2$, i.e., $K_{1,n}\Box P_2$. We call every $C_4$ in book graph $B_n$ a *page* of $B_n$. All pages in $B_n$ have a common side $v_0w_0$. The distinguishing number of $B_n$ was computed in [2], and we shall show that $D(B_n) = D_l(B_n)$.

THEOREM 2.4. *For every $n \geq 2$, $D_l(B_n) = D(B_n) = \lceil \sqrt{n} \rceil$.*

In the rest, we try to obtain a necessary and sufficient condition for a graph $G$ such that $D(G) = D_l(G)$. To do this, first we need to state some notation and results from set theory. Let $G$ be a graph with $V(G) = \{a_1, \ldots, a_n\}$ and $D(G) = d$. Let $L = \{L_i\}_{i=1}^n$ be an arbitrary sequence such that $|L_i| = d$ and $L_i \subseteq \{1, \ldots, m\}$ for some $m \geq d$ and every $1 \leq i \leq n$. If $L$ is a distinguishing $L$-labeling of $G$ then there exists a distinguishing labeling $C$ of vertices of $G$ such that $C(v_i) \in L_i$ for all $1 \leq i \leq n$. On the other hand, for every distinguishing labeling $C$, we can construct $\binom{m-1}{d-1}^n$ sequences $L^{(C)} = \{L_i^{(C)}\}_{i=1}^n$ such that $C(v_i) \in L_i^{(C)}$, $|L_i^{(C)}| = d$ and $L_i^{(C)} \subseteq \{1, \ldots, m\}$ for every $1 \leq i \leq n$. We call such sequences the $(m, d)$*-related sequences* to $C$. If we denote the set of all related sequences to $C$ by $\mathcal{L}_{(m,d)}^{\{a_1,\ldots,a_n\}}(C)$, then $|\mathcal{L}_{(m,d)}^{\{a_1,\ldots,a_n\}}(C)| = \binom{m-1}{d-1}^n$. Let $\mathcal{L}(G, m)$ be the set of all distinguishing labeling of $G$ with at most $m$ labels $\{1, \ldots, m\}$. Set $\mathcal{L}(G, m) = \{C_1, \ldots, C_{t_m}\}$. We suppose that $B_{(m,d)}^{\{a_1,\ldots,a_n\}}(C_1, \ldots, C_{t_m})$ is the set of all those sequences $L = \{L_i\}_{i=1}^n$ such that $|L_i| = d$ and $L_i \subseteq \{1, \ldots, m\}$ which are constructed using the distinguishing labelings in $\mathcal{L}(G, m)$, i.e., $B_{(m,d)}^{\{a_1,\ldots,a_n\}}(C_1, \ldots, C_{t_m}) = \bigcup_{i=1}^{t_m} \mathcal{L}_{(m,d)}^{\{a_1,\ldots,a_n\}}(C_i)$. By these statements we have the following theorem:

THEOREM 2.5. *Let $G$ be a graph with $V(G) = \{a_1, \ldots, a_n\}$ and the distinguishing number $D(G) = d$. Let $\mathcal{L}(G, m) = \{C_1, \ldots, C_{t_m}\}$ be the set of all distinguishing labeling of $G$ with at most $m$ labels $\{1, \ldots, m\}$ where $m \geq d$. An arbitrary sequence $L = \{L_i\}_{i=1}^n$ with $|L_i| = d$ and $L_i \subseteq \{1, \ldots, m\}$ for every $1 \leq i \leq n$, is a distinguishing $L$-labeling of $G$, if and only if $L \in B_{(m,d)}^{\{a_1,\ldots,a_n\}}(C_1, \ldots, C_{t_m})$.*

## References


1. M. O. Albertson and K. L. Collins, *Symmetry breaking in graphs*, Electron. J. Combin. **3** (1996) R18.
2. S. Alikhani and S. Soltani, *Distinguishing number and distinguishing index of certain graphs*, Filomat **31** (14) (2017) 4393–4404.
3. M. Chan, *The distinguishing number of the augmented cube and hypercube Powers*, Discrete Math. **308** (2008) 2330–2336.
4. L. S. Chandran, S. Padinhatteeri and K. Ravi Shankar, *List Distinguishing Number of $p^{th}$ Power of Hypercube and Cartesian Powers of a Graph.* In: Changat M., Das S. (eds) Algorithms and Discrete Applied Mathematics, CALDAM 2020, Lecture Notes in Computer Science, vol 12016. Springer, Cham.
5. C. T. Cheng, *On computing the distinguishing numbers of trees and forests*, Electron. J. Combin. **13** (2006) R11.
6. M. Ferrara, B. Flesch and E. Gethner, *List-distinguishing colorings of graphs*, Electron. J. Combin. **18** (2011) 161.
7. M. Ferrara, E. Gethner, S. G. Hartke, D. Stolee and P. S. Wenger, *List distinguishing parameters of trees*, Discrete Appl. Math. **161** (6) (2013) 864–869.
8. P. Immel and P. S. Wenger, *The list distinguishing number equals the distinguishing number for interval graphs*, Discuss. Math. Graph Theory. **37** (2017) 165–174.
9. D. Kim, Y. S. Kwon and J. Lee, *The distinguishing numbers of Merged Johnson graphs*, Bull. Korean Math. Soc. **52** (2015) 395–408.


E-mail: alikhani@yazd.ac.ir
E-mail: s.soltani1979@gmail.com

# Relative Isosuperfluous Submodules

Mostafa Amini*
Faculty of Mathematical Sciences, University of Payame Noor, Tehran, Iran

Abstract. We introduce isosuperfluous $R$-submodules and then we examine some characteristics of these modules on max rings. Also, we introduce and study the notions of isoprojective cover modules and isosemiperfect rings by using the notion of isosuperfluous submodules. Finally, we investigate some properties of these modules on isoartinian rings.
**Keywords:** Isosuperfluous submodule, Isoprojective cover, Strongly superfluous submodule, Isoartinian modules.
**AMS Mathematical Subject Classification [2010]:** 16D10, 16D99, 13C13.

## 1. Introduction

Throughout this paper, all rings are associative rings with identity, and modules are unitary right modules. A submodule $N$ of an $R$-module $M$ is superfluous in $M$ and denoted by $N \ll M$, in case for any submodule $L$ of $M$, $L + N = M$ implies $L = M$. Recently, Babak Amini and Afshin Amini in [2] introduced the notions of strongly superfluous submodule, and then the basic properties of strongly superfluous submodules on max rings are investigated. A submodule $K$ of an $R$-module $M$ is said to be *strongly superfluous* in $M$, denoted by $K \leq_{ss} M$, if $\bigoplus_{i\in I} K \ll \bigoplus_{i\in I} M$ for any index set $I$. Also in 2016, Facchini and Nazemian introduced the notions of isoartinian and isonoetherian modules. A module $M$ is said to be *isoartinian* if, for every descending chain $M \geq M_1 \geq M_2 \geq \cdots$ of submodules of $M$, there exists an index $n \geq 1$ such that $M_n$ is isomorphic to $M_i$ for every $i \geq n$. Dually, $M$ is called *isonoetherian* if, for every ascending chain $M_1 \leq M_2 \leq \cdots$ of submodules of $M$, there exists an index $n \geq 1$ such that $M_n \cong M_i$ for every $i \geq n$. A module $M$ is *isosimple* if it is non-zero and every non-zero submodule of $M$ is isomorphic to $M$ (See [4]).

In this paper, we introduce and study isosuperfluous submodules and isoprojective cover modules and then, we examine some properties of those modules on max rings and isoartinian rings, respectively. A submodule $N$ of a module $M$ is *isosuperfluous* in $M$ and denoted by $N \leq_{iso} M$, in case for any submodule $L$ of $M$, $L + N = M$ implies $L \cong M$. A module $M$ is said to be *isoprojective cover* of module $B$ if $M$ is projective and $\phi : M \to B$ is a surjective map with $\ker\phi \leq_{iso} M$. A ring $R$ is called right *isosemiperfect* if every finitely generated right $R$-module has a isoprojective cover. Also, examples are given showing that every isosuperfluous submodule is not superfluous and strongly superfluous and every isoprojective cover module is not projective cover.

*Speaker

## 2. Main Results

We begin this section by recalling the following definition.

Definition 2.1. A submodule $N$ of an $R$-module $M$ is isosuperfluous in $M$ and denoted by $N \leq_{iso} M$, in case for any submodule $L$ of $M$, $L+N=M$ implies $L \cong M$.

Clearly, any superfluous submodule is isosuperfluous but not conversely, for example, submodule $2\mathbb{Z}$ of $\mathbb{Z}$ is isosuperfluous but $2\mathbb{Z}$ is not superfluous and strongly superfluous in $\mathbb{Z}$, since $\mathbb{Z}$ is isosimple $\mathbb{Z}$-module by [4, Remark 2.2].

Proposition 2.2. *Let $M$ be a module with submodules $L, K$ and $N_i$ for any $i \in I$. The following statements hold true.*

i) *If $L+K \leq_{iso} M$, then $L \leq_{iso} M$ and $K \leq_{iso} M$.*
ii) *If $L \ll M$ and $K \leq_{iso} M$, then $L+K \leq_{iso} M$.*
iii) *If $M$ is finitely generated and $N_i \ll M$ for any $i \in I$, then $\oplus N_i \leq_{iso} M$.*

Proof. (i) Let, for submodule $D$ of $M$, $D+L=M$. Since $D+L+K=M$ and $L+K \leq_{iso} M$, we have $D \cong M$. Therefore, $L \leq_{iso} M$ and also similarly $K \leq_{iso} M$.

(ii) Let, for submodule $D$ of $M$, $D+L+K=M$. Since $K \ll M$, we have $D+L=M$. By hypothesis, $L \leq_{iso} M$ and so $D \cong M$.

(iii) Assume that $N_i \ll M$ for any $i \in I$. If $\oplus N_i \nleq_{iso} M$, then $\bigoplus_{i\in I} N_i$ is not superfluous in $M$. Thus, if for a submodule $D$ of $M$, $D + \bigoplus_{i\in I} N_i = M$, then $D \neq M$ and so $\frac{M}{D} \neq 0$. As $\frac{M}{D}$ is finitely generated, $M/D$ contains a maximal submodule $X$ such taht $D \subseteq X$. But $N_i \subseteq X$ for any $i \in I$ (if $N_i \nsubseteq X$, we have $N_i + X = M$ which implies $M = X$, a contradiction). Therefore, any $N_i \subseteq X$ and so from $D + \bigoplus_{i\in I} N_i = M$, it follows that $M \subseteq X$, which is a contradiction. Consequently $\oplus N_i \leq_{iso} M$. □

Recall that a ring $R$ is said to be right max in case every nonzero right $R$-module has a maximal submodule.

Proposition 2.3. *Let $R$ be a ring and $M$ an $R$-module. Then, the following statements are equivalent.*

i) *$R$ is a right max ring.*
ii) *Let $\{N_f\}_{f\in F}$ be a family of nonzero right $R$-submodules of $M$ and $F = I \cup \{j\}$. Then $\bigoplus_{f\in F} N_f \leq_{iso} M$ and $\bigoplus_{i\in I} N_i \ll M$ if and only if $N_i \ll M$ and $N_j \leq_{iso} M$.*
iii) *Let $\{N_f\}_{f\in F}$ be a family of nonzero right $R$-submodules of $M$ and $F = I \cup \{j\}$. Then $\sum_{f\in F} N_f \leq_{iso} M$ if and only if $i \in I$, $N_i \ll M$ and $N_j \leq_{iso} M$.*

Proof. (i) $\Longrightarrow$ (ii) By [2, Theorem 2.8], if $M$ is a nonzero right $R$-module, then $N_i \ll M$ if and only if $\bigoplus_{i\in I} N_i \ll M$ for any $i \in I$ and so, by Proposition 2.2, $\bigoplus_{f\in F} N_f = \bigoplus_{i\in I} N_i + N_j \leq_{iso} M$. If $\bigoplus_{f\in F} N_f = \bigoplus_{i\in I} N_I + N_j \leq_{iso} M$, then $N_j \leq_{iso} M$ by Proposition 2.2.

(ii)$\Longrightarrow$ (iii) By (ii), $\bigoplus_{i\in I} N_i \ll M$ if and only if $N_i \ll M$ for any $i \in I$. Since $\bigoplus_{i\in I} N_i \subseteq M \subseteq \bigoplus_{i\in I} M$, by [5, Lemma 4.59], $\bigoplus_{i\in I} N_i \ll \bigoplus_{i\in I} M$. On the other hand, $\phi : \bigoplus_{i\in I} M \to M$ is epimorphism. Hence, by [1, Lemma 5.18], $\sum_{i\in I} N_i = \phi(\bigoplus_{i\in I} N_i) \ll M$. Thus, by Proposition 2.2, $\sum_{i\in F} N_i = \sum_{i\in I} N_i + N_j \leq_{iso} M$.

(iii) $\Longrightarrow$ (i) Let $M$ be a nonzero right $R$-module. By [1, Proposition 9.13], $Rad(M) = \sum\{N \mid N \textit{ is superfluous in } M\}$. As every superfluous submodule is isosuperfluous, by (iii), $Rad(M) = \sum_{i\in I} N_i \leq_{iso} M$. We claime that $Rad(M) \neq M$. If $Rad(M) = M$, then $Rad(M) + N = M$ for any submodule $N$ of $M$. Hence, by Definition 2.1, $N \cong M$ so that $M$ is isosimple. Thus, by [4, Remark 2.2], $M$ is finitely generated which is a contradiction. Therefore, $Rad(M) \neq M$ and any nonzero right $R$-modules $M$ has a maximal submodule. □

Definition 2.4. An $R$-module $M$ is callled isoprojective cover of a module $B$ if $M$ is projective and $\phi : M \to B$ is a surjective map with $\ker\phi \leq_{iso} M$. Also, a ring $R$ is called right isosemiperfect if every finitely generated right $R$-module has a isoprojective cover.

It is clear that any projective cover is isoprojective cover but not conversely. For example, [5, Example 4.61], let $R = \mathbb{Z} = M$ and $B = \mathbb{Z}_2$. It is clear that $\phi : \mathbb{Z} \to \mathbb{Z}_2$ is a surjective map with $\phi(x) = y$, where $\mathbb{Z}_2 =< y >$. Hence $\phi(3x) = y$ and $\mathbb{Z} = \ker\phi+ < 3x >$ so that $\mathbb{Z} \cong< 3x >$. Therefore, $\ker\phi \leq_{iso} \mathbb{Z}$ and so $\mathbb{Z}$ is a isoprojective cover of $\mathbb{Z}_2$. But $\mathbb{Z}$ is not isoprojective cover of $\mathbb{Z}_2$.

Proposition 2.5. *Let $R$ be a ring. Then the following statements are equivalent.*

i) *$R$ is a right max ring;*
ii) *Let $\{N_f\}_{f\in F}$ be a family of nonzero projective $R$-submodule of $M$ and $F = I \cup \{j\}$. Then $(M, \sum_{f\in F} \phi_f)$ is isoprojective cover and $(M, \sum_{i\in I} \phi_i)$ is projective cover if and only if $(M, \phi_i)$ is projective cover for any $i \in I$ and $(M, \phi_j)$ is isoprojective cover;*
iii) *If $P/Rad(P)$ is semisimple for every projective $R$-module $P$, then any nonzero $R$-module has a maximal submodule.*

Proof. (iii)$\Longrightarrow$ (i) and (i)$\Longrightarrow$(ii) is clear by Proposition 2.3.

(ii) $\Longrightarrow$ (iii) For every nonzero $R$-modules $M$, there exists an epimorphism $f : P \to M$, where $P$ is projective. Then, By [1, Exercises 9, pp. 122], $f(Rad(P)) = Rad(M)$. By (ii), $Rad(P) \ll P$ and so, by [1, Lemma 5.18], $Rad(M) \leq_{iso} M$. Thus, $Rad(M) \neq M$ so that $M$ has a maximal submodule. □

Corollary 2.6. *Let $R$ be a ring. Then the following statements are equivalent.*

i) *$R$ is a right max ring.*
ii) *Let $N_i$ be a nonzero $R$-submodule of $M_i$ for any $i \in I$. Then $(\oplus_{i\in I} M_i, \oplus_{i\in I}\phi_i)$ is projective cover if and only if $(M_i, \oplus_{i\in I}\phi_i)$ is projective cover.*

Theorem 2.7. *Let $D$ on $M_n(D)$ be a right $V$-domain. Then every isoartinian semiprime Noetherian ring is isosemiperfect.*

Proof. We only need to prove that every finitely generated $R$-module has a isoprojective cover. Let $R$ be a right isoartinian semiprime right Noetherian ring. By [4, Teorem 4.7], $R \cong \prod_{i=1}^{k} M_{n_i}(D_i)$, where any $D_i$ is a $PRID$. Thus for any finitely generated right $R$-module $M$, by [3, Theorem 3.4], we have $M = \oplus T_i$, where any $T_i$ is either simple left $R$-module or isosimple direct summand of $R_R$. Let every $T_i$ be a simple module. As $\frac{R}{Jac(R)}$ is finitely generated, $\frac{R}{Jac(R)}$ is semisimple which is a contradiction; because $R = Z$ is a right isoartinian semiprime right Noetherian ring that it is not semisimple. Therefore, any $T_i$ is isosimple direct summand of $R_R$ and so $M = \oplus T_i$ is projective. By Definition 2.1 and [5, Lemma 4.60], $Jac(R)M \leq_{iso} M$. Therefore, if for a submodule $S$ of $M$, $Jac(R)M + S = M$, then $M \cong S$. Since $M$ is projective and $f : M \to S$ is isosuperfluous, we deduce that $S$ has a isoprojective cover. Hence $M$ has a isoprojective cover. □

## References


1. F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Spring-Verlag, New York, 1974.
2. B. Amini and A. Amini, *On strongly superfluous submodules*, Comm. Algebra **40** (8) (2012) 2906–2919.
3. M. Behboodi, A. Daneshvar and M. R. Vedadia, *Two generalizations of the Wedderburn-Artin theorem with applications* (2016). `arXiv:1605.09740`
4. A. Facchini and Z. Nazemian, *Modules with chain conditions up to isomorphism*, J. Algebra **453** (2016) 578–601.
5. J. J. Rotman, *An Introduction to Homological Algebra*, Academic Press, New York, 2009.


E-mail: amini.pnu1356@gmail.com

# Semi-Symmetric Graphs of Certain Orders

Mohammad Reza Darafsheh*
School of mathematics, Statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran

ABSTRACT. A connected simple graph $\Gamma$ is called semi-symmetric if $\mathbb{A}ut(\Gamma)$ acts transitively on the edge-set of $\Gamma$ but intransitive on its vertices. If $\Gamma$ is regular of degree 3, then it is called cubic. We classified all semi-symmetric cubic graphs of certain orders, which are presented here.
**Keywords:** Semi-symmetric graph, Edge-transitive graph, Cubic graph.
**AMS Mathematical Subject Classification [2010]:** 20B25, 20C10.

## 1. Introduction

We assume $\Gamma = (V, E)$ is a finite simple connected graph with vertex set $V$ and edge set $E$. The full automorphism group of $\Gamma$ is denoted by $A = \mathbb{A}ut(\Gamma)$ and the edge joining $u, v \in V$ is denoted by $uv$. An $s$-arc in $\Gamma$ is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices in $V$ such that $v_{i-1}$ is adjacent to $v_i$ for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$. The set of all $s$-arcs in $\Gamma$ is denoted by $s$-Arc.

For a graph $\Gamma = (V, E)$ and a subgroup $G \leq A$, $\Gamma$ is said to be $G$-vertex transitive, $G$-edge-transitive or $G$-$s$-arc transitive if $G$ acts transitively on $V$, $E$ or $s$-Arc respectively. A graph $\Gamma = (V, E)$ is called $G$-semisymmetric if it is $G$-edge transitive but not $G$-vertex transitive. If $G = A$, then the term $G$ is omitted in the above notations.

If $s = 1$, then 1-arc-transitive means arc-transitive or simply symmetric.

It can be shown that a $G$-edge transitive but not vertex-transitive graph is necessarily bipartite, where the two bipartite parts are orbits of $G$ on $V$ and if $\Gamma$ is regular, then the two partites have the same cardinality.

The class of semi-symmetric graphs was first introduction by Folkman [4], in which several infinite families of such graphs were constructed and eight open problems were posed. If $p$ is an odd prime then Folkman proved there is no semi-symmetric graph of order $2p^2$. In [3] semi-symmetric graph of order $2pq$, where $p$ and $q$ are distinct primes was classified, while semi-symmetric graphs of order $2p^3$, $p$ prime, were classified in [7]. Classification of cubic semi-symmetric graphs of various order such as $6p^3$, $28p^2$, $18p^n$, $4p^3$, $6p^2$, $6p^3$, $8p^2$, $10p^3$, where $p$ is a prime number, was considered by several authors.

## 2. Preliminary Results

In the following, some results which are used to prove our main results are listed.

*Speaker

Theorem 2.1. [5] *Let $\Gamma$ be a connected cubic semi-symmetric graph and $G \leq \mathbb{A}ut(\Gamma)$. Then the vertex stabilizer of $G$ has order $2^r \cdot 3$, where $0 \leq r \leq 7$.*

Theorem 2.2. [6] *Let $\Gamma = (V, E)$ be a connected cubic semi-symmetric graph with bipartite set $V = U \cup W$. Let $N$ be a normal subgroup of $A = \mathbb{A}ut(\Gamma)$. If $N$ is intransitive on both $U$ and $W$, then $N$ acts semi-regularly on both $U$ and $W$ and $\Gamma$ is an $N$-regular covering of an $\frac{A}{N}$ semi-symmetric graph.*

## 3. Main Results

Our aim is to present our results on cubic semi-symmetric graphs of order $14p^2$, $20p$, $34p^3$, $20p^2$ and $12p^3$.

Theorem 3.1. [2] *If $\Gamma$ is a cubic semi-symmetric graph of order $14p^2$, $p$ prime, then $p = 3$ and $\Gamma$ is the Tuttss 12-cage.*

Theorem 3.2. [8] *If $\Gamma$ is a cubic semi-symmetric graph of order $20p$, $p$ prime, then $p = 11$.*

Theorem 3.3. [9] *There is no cubic semi-symmetric graph of order $20p^2$, $p$ prime. Therefore, every cubic edge-transitive graph of order $20p^2$ is necessarily symmetric.*

But further investigations on semi-symmetric graphs of order $34p^3$ and $12p^3$, $p$ prime, yield the following results which are still under review.

Theorem 3.4. *If $\Gamma$ is a semi-symmetric cubic graph of order $34p^3$, $p$ prime, then $p = 17$.*

Theorem 3.5. *If $\Gamma$ is a semi-symmetric cubic graph of order $12p^3$, $p$ prime, then $p = 5$ or $p = 7$.*

## 4. Proofs

Here we outline the proof of Theorem 3.1.

Lemma 4.1. *Let $\Gamma$ be a connected cubic semi-symmetric graph of order $14p$, $p \neq 7$ and odd prime, then $p = 13$ and $\Gamma$ is the graph S182 in Conder et al. list [1].*

Proof. Let $\Gamma = (V, E)$ be a connected cubic semi-symmetric graph of order $14p$ and let $A = \mathbb{A}ut(\Gamma)$. Then $\Gamma$ is bipartite. Let $U$ and $W$ be its two parts. Then $|U| = |W| = 7p$. If $A = \mathbb{A}ut(\Gamma)$, then, by Theorem 2.1, we have $|A| = 2^r \cdot 3 \cdot 7 \cdot p$ with $0 \leq r \leq 7$. By [1], if $p \leq 53$, then such graphs exist only when $p = 13$. Now we may assume $p > 53$.

We distinguish two cases.

Case 1. $N$ is not solvable. In this case, $N$ itself must be a simple group. Because of $|N| \mid 2^r \cdot 3 \cdot 7 \cdot p$, $N$ must be a $K_3$ or a $K_4$-group. If $N$ is a $K_3$-group, then $N \cong \mathbb{A}_5$, $\mathbb{A}_6$, $L_2(7)$, since we have assumed $p > 53$, none of the above cases are possible. If $N$ is a $K_4$-group, then again we do not obtain a possibility for $N$. This is because $|N| \mid 2^5 \cdot 3 \cdot 7 \cdot p$ and examination of groups in the list of $K_4$-groups rules out $N$.

Case 2. $N$ is solvable. In this case $N \cong \mathbb{Z}_t^k$, $|U| = |V| = 7p$ implying that $N$ is intransitive. $t^k|7p$, hence $r = 7$ or $t$. Let $N \cong \mathbb{Z}_7$, consider the quotient graph $\Gamma_N = \frac{\Gamma}{N}$ of $\Gamma$ relative to $N$, where $\Gamma_N$ is a cubic $\frac{A}{N}$-semi-symmetric graph of order $2p$. But, by [4], such a graph does not exist. Let $N \cong \mathbb{Z}_p$. Then $\Gamma_N$ is a cubic $\frac{A}{N}$-semi-symmetric graph of order 14. But such a graph does not exist by [1].

□

THEOREM 4.2. *Let $\Gamma$ be a cubic semi-symmetric graph of order $14p^2$, where $p \neq 7$ odd prime. Then $p = 3$ and $\Gamma$ is isomorphic to the Tuttes 12-cage.*

PROOF. By [1], we may assume that $p > 7$. For $p \leq 7$ only for $p = 3$ the Tuttes 12-cage is a connected cubic semi-symmetric graph of order $14 \times 3^2 = 126$. Since $\Gamma = (V, E)$ is a connected semi-symmetric graph of order $14p^2$, $\Gamma$ is bipartite with parts $U$ and $W$, $|U| = |W| = 7p^2$. We set $A = \mathbb{A}ut(\Gamma)$. By Theorem 2.1, $|A| = 2^r \cdot 3 \cdot 7 \cdot p^2$. Let $N$ be a minimal normal subgroup of $A$. Then $|N| \mid 2^r \cdot 3 \cdot 7 \cdot p^2$. $N$ is a product of isomorphic simple groups.

Case 1. $N$ is not solvable. Then $N$ is a simple non-abelian group. If $N$ is not transitive on $U$ and $W$, then $N$ acts semi-regularly on both $U$ and $W$. Hence $|N| \mid 14p^2$, a contradiction because $4 \mid |N|$. Therefore, $N$ is transitive on at least one of $U$ or $W$ implying $7p^2 \mid |N|$. Therefore $|N| = 2^s \cdot 7 \cdot p^2$ or $2^5 \cdot 3 \cdot 7 \cdot p^2$, where $0 \leq s \leq r$. Hence $N$ is a $K_3$ or a $K_4$ simple group. If $N$ is a $K_3$-group, then only $N \cong PSL_2(8)$ of order $2^3 \cdot 3^2 \cdot 7$ with $p = 3$ is possible which not the case because we have assumed $p > 7$. If $N$ is a $K_4$-group of order $2^s \cdot 3 \cdot 7 \cdot p^2$, $0 < s \leq r \leq 7$ no possibility arises.

Case 2. $N$ is solvable group. Hence $N \cong \mathbb{Z}_r^k$, where $r$ is a prime number. Since $|U| = |W| = 7p^2$, $N$ is in transitive on both $U$ and $W$ and is semi-regular on $U$ and $W$. Therefore $r^k \mid 7p^2$, hence $r = 7$ or $p$. If $N \cong \mathbb{Z}_7$, then the quotient graph $\Gamma_N$ is a cubic $\frac{A}{N}$-semi-symmetric graph of order $2p^2$, a contradiction because by [4] such graphs dont exist. If $N \cong \mathbb{Z}_p$, then $\Gamma_N$ is a cubic $\frac{A}{N}$-semi-symmetric graph of order $14p$. Now, by Lemma 4.1, $p = 13$. Therefore $\Gamma$ is a connected cubic semi-symmetric graph of order $14 \cdot 13^2 = 2366$ which can be proved it does not exist. This is by an unpublished result of M. Conder and P. Potonik who obtain a list of cubic semi-symmetric graphs of order up to 10000.

If $N \cong \mathbb{Z}_{p^2}$, then $\Gamma_N$ is cubic $\frac{A}{N}$-semi-symmetric graph of order 14, which, by [1], does not exist. □

## References


1. M. Conder, A. Malnič, D. Marušič and P. Potočnik, *A census of semi-symmetric cubic graph on up to 768 vertices*, J. Algebraic Combin. **23** (3) (2006) 255–294.
2. M. R. Darafsheh and M. Shahsavaran, *Cubic semi-symmetric graphs of order $14p^2$, where p is a prime,* Ars combin. accepted.
3. S. Du and M. Xu, *A classification of semisymmetric graphs of order 2pq*, Comm. Algebra **28** (6) (2000) 2685–2715.
4. J. Folkman, *Regular line-symmetric graphs*, J. Combin. Theory **3** (1967) 215–232.

5. D. M. Goldschmidt, *Automorphisms of trivalent graphs*, Ann. of Math. **111** (2) (1980) 377–406.
6. Z. Lu, C. Wang and M. Xu, *On semisymmetric cubic graphs of order* $6p^2$, Sci. China Ser. A **47** (1) (2004) 1–17.
7. A. Malnic, D. Marusic and C. Wang, *Cubic edge-transitive graphs of order* $2p^3$, Discrete Math. **274** (1-3) (2004) 187–198.
8. M. Shahsavarav and M. R. Darafsheh, *Classifying semi-symmetric cubic graphs of order* $20p$, Turk. J. Math. **43** (6) (2019) 2755–2766.
9. M. Shahsavarav and M. R. Darafsheh, *On semi-symmetric cubic graphs of order* $20p^3$*, p prime*, Discuss. Math. Graph Theory (2019). DOI:10.7151/dmgt.2213.

E-mail: darafsheh@ut.ac.ir

# On (Quasi-)Morphic Rings

Rasoul Moradi
Faculty of Intelligent Systems Engineering and Data Science, Persian Gulf University, Bushehr, Iran
and Najmeh Dehghani*
Faculty of Intelligent Systems Engineering and Data Science, Persian Gulf University, Bushehr, Iran

Abstract. The main objective of this work is to study (quasi-)morphic property for skew polynomial rings. Let $R$ be a ring and $\sigma$ be a ring homomorphism on $R$. We show that if $R[x,\sigma]/(x^{n+1})$ $(n \geq 1)$ is quasi-morphic then so is $R$. It is also proved that $R$ is a regular ring provided that $R[x;\sigma]/(x^{n+1})$ is morphic. Some applications of our results are provided.
**Keywords:** Annihilator, Morphic ring, Quasi-morphic ring, Regular, Unit-regular.
**AMS Mathematical Subject Classification [2010]:** 16E50, 16S70.

## 1. Introduction

Throughout this paper we assume that $R$ is an associative ring (not necessarily commutative) with unity. If $X \subseteq R$ then the notations $\mathrm{r.ann}_R(X)$ $(\mathrm{l.ann}_R(X))$ denotes the right (left) annihilator of $X$ with elements from $R$ and it is defined by $\{r \in R \mid Xr = 0\}$ $(\{r \in R \mid rX = 0\})$. Nicholson and Campos, in 2004 [9], called a ring $R$ *left morphic* if for any $a \in R$, there exists an element $b \in R$ such that $\mathrm{l.ann}_R(a) = Rb$ and $Ra = \mathrm{l.ann}_R(b)$. Equivalently, a ring $R$ is left morphic if and only if for every $a \in R$, $R/Ra \simeq \mathrm{l.ann}_R(a)$. Camillo and Nicholson, in 2007 [2], generalized this concept to the quasi-morphic ring. They called a ring $R$ *left quasi-morphic* provided that for any $a \in R$, there exist elements $b, c \in R$ such that $\mathrm{l.ann}_R(a) = Rb$ and $Ra = \mathrm{l.ann}_R(c)$. Right (quasi-)morphic rings are defined in the same way. A left and right (quasi-)morphic ring is called *(quasi-)morphic*. These concepts have been of interest to a number of researchers, for example see [1, 3] and [4]. Clearly, every left morphic ring is left quasi-morphic however the converse is false. While for a commutative ring $R$, these two concepts coincide. Recall that a ring $R$ is said to be *(unit-)regular* if for every $x \in R$, there exists $u \in R$ $(u \in \mathrm{U}(R))$ such that $a = aua$. For more information on the theory of regular rings, see [6]. Every regular (resp., unit-regular) ring is quasi-morphic (resp., morphic) however the converse does not hold true. It is proved that unit regular rings are precisely regular and (left)morphic rings. For more details, see [2, 5] and [9].

*Speaker

The relations between regular (resp., unit-regular) rings and quasi-morphic (resp., morphic) rings have been focus of the mathematicians. For instance, it has been proved that if $R$ is a regular ring then for any $n \geq 1$, $R[x]/(x^{n+1})$ $(n \geq 1)$ is quasi-morphic [8, Theorem 4] and the converse has been asked as the following question in [8, Question 1]:

Question 1.1. Let $n \geq 1$ be an integer and $R[x]/(x^{n+1})$ is left and right quasi-morphic. Is it true that $R$ is a regular ring?

Moreover, if $R[x]/(x^{n+1})$ is left (quasi-)morphic where $n \geq 1$, then $R$ has also the property [8, Lemma 10]. It has been shown that for an integer $n \geq 1$, a ring $R$ is unit regular if and only if $R[x]/(x^{n+1})$ is morphic [8, Theorem 11]. Moreover, by [7, Corollary 3], if $R$ is a unit-regular ring and $\sigma : R \to R$ is an endomorphism such that $\sigma(e) = e$ for all $e^2 = e \in R$, then $R[x;\sigma]/(x^{n+1})$ $(n \geq 0)$ is left morphic. These motivate us to study (quasi-)morphic property for the skew polynomial ring $R[x;\sigma]/(x^{n+1})$ where $\sigma$ is a ring homomorphism on $R$. We show that if $n \geq 1$ and $R[x;\sigma]/(x^{n+1})$ is left quasi-morphic, then $R$ is also left quasi-morphic. Besides, it will be shown that this result also is true for the morphic's case provided that $\sigma$ is an isomorphism. Moreover, we will prove that a ring $R$ is regular provided that $R[x;\sigma]/(x^{n+1})$ is left and right morphic for some $(n \geq 1)$. As an application, some of results in [8] are generalized.

## 2. Main Results

Let $R$ be a ring. We remind that the ring of polynomials in indeterminate $x$ over $R$ is denoted by $R[x]$. Let $\sigma : R \to R$ be a ring homomorphism. The skew polynomial ring $R[x;\sigma]$ is defined to be the set of all left polynomials of the form $a_0 + a_1x + \cdots + a_nx^n$ with coefficients $a_0, \ldots, a_n \in R$. Addition is defined as usual, and multiplication is defined by using the relation $xr = \sigma(r)x$ where $r \in R$. Let $n \geq 0$ and $S := R[x;\sigma]/(x^{n+1})$. In whole of the paper, note that for any $\alpha = \sum_{i=0}^{t} a_ix^i \in R[x;\sigma]$, we let $\bar{\alpha} = \sum_{i=0}^{n} a_ix^i \in S$ be the image of $\alpha$.
In [8], authors have been studied the (quasi-)morphicness of the ring $R[x]/(x^{n+1})$ $(n \geq 1)$. Here we investigate relation between quasi-morphic property for the skew polynomial ring $R[x;\sigma]/(x^{n+1})$ $(n \geq 0)$ and (regularity) quasi-morphicness of the ring $R$. First we prove the following proposition.

Proposition 2.1. *Let $R$ be a ring, $\sigma : R \to R$ be an endomorphism and $n \geq 0$ be an integer. If $R[x;\sigma]/(x^{n+1})$ is left (right) quasi-morphic then so is $R$.*

Proof. Assume that $S := R[x;\sigma]/(x^{n+1})$ is left quasi-morphic and $a$ be any nonzero arbitrary element of $R$. Therefore there exists an element $\alpha = \sum_{i=0}^{n} a_ix^i \in S$ such that $\mathrm{l.ann}_S(a) = S\alpha$. It is easy to see that $\mathrm{l.ann}_R(a) = Ra_0$. By our assumption on $S$, $Sax^n = \mathrm{l.ann}_S(\beta)$ where $\beta = \sum_{i=0}^{n} b_ix^i \in S$. Thus $ax^n\beta = 0$ and so $\sum_{i=0}^{n} a\sigma^n(b_i)x^{n+i} = 0$. Thus $a\sigma^n(b_0) = 0$ and so $Ra \subseteq \mathrm{l.ann}_R(\sigma^n(b_0))$. Let $r \in \mathrm{l.ann}_R(\sigma^n(b_0))$. Therefore $rx^n\beta = \sum_{i=0}^{n} r\sigma^n(b_i)x^{n+i} = r\sigma^n(b_0)x^n = 0$. Thus $rx^n \in \mathrm{l.ann}_S(\beta) = Sax^n$. Therefore $rx^n = \gamma ax^n$ where $\gamma = \sum_{i=0}^{n} c_ix^i \in S$. Hence $rx^n = c_0ax^n$ and so $r = c_0a \in Ra$. Therefore $\mathrm{l.ann}_R(\sigma^n(b_0)) \subseteq Ra$. Thus $Ra = \mathrm{l.ann}_R(\sigma^n(b_0))$ which proves the theorem. The proof of right quasi-morphic is similar. $\square$

We note that by the following example the converse of Proposition 5 does not hold in general even the case $\sigma$ is an isomorphism on $R$.

EXAMPLE 2.2. Assume that $R = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\sigma : R \to R$ is defined by $\sigma(a, b) = (b, a)$. We note that $R$ is a regular ring and $\sigma$ is a ring isomorphism. Therefore $R$ is right and left quasi-morphic [2]. We show that $S := R[x; \sigma]/(x^2)$ is not left quasi-morphic. To see it, let $b = (0, 1) \in R$. On the contrary, suppose that $S$ is left quasi-morphic. Therefore there exists $a+dx \in S$ such that $\mathrm{l.ann}_S(bx) = S(a+dx)$. Thus $ab = 0$ and so $a = (a_1, 0)$ where $a_1 \in \mathbb{Z}_2$. Since $\sigma(b)b = 0$, $\sigma(b) \in \mathrm{l.ann}_S(bx)$. This shows that $a \neq 0$ and so $a = (1, 0)$. On the other hand, $x \in \mathrm{l.ann}_S(bx)$. Thus $x = (s_1 + s_2x)(a + dx)$ where $s_1 = (t_1, w_1) \in R$ and $s_2 = (t_2, w_2) \in R$. Therefore $s_1a = 0$ and $s_1d+s_2\sigma(a) = 1$. Hence $a = a(s_1d+s_2\sigma(a)) = as_2\sigma(a) = s_2a\sigma(a) = 0$. It is a contradiction.

PROPOSITION 2.3. *Let $R$ be a ring and $\sigma : R \to R$ be a ring isomorphism. If $R[x; \sigma]/(x^{n+1})$ $(n \geq 0)$ left morphic then $R$ is also left morphic.*

PROOF. Assume that $n \geq 0$ and $S := R[x; \sigma]/(x^{n+1})$ is left morphic. Let $a$ be any nonzero arbitrary element in $R$. Thus there exists $\alpha = \sum_{i=0}^{n} r_ix^i \in S$ such that $\mathrm{l.ann}_S(\alpha) = Sa$ and $\mathrm{l.ann}_S(a) = S\alpha$. Therefore $a\alpha = \alpha a = 0$ and so $ar_0 = r_0a = 0$. Hence $Ra \subseteq \mathrm{l.ann}_R(r_0)$ and $Rr_0 \subseteq \mathrm{l.ann}_R(a)$. It is easy to see that $\mathrm{l.ann}_R(a) = Rr_0$. Now assume that $r \in \mathrm{l.ann}_R(r_0)$. Therefore $x^nr\alpha = \sigma^n(rr_0)x^n = 0$ and so $x^nr \in \mathrm{l.ann}_S(\alpha) = Sa$. Thus there exists $\beta = \sum_{i=0}^{n} b_ix^i \in S$ such that $x^nr = \beta a$ and it shows that $\sigma^n(r) = b_n\sigma^n(a)$. Since $\sigma$ is an isomorphism, $\sigma^n(s) = b_n$ for some $s \in R$. Therefore $\sigma^n(r) = \sigma^n(sa)$ and so $r = sa \in Ra$. Thus $\mathrm{l.ann}_R(r_0) = Ra$. The proof is now completed. □

We note that the converse of the above proposition does not hold true. To see it, consider the ring $R$ and endomorphism $\sigma$ mentioned in Example 2.2. In fact $R$ is unit-regular and so morphic while $R[x; \sigma]/(x^2)$ is not even left quasi-morphic.

As an application of Propositions 2.3 and 5, we can deduce the following corollary which is proved in [8, Lemma 10].

COROLLARY 2.4. *Let $n \geq 0$ be an integer. If $R[x]/(x^{n+1})$ is left quasi-morphic (resp., left morphic), then so is $R$.*

PROOF. It follows from Propositions 2.3 and 5 by setting $\sigma = 1$. □

In the next we investigate morphic property for $R[x; \sigma]/(x^{n+1})$ without the assumption that "$\sigma$ is an isomorphism".

THEOREM 2.5. *Let $R$ be a ring, $\sigma : R \to R$ be an endomorphism and $n \geq 1$ be an integer. If $R[x; \sigma]/(x^{n+1})$ is morphic then $R$ is regular.*

PROOF. Let $S := R[x; \sigma]/(x^{n+1})$ be morphic. Then by Proposition 5, $R$ is quasi morphic. Let $a \in R$ be any nonzero element. Therefore there exists an element $b \in R$ such that $Ra = \mathrm{l.ann}_R(b)$. Let $\alpha := bx^n$. Since $S$ is left morphic, there exists $\beta = \sum_{i=0}^{n} b_ix^i \in S$ such that $\mathrm{l.ann}_S(\alpha) = S\beta$ and $S\alpha = \mathrm{l.ann}_S(\beta)$. Since $S$ is also right morphic, there exists $\gamma \in S$ such that $\beta S = \mathrm{r.ann}_S(\gamma)$. Therefore

$$\mathrm{r.ann}_S(\alpha) = \mathrm{r.ann}_S(\mathrm{l.ann}_S(\beta)) = \mathrm{r.ann}_S(\mathrm{l.ann}_S(\mathrm{r.ann}_S(\gamma))) = \mathrm{r.ann}_S(\gamma) = \beta S.$$

We note that $x\alpha = \sigma(b)x^{n+1} = 0$ and also $\alpha x = 0$. Thus $x \in \mathrm{l.ann}_S(\alpha) = S\beta$ and $x \in \mathrm{r.ann}_S(\alpha) = \beta S$. Therefore there exist $\sum_{i=0}^{n} r_i x^i$ and $\sum_{i=0}^{n} s_i x^i$ in $S$ such that $x = (\sum_{i=0}^{n} r_i x^i)(\sum_{i=0}^{n} b_i x^i)$ and $x = (\sum_{i=0}^{n} b_i x^i)(\sum_{i=0}^{n} s_i x^i)$. Thus $r_0 b_0 = 0$, $r_0 b_1 + r_1\sigma(b_0) = 1$, $b_0 s_0 = 0$ and $b_0 s_1 + b_1\sigma(s_0) = 1$. Now we have the following:

$$r_0 = r_0(b_0 s_1 + b_1\sigma(s_0)) = r_0 b_1 \sigma(s_0),$$
$$\sigma(s_0) = (r_0 b_1 + r_1\sigma(b_0))\sigma(s_0) = r_0 b_1 \sigma(s_0).$$

Thus $r_0 = \sigma(s_0)$ and so $b_0 = (b_0 s_1 + b_1\sigma(s_0))b_0 = b_0 s_1 b_0 + b_1 r_0 b_0 = b_0 s_1 b_0$. Therefore $b_0$ is regular. Since $\mathrm{l.ann}_S(\alpha) = S\beta$, it is routine to see that $Rb_0 = \mathrm{l.ann}_R(b) = Ra$. We show that $a$ is regular. To see it, let $e := s_1 b_0$. It is easy to see that $e^2 = e$ and $Rb_0 = Re$. Therefore $Ra = Re$ and so $a = ae = as_1 b_0$. Since $b_0 \in Ra$, $b_0 = ta$ where $t \in R$. Therefore $a = as_1 ta$ and so $a$ is regular, as desired. □

Corollary 2.6. *Let $R$ be a ring, $\sigma : R \to R$ be a ring homomorphism. If $R[x;\sigma]/(x^{n+1})$ is morphic (for some $n \geq 1$), then the following statements hold:*

1) *If $\sigma$ is an isomorphism then $R$ is unit regular.*
2) *If $R$ is commutative then $R$ is unit regular.*

Proof. Since $R[x;\sigma]/(x^{n+1})$ is morphic and by Theorem 2.5, $R$ is a regular ring.

1) By Theorem 2.3, $R$ is morphic. We note that a morphic and regular ring $R$ is unit-regular [9, Proposition 5].
2) We just note that every commutative regular ring is unit regular.

□

We end the paper with the following corollary which is proved in [8, Theorem 11], as an application of Theorem 2.5. This is also a partial an answer to a question 1.1 raised in [8, Question 1].

Corollary 2.7. *Let $R$ be a ring and $n \geq 1$. If $R[x]/(x^{n+1})$ is morphic then $R$ is unit-regular.*

Proof. Let $\sigma$ be an identity homomorphism on $R$. Now apply Corollary 2.6. □

## References


1. L. V. An, T. G. Nam and N. S. Tung, *On quasi-morphic rings and related problems*, Southeast Asian Bull. Math. **40** (1) (2016) 23–34.
2. V. Camillo and W. K. Nicholson, *Quasi-morphic rings*, J. Algebra Appl. **6** (5) (2007) 789–799.
3. V. Camillo, W. K. Nicholson and Z. Wang, *Left quasi-morphic rings*, J. Algebra Appl. **7** (6) (2008) 725–733.
4. H. Chen, *Regular elements in quasi-morphic rings*, Comm. Algebra **39** (4) (2011) 1356–1364.
5. G. Erlich, *Units and one-sided units in regular rings*, Trans. Amer. Math. Soc. **216** (1976) 81–90.
6. K. R. Goodearl, *Von Neumann Regular Rings*, Monographs and Studies in Mathematics, 4. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1979.
7. T. K. Lee and Y. Zhou, *A theorem on unit regular rings*, Canad. Math. Bull. **53** (2) (2010) 321–326.

8. T. K. Lee and Y. Zhou, *Regularity and morphic property of rings*, J. Algebra **322** (4) (2009) 1072–1085.
9. W. K. Nicholson and E. S. Campos, *Rings with the dual of the isomorphism theorem*, J. Algebra **271** (1) (2004) 391–406.

E-mail: n.dehghani@pgu.ac.ir
E-mail: rasoulmoradi1376@gmail.com

# On Injectivity of Certain Gorenstein Injective Modules

Hossein Eshraghi*
Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Kashan, Kashan, Iran

Abstract. In this note, we will be concerned with injectivity of Gorenstein injective modules over certain rings. Specifically, we will show that if $R$ is a complete local $d$-Gorenstein ring and $M$ is a Gorenstein injective $R$-module possessing a syzygy $K_n$, $n \geq d$ such that ${}^{\perp}K_n \cap K_n^{\perp} = \mathrm{Add}\,(K_n) \cup \mathrm{Inj}(R)$, then $M$ is injective. This is particularly related to the dual notion of the famous Auslaner-Reiten Conjecture recently posed.
**Keywords:** Gorenstein injective module, Gorenstein ring, Ordinal number.
**AMS Mathematical Subject Classification [2010]:** 18G20, 18G25, 13H10, 03E10.

## 1. Introduction

The main theme of this paper is to deal with situations under which certain Gorenstein injective modules are injective. To give a more precise description, let us track back to the well-known paper by M. Auslander and M. Bridger [1] where they defined the notion of modules of $G$-dimension zero. Over commutative Gorenstein local rings, these modules coincide with (maximal) Cohen-Macaulay modules.

Several decades later, E.E. Enochs and O.M.G. Jenda introduced a framework that was able to pass the definition of zero $G$-dimension modules to the setting of non-commutative rings [4]. This attempt led in defining the so-called Gorenstein modules; namely, Gorensein projective, Gorenstein injective, and Gorenstein flat modules. Now a days, Gorensein modules are known to play significant role in various branches of algebra, e.g. from representation theory of finite dimensional algebras, where they emerge under different names, to relative homological algebra [5], etc.

Identifying Gorenstein modules in categories other than module categories has also been an active framework of research during last decade. In this regard, we want to mention the papers [2] where Gorenstein projective and injective objects in the category of (possibly infinite) quiver representations has been considered.

The importance of dealing with these Gorenstein modules may also be viewed from several other perspectives, one of which is the view-point of homological conjectures, particularly those appearing in representation theory of finite dimensional algebras. One of the most long-standing conjectures in this field is the so-called Auslander-Reiten Conjecture, asserting that any finitely generated module $M$ over a finite dimensional algebra $\Lambda$ satisfying $\mathrm{Ext}^i_{\Lambda}(M, M \oplus \Lambda) = 0$ for $i \geq 1$ is projective. The conjecture, being possible to be formulated in terms of Gorenstein

*Speaker

projective modules, also has parallel statements in commutative algebra and has recently been considered in a stronger dual sense [6]. Being involved with Gorenstein injective Artinian modules, this dual statement is another motivation for us to deal with Gorenstein modules.

## 2. Main Results

Let us firstly fix some notation: Throughout the paper, $(R, \mathfrak{m})$ is a commutative local Noetherian ring whose unique maximal ideal is $\mathfrak{m}$. We assume further that $R$ is $d$-Gorenstein, $d \geq 0$, in the sense that it has finite self injective dimension equal to $d$ [8]. For an $R$-module $M$, $\mathrm{Add}\,(M)$ denotes the big additive closure of $M$ whose objects are all $R$-modules that are isomorphic to a direct summand of a direct sum of (probably infinite) copies of $M$. Also, $M$ is said to be self-orthogonal provided it has no self extensions, that is to say, $\mathrm{Ext}^1_R(M, M) = 0$. Moreover, $\mathrm{Inj}(R)$ denotes the class of injective $R$-modules.

Definition 2.1. For an $R$-module $M$, let ${}^{\perp}M$, the left orthogonal class to $M$, be the class of all $R$-modules $N$ with $\mathrm{Ext}^1_R(N, M) = 0$. The notion of $M^{\perp}$, the right orthogonal class to $M$, is defined dually.

We start by recalling the definition of a Gorenstein injective $R$-module.

Definition 2.2. An $R$-module $M$ is said to be Gorenstein injective provided it is a syzygy of an exact complex of injective $R$-modules

$$\cdots \to I_1 \to I_0 \to I_{-1} \to \cdots ,$$

that remains exact after applying the functor $\mathrm{Hom}_R(E, -)$ for all injective $R$-module $E$.

Such a complex is reffered to as a complete resolution of $M$ and the kernels of the positive differentials are sometimes called the syzygies of $M$. (This causes no ambiguity since we do not work with projective resolutions, the setting in which the term "syzygy" is very often used.)

It is clear that injective modules are Gorenstein injective. We note that Gorenstein projective modules are defined dually and it is also well-known that this notion runs in a parallel way to that of the so-called moduels of zero $G$-dimension, defined by Auslander and Bridger in [1]. For basic properties of Gorenstein injective modules and their projective and flat counterparts, we refer to the classical book [5]. We also require some elementary properties of ordianl numbers, for which we refer to any classical text book on set theory, e.g. [7].

Definition 2.3. Let $\lambda$ be an ordinal number. A family of submodules $\{M_\alpha\}_{\alpha<\lambda}$ of an $R$-module $M$ is said to be continuous if $M_\alpha \subset M_\beta$ for $\alpha \leq \beta < \lambda$ and every limit ordinal $\beta < \lambda$ satisfies $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$.

The following lemma, due essentially to Eklof and Trlifaj, is crucially used in this paper. For its proof and the notions used therein, we refer to [3].

Lemma 2.4. *Let $M$ and $N$ be $R$-modules such that $M$ can be written as the union of a continuous chain $\{M_\alpha\}_{\alpha<\lambda}$ of its submodules. Assume that $\mathrm{Ext}^1_R(M_0, N) = 0 = \mathrm{Ext}^1_R(\frac{M_{\alpha+1}}{M_\alpha}, N)$ for every $\alpha + 1 < \lambda$. Then $\mathrm{Ext}^1_R(M, N) = 0$.*

**Construction.** Let $M$ be a Gorenstein injective $R$-module. Assume further that for some $n \geq d$, $M$ has a syzygy $K_n$ (automatically Gorenstein injective) that is self-orthogonal and satisfies ${}^{\perp}K_n \cap K_n^{\perp} = \mathrm{Add}\,(K_n) \cup \mathrm{Inj}(R)$. Hence there exists a minimal complete resolution

$$\cdots \to I_1 \to I_0 \to I_{-1} \to \cdots ,$$

of $M$ as stated above, with $M = \mathrm{Ker}(I_{-1} \to I_{-2})$ and $K_n = \mathrm{Ker}(I_{n-1} \to I_{n-2})$; here minimal means that the left part of the resolutions comes up by using consecutive injective covers [5, Theorem 5.4.1]. Consider the short exact sequence $0 \to K_{n+1} \to I_n \to K_n \to 0$ and set $M_0 = E(\frac{R}{\mathfrak{m}})$, the injective envelope of the $R$-module $\frac{R}{\mathfrak{m}}$. Using transfinite induction, we construct a continuous chain of modules $\{M_\alpha\}_{\alpha<\lambda}$, for any ordinal number $\lambda$, with $C = \bigcup_{\alpha<\lambda} M_\alpha$ such that $\frac{M_{\alpha+1}}{M_\alpha} \simeq \bigoplus_J K_n$ for some index set $J$, and such that for any $\alpha + 1 < \lambda$, any $R$-homomorphism $K_{n+1} \to M_\alpha$ may be extended to an $R$-homomorphism $I_n \to M_{\alpha+1}$. In view of [5, Corollary 7.3.2], this implies that any $R$-homomorphism $K_{n+1} \to C$ has an extension $I_n \to C$ or, equivalently, $\mathrm{Ext}^1_R(K_n, C) = 0$. This means that $C \in K_n^{\perp}$.

On the other hand, since $K_n$ is Gorenstein injective, one has $\mathrm{Ext}^1_R(M_0, K_n) = 0$ according to [5, Theorem 10.1.3]. Also

$$\begin{aligned}\mathrm{Ext}^1_R(\frac{M_{\alpha+1}}{M_\alpha}, K_n) &\simeq \mathrm{Ext}^1_R(\bigoplus_J K_n, K_n) \\ &\simeq \prod_J \mathrm{Ext}^1_R(K_n, K_n) \\ &= 0,\end{aligned}$$

because $K_n$ was supposed to be self-orthogonal. Hene, by Lemma 2.4, $\mathrm{Ext}^1_R(C, K_n) = 0$ which means $C \in {}^{\perp}K_n$. So finally our hypothesis reveals that $C \in \mathrm{Add}\,(K_n) \cup \mathrm{Inj}(R)$.

LEMMA 2.5. *Under the hypothesis of the Construction, $I_n$ has no direct summands isomorphic to $E(\frac{R}{\mathfrak{m}})$.*

The proof of this lemma is based mainly on the aforementioned Construction and, in particular, on the observation that $C \in \mathrm{Add}\,(K_n) \cup \mathrm{Inj}(R)$. We also need the following interesting lemma.

LEMMA 2.6. *Suppose $\mathfrak{p}$ and $\mathfrak{q}$ are two prime ideals of $R$. Then*

$$\mathrm{Hom}_R(E(\frac{R}{\mathfrak{p}}), E(\frac{R}{\mathfrak{q}})) \neq 0,$$

*if and only if $\mathfrak{p} \subseteq \mathfrak{q}$.*

PROOF. This is taken from [5, Theorem 3.3.8]. □

Having proved the couple of lemmas, we are now in the position to state and prove the main result of the paper.

THEOREM 2.7. *Let $(R, \mathfrak{m})$ be a complete local $d$-Gorenstein ring and let $M$ be an Artinian Gorenstein injective $R$-module admitting a self-orthogonal syzygy $K_n$, $n \geq d$, such that ${}^{\perp}K_n \cap K_n^{\perp} = \mathrm{Add}\,(K_n) \cup \mathrm{Inj}(R)$. Then $M$ is injective.*

**Sketch of The Proof.** Take the left part of the aforementioned complete resolution of $M$, that is,

$$\cdots \to I_{n+1} \to I_n \to \cdots \to I_1 \to I_0 \to M \to 0,$$

and apply $\mathrm{Hom}_R(E(\frac{R}{\mathfrak{m}}), -)$. By the definition, one obtains the exact complex

$$\cdots \to \mathrm{Hom}_R(E(\frac{R}{\mathfrak{m}}), I_{n+1}) \to \mathrm{Hom}_R(E(\frac{R}{\mathfrak{m}}), I_n) \to \cdots \to \mathrm{Hom}_R(E(\frac{R}{\mathfrak{m}}), I_0) \to \mathrm{Hom}_R(E(\frac{R}{\mathfrak{m}}), M) \to 0.$$

Since $R$ is Noetherian, the structure of injective $R$-modules [5, Theorem 3.3.10] in conjunction with Lemma 2.5 yields that $I_n$ decomposes as a direct sum of injective modules of the form $E(\frac{R}{\mathfrak{p}})$ for non-maximal prime ideals $\mathfrak{p}$ of $R$. Therefore Lemma 2.6 gives $\mathrm{Hom}_R(E(\frac{R}{\mathfrak{m}}), I_n) = 0$ so that one gets an exact sequence

$$0 \to \mathrm{Hom}_R(E(\frac{R}{\mathfrak{m}}), I_{n-1}) \to \mathrm{Hom}_R(E(\frac{R}{\mathfrak{m}}), I_{n-2}) \to \cdots \to \mathrm{Hom}_R(E(\frac{R}{\mathfrak{m}}), I_0) \to \mathrm{Hom}_R(E(\frac{R}{\mathfrak{m}}), M) \to 0.$$

Taking into account that $R$ is complete, another application of Lemma 2.6 to this sequence settles that the $R$-module $\mathrm{Hom}_R(E(\frac{R}{\mathfrak{m}}), M)$ is of finite projective dimension. Moreover, by [5, Ex. 8, p. 252], this module is also Gorenstein projective. Thus it is a free module by [5, Proposition 10.2.3]. Finally, [6, Proposition 2.4] gives that $M$ is injective, as required.

## References


1. M. Bridger and M. Auslander, *Stable Module Theory,* Mem. Amer. Math. Soc. 94, Providence, RI, 1969.
2. E. E. Enochs, S. Estrada and J. R. García Rozas, *Injective representations of infinite quivers. applications,* Canad. J. Math. **61** (2) (2009) 315–335.
3. P. C. Eklof and J. Trlifaj, *How to make Ext vanish,* Bull. Lond. Math. Soc. **33** (1) (2001) 41–51.
4. E. E. Enochs and O. M. G. Jenda, *Gorenstein injective and projective modules,* Math. Z. **220** (1995) 611–633.
5. E. E. Enochs and O. M. G. Jenda, *Relative Homological Algebra,* Walter de Gruynter, Berlin, New York, 2000.
6. H. Eshraghi and A. Mahin Fallah, *Auslander-reiten conjecture in a dual vein*, Algebr. Represent. Theory (2020). `DOI: 10.1007/s10468-020-09990-w`
7. C. C. Pinter, *A Book of Set Theory,* Dover Publications, Inc. Mineola, New York, 2014.
8. J. J. Rotman, *An Introduction to Homological Algebra,* 2nd ed., Universitext, Springer, New York, 2009.


E-mail: eshraghi@kashanu.ac.ir

# $k-$Numerical Range of Quaternion Matrices with Respect to Nonstandard Involutions

Narges Haj Aboutalebi*
Department of Mathematics, Shahrood Branch, Islamic Azad University, Shahrood, Iran

Abstract. Let $\phi$ be a nonstandard involution on the set of all quaternions and $\alpha$ be a quaternion such that $\phi(\alpha) = \alpha$. In this paper, the notion of $k-$numerical range of quaternion matrices with respect to $\phi$ is introduced. Some basic algebraic properties are investigated.
**Keywords:** Quaternion matrices, Nonstandard involution, $k-$Numerical range.
**AMS Mathematical Subject Classification [2010]:** 15A60, 15B33, 15A18.

## 1. Introduction

Let the set of $\mathbb{R}$ and $\mathbb{C}$ be real and complex numbers, respectively. The four-dimensional algebra over $\mathbb{R}$ with the standard basis $\{1, i, j, k\}$ is denoted by $\mathbb{H}$. An ordered triple $(q_1, q_2, q_3)$ of quaternions, where $q_1^2 = q_2^2 = q_3^2 = -1$, $q_1q_2 = q_3 = -q_2q_1$, $q_2q_3 = q_1 = -q_3q_2$, $q_3q_1 = q_2 = -q_1q_3$ and $1q = q1 = q$ for all $q \in \{q_1, q_2, q_3\}$ is said a units triple. So, the triple $(i, j, k)$ is a units triple of quaternions and it is called the standard triple. If $q \in \mathbb{H}$, then there are unique $a_0, a_1, a_2, a_3 \in \mathbb{R}$ such that $q = a_0 + a_1q_1 + a_2q_2 + a_3q_3$. Let $q_i = p_{1,i}i + p_{2,i}j + p_{3,i}k \in \mathbb{H}$, where $i = 1, 2, 3$. The ordered triple $(q_1, q_2, q_3)$ is a units triple if and only if the matrix $P = (p_{ij})$ is orthogonal and $det(P) = 1$ [3, Proposition 2.4.2].

A map $\phi : \mathbb{H} \to \mathbb{H}$ is called an involution if $\phi(x + y) = \phi(x) + \phi(y)$, $\phi(xy) = \phi(y)\phi(x)$ and $\phi(\phi(x)) = x$ for all $x, y \in \mathbb{H}$. One can easily see that $\phi$ is one-to-one and onto. Also, the $4 \times 4$ matrix responding of $\phi$, with respect to the standard basis of $\mathbb{H}$ , is $diag(1, T)$, where $T = -I$ or $T$ is a $3 \times 3$ real orthogonal symmetric matrix with eigenvalues $1, 1, -1$. $\phi$ is called the standard involution for $T = -I$ and for other case, $\phi$ is called a nonstandard involution [3, Definition 2.4.5]. The set of all quaternions that are invariant by $\phi$ is defined and denoted by

$$Inv(\phi) = \{q \in \mathbb{H} : \ \phi(q) = q\}.$$

Let $\mathbb{H}^n$ be the collection of all $n-$column vectors and $M_{m\times n}(\mathbb{H})$ be the set of all $m \times n$ matrices with entries in $\mathbb{H}$. For the case $m = n$, $M_{m\times n}(\mathbb{H})$ is denoted by $M_n(\mathbb{H})$. Let $A \in M_{m\times n}(\mathbb{H})$, the $n \times m$ matrix $A_\phi$ is obtained by applying $\phi$ entrywise to $A^T$. Let $A \in M_n(\mathbb{H})$ and $\alpha \in Inv(\phi)$, the numerical range of $A$ with respect to $\phi$ is defined and denoted by

$$W_\phi^{(\alpha)}(A) = \{x_\phi Ax : \ x \in \mathbb{H}^n, \ x_\phi x = \alpha\}.$$

*Speaker

To access more information about some known result see [1, 3].

In this paper, we are going to introduce and study the $k-$numerical range of quaternion matrices with respect to nonstandard involutions.

## 2. Main Results

In this section, we assume that $k$ and $n$ are positive integers such that $k \leq n$. Also, let $I_k$ denotes the $k \times k$ identity matrix. The relation $\sim_\phi$ on $\mathbb{H}$ is defined by

$$\lambda \sim_\phi \mu \Longleftrightarrow \exists \beta \in \mathbb{H} \setminus \{0\} \ s.t. \ \lambda = \beta_\phi \mu \beta,$$

where $\lambda, \mu \in \mathbb{H}$. It is clear that $\sim_\phi$ is an equivalent relation on the quaternions. For every $\lambda \in \mathbb{H}$, the $\phi-$class of $\lambda$ is defined by

$$[\lambda]_\phi = \{\beta_\phi \lambda \beta : \beta \in \mathbb{H}, \ \beta \neq 0\}.$$

Definition 2.1. Let $A \in M_n(\mathbb{H})$ and $\phi : \mathbb{H} \to \mathbb{H}$ be an involution. Also, let $\alpha \in Inv(\phi)$ and $1 \leq k \leq n$. The $k-$numerical range of $A$ with respect to $\phi$ is defined and denoted by

$$W_\phi^{(\alpha,k)}(A) = \{\frac{1}{k} tr(X_\phi A X) : \ X \in M_{n\times k}(\mathbb{H}), \ X_\phi X = \alpha I_k\}.$$

Remark 2.2. Let $A \in M_n(\mathbb{H})$ and $\phi : \mathbb{H} \to \mathbb{H}$ be an involution. Moreover, let $\alpha \in Inv(\phi)$, $1 \leq k \leq n$. For every $X = [x_1, \ldots, x_k]$ with $X_\phi X = \alpha I_k$, we have for all $i, j = 1, \ldots, k$

$$(x_i)_\phi x_i = \left\{ \begin{array}{ll} \alpha, & i = j, \\ 0, & i \neq j. \end{array} \right.$$

Then by Definition 2.1, we have

$$W_\phi^{(\alpha,k)}(A) = \{\frac{1}{k}\sum_{i=1}^{k}(x_i)_\phi A x_i : \ \{x_1, \ldots, x_k\} \text{ is a set in } \mathbb{H}^n \text{ such that } (x_i)_\phi x_j = \alpha\delta_{ij} \ \},$$

where $\forall i, j = 1, \ldots, k$. Recall that

$$\delta_{ij} = \left\{ \begin{array}{ll} 1, & i = j, \\ 0, & i \neq j. \end{array} \right.$$

It is clear that if $k = 1$, then we have

$$W_\phi^{(\alpha,1)}(A) = \{x_\phi A x : \ x \in \mathbb{H}^n, \ x_\phi x = \alpha\} = W_\phi^{(\alpha)}(A).$$

So, the notion of $k-$numerical range of $A$ with respect to $\phi$ is a generalization of the numerical range of $A$ with respect to $\phi$. Also, if in Definition 2.1, the units triple $(q_1, q_2, q_3)$ is the standard triple, i.e. $(q_1, q_2, q_3) = (i, j, k)$, $\alpha = 1$ and $\phi$ is the standard involution, then we have

$$W_\phi^{(1,k)}(A) = W^k(A) = \{\frac{1}{k}\sum_{i=1}^{k} x_i^* A x_i : \ \{x_1, \ldots, x_k\} \text{ is an othonormal set in } \mathbb{H}^n\}.$$

To access more details, see [2].

Definition 2.3. Let $\phi : \mathbb{H} \to \mathbb{H}$ is an involution. Also let $U \in M_n(\mathbb{H})$. $U$ is called $\phi-$unitary if $U_\phi U = U U_\phi = I_n$ and the set of all $n \times n$ $\phi-$unitary matrices is denoted by $\mathcal{U}_n$.

In this paper, we assume that $\phi$ is a nonstandard involution on $\mathbb{H}$ such that $\phi(1) = 1,\ \phi(q_1) = -q_1,\ \phi(q_2) = q_2, \phi(q_3) = q_3$. In this case, we have $Inv(\phi) = Span_{\mathbb{R}}\{1, q_2, q_3\}$.

EXAMPLE 2.4. Let $A = \begin{bmatrix} q_1 & 0 \\ 0 & -q_1 \end{bmatrix}$. Then $W_\phi^{(0,2)}(A) = \{0\}$.

In the following theorem, we state some basic properties of the $k-$numerical range of quaternion matrices with respect to $\phi$.

THEOREM 2.5. *Let $A \in M_n(\mathbb{H})$. Then the following assertions are true:*

(a) $W_\phi^{(\alpha,k)}(rA + sI) = rW_\phi^{(\alpha,k)}(A) + s\alpha$ *and* $W_\phi^{(\alpha,k)}(A + B) \subseteq W_\phi^{(\alpha,k)}(A) + W_\phi^{(\alpha,k)}(B)$*, where* $r, s \in \mathbb{R}$ *and* $B \in M_n(\mathbb{H})$*;*
(b) $W_\phi^{(\alpha,k)}(U_\phi AU) = W_\phi^{(\alpha,k)}(A)$*, where* $U \in \mathcal{U}_n$*;*
(c) $W_\phi^{(\alpha,k+1)}(A) \subseteq conv(W_\phi^{(\alpha,k)}(A))$*, where* $k < n$*;*
(d) *If* $\lambda \in W_\phi^{(0,k)}(A)$*, then* $[\lambda]_\phi \subseteq W_\phi^{(0,k)}(A)$*;*
(e) $W_\phi^{(\alpha,k)}(A_\phi) = (W_\phi^{(\alpha,k)}(A))_\phi$*.*

Let $S \subseteq \mathbb{H}$. Then $S$ is called a radial set in $\mathbb{H}$ if $\lambda \in S$ implies that $t\lambda \in S$ for all $t > 0$. In the following proposition, we show that $W_\phi^{(0,k)}(A)$ is a radial set in $\mathbb{H}$.

PROPOSITION 2.6. *Let $A \in M_n(\mathbb{H})$ and $1 \leq k \leq n$. Then $W_\phi^{(0,k)}(A)$ is a radial set in $\mathbb{H}$.*

PROOF. Let $\lambda \in W_\phi^{(0,k)}(A)$ and $t > 0$ be given. Therefore, there is a $X \in M_{n\times k}(\mathbb{H})$ such that $X_\phi X = 0.I_k$ and $\lambda = \frac{1}{k}tr(X_\phi AX)$. Since $t > 0$, we have $t\lambda = \frac{1}{k}tr(\sqrt{t}X_\phi A\sqrt{t}X)$. Then by putting $Y = \sqrt{t}X$, we have $Y_\phi Y = 0.I_k$ and $t\lambda = \frac{1}{k}tr(Y_\phi AY)$. Hence, $t\lambda \in W_\phi^{(0,k)}(A)$. This completes the proof. □

A matrix $A \in M_n(\mathbb{H})$ is called $\phi-$Hermitian if $A = A_\phi$ and $\phi-$skewHermitian if $A = -A_\phi$. Now, we state the following theorem.

THEOREM 2.7. *Let $A \in M_n(\mathbb{H})$. Then the following assertions are true:*

(a) *If $A$ is a $\phi-$Hermitian matrix, then* $W_\phi^{(\alpha,k)}(A) \subseteq Span_{\mathbb{R}}\{1, q_2, q_3\}$*;*
(b) *If $A$ is a $\phi-$skewHermitian matrix, then* $W_\phi^{(\alpha,k)}(A) \subseteq Span_{\mathbb{R}}\{q_1\}$*.*

PROOF. Let $\mu \in W_\phi^{(\alpha,k)}(A)$ be given. Then there is a $X \in M_{n\times k}(\mathbb{H})$ such that $\mu = \frac{1}{k}tr(X_\phi AX)$ and $X_\phi X = \alpha I_k$. So, we have

$$\mu_\phi = \frac{1}{k}tr(X_\phi A_\phi X) = \frac{1}{k}tr(X_\phi AX).$$

Therefore, $\mu_\phi = \mu$. Hence, $\mu \in Span_{\mathbb{R}}\{1, q_2, q_3\}$. The proof of (b) is similar to (a). □

## References

1. Gh. Aghamollaei and M. Rahjoo, *On quaternionic numerical ranges with respect to nonstandard involutions*, Linear Algebra Appl. **540** (2018) 11–25.
2. N. Haj Aboutalebi, Gh. Aghamollaei and H. Momenaee Kermani, *Higher numerical ranges of quaternion matrices*, Electron. J. Linear Algebra **30** (2015) 889–904.
3. L. Rodman, *Topics in Quaternion Linear Algebra*, Princeton University Press, Princeton, 2014.

E-mail: aboutalebi.n@yahoo.com

# Adaptive Simpler GMRES Based on Tensor Format for Sylvester Tensor Equation

Ali Iranmanesh*
Department of Applied Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran
and Azita Tajaddini
Department of Applied Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran

Abstract. The problem of Sylvester tensor equations is a crucial issue in several research applications. Krylov subspace methods are very effective approaches to solve this problems due to their merits in large and sparse problems. We present an adaptive simpler GMRES method for solving the Sylvester tensor equation and then obtain an upper bound for condition number of the basis matrix. Eventually, a numerical example is conducted to illustrate the effectiveness of the method.
**Keywords:** Tensor Krylov subspace, Adaptive simpler GMRES.
**AMS Mathematical Subject Classification [2010]:** 15A69, 65F10, 65F15.

## 1. Introduction

In this paper, we consider the Sylvester tensor equation

$$\mathcal{X} \times_1 A^{(1)} + \mathcal{X} \times_2 A^{(2)} + \cdots + \mathcal{X} \times_N A^{(N)} = \mathcal{D}, \tag{1}$$

where the matrices $A^{(j)} \in \mathbb{R}^{I_j \times I_j}$, for $j = 1, 2, \ldots, n$ and the right-hand side tensor $\mathcal{D} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ are given while the tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ is unknown and should be estimated. Furthermore, notation $\times_n$ denotes $n$-mode product which is defined in the preliminaries section.

Recently, tensor Sylvester equations have received a great deal of attention in the real-world applications, for example image restoration, machine learning [6, 10] and the problems which are obtained from discretization of a linear partial differential equation in high dimension by finite element, finite difference or spectral methods [1, 3, 8].

In the following, we review some research works in the field of the Krylov subspace methods to solve the Sylvester tensor Eq. (1). For instance, Heyouni et al. [4] proposed the tensor format of the Hessenberg based methods, such as Hessenbrg_BTF and CMRH_BTF. These methods are constructed based on Petrov-Galerkin and minimal residual norm conditions, respectively. In [2], Bentbib et al. applied the block and global Arnoldi-based Krylov projection approaches

*Speaker

to the coefficient matrices in order to transform the original Sylvester tensor equation with low rank right-hand side to a low dimensional Sylvester equation which can be solved by any tensor Krylov subspace method.

In the past decade, the GMRES method have been taken into account as the one of the most popular algorithms for solving linear system of equations with single right-hand side and multiple right-hand sides and so matrix equations. In this algorithm, it requires that an upper Hessenberg least-squares problem is solved. In order to reduce the computational cost, Walker et al. [11] suggested the simpler GMRES approach. Although it diminishes the computational cost, it suffers from a numerical unstability. Because, the condition number of the matrix whose columns are a basis for the search subspace is closely related to the residual norm. This means that when the condition number of the basis matrix increases, the residual norm decreases at the same time or in the some sense, the basis matrix which is constructed by the simpler GMRES algorithm is well-conditioned if and only if either stagnation occurs or convergence slows down. To overcome this problem, Jiránek et al. [5] proposed a version of the simpler GMRES which generates a basis of Krylov subspace in such a way that the condition number of basis matrix is retained in a satisfactory level. Eventually, it called Adaptive simpler GMRES (in short Ad-SGMRES). Inspired by this idea, we develop the Adaptive simpler GMRES based on tensor format (Ad-SGMRES_BTF) for solving the Sylvester tensor Eq. (1). Then we obtain an upper bound for condition number of the basis matrix. Finally, to evaluate the efficiency of the proposed method, a numerical example is given.

## 2. Preliminaries

In this section, some basic definitions of tensors are summarized. A tensor is known as a multi-mode array. For example, a vector or a matrix can be considered as a 1-mode tensor or a 2-mode tensor, respectively. Throughout the paper, vectors, matrices and tensors are shown by lower-case letters (e.g. $a$), upper-case letters (e.g. $A$) and calligraphic letters (e.g. $\mathcal{A}$), respectively. An $N$-mode tensor $\mathcal{A}$ is represented as $\mathcal{A} \in \mathbb{R}^{I_1\times I_2\times\ldots\times I_N}$ in which each $I_k$ (for $k = 1,\ldots,N$) indicates the $k$-mode of $\mathcal{A}$. The $k$-th frontal slices of an $N$-mode tensor $\mathcal{A}$ are indicated by $\mathcal{A}_k$, for $k = 1,\ldots,I_N$. The inner product of two tensors $\mathcal{X},\mathcal{Y} \in \mathbb{R}^{I_1\times I_2\times\ldots\times I_N}$ is defined by $\langle\mathcal{X},\mathcal{Y}\rangle = \sum_{i_1=1}^{I_1}\sum_{i_2=1}^{I_2}\cdots\sum_{i_N=1}^{I_N} x_{i_1i_2\cdots i_N}y_{i_1i_2\cdots i_N}$. Also, the corresponding norm of the tensor $\mathcal{X}$ is given by $\|\mathcal{X}\| = \sqrt{\langle\mathcal{X},\mathcal{X}\rangle}$. The notation $I^{(m)}$ stands for the identity matrix of size $m$. Also, condition number of the matrix $C$ is denoted by $\kappa_2(C) = \|C\|_2\,\|C^{-1}\|_2$.

In the sequel, three essential tensor multiplications are described:

Definition 2.1. [7] Let $\mathcal{X} \in \mathbb{R}^{I_1\times\cdots\times I_n\times\cdots\times I_N}$ and $\mathcal{Y} \in \mathbb{R}^{I_1\times I_2\times\cdots I_{N-1}\times I_M}$ be two $N$-mode and $M$-mode tensors, respectively, $t \in \mathbb{R}^{I_n}$ and $U \in \mathbb{R}^{J\times I_n}$, then

• The $n$-mode vector product of a tensor $\mathcal{X}$ with a vector $t$ is indicated by $\mathcal{X}\bar{\times}_n t \in \mathbb{R}^{I_1\times\cdots\times I_{n-1}\times I_{n+1}\cdots\times I_N}$ and its elements are

$$(\mathcal{X}\bar{\times}_n t)_{i_1\cdots i_{n-1}i_{n+1}\cdots i_N} = \sum_{i_n=1}^{I_n} a_{i_1i_2\cdots i_N}t_{i_n}.$$

• The $n$-mode matrix product of a tensor $\mathcal{X}$ with a matrix $U$ is denoted by $\mathcal{A}\times_n U \in \mathbb{R}^{I_1\times\cdots\times I_{n-1}\times J\times I_{n+1}\times\cdots\times I_N}$ and its elements are

$$(\mathcal{X}\times_n U)_{i_1\cdots i_{n-1}ji_{n+1}\cdots i_N} = \sum_{i_n=1}^{I_n} a_{i_1i_2\cdots i_N}u_{ji_n}.$$

• The $\boxtimes^{(N)}$-product between two tensors $\mathcal{X}$ and $\mathcal{Y}$ is denoted by $\mathcal{X} \boxtimes^{(N)} \mathcal{Y} \in \mathbb{R}^{I_N \times I_M}$ and its elements

$$\left[\mathcal{X} \boxtimes^{(N)} \mathcal{Y}\right]_{i,j} = trace\left(\mathcal{X}_i \boxtimes^{(N-1)} \mathcal{Y}_j\right), \quad i = 1, \dots I_N, j = 1, \dots, I_M,$$

in which $\mathcal{X}_i$ and $\mathcal{Y}_j$ are the $i$-th and $j$-the column slices of $\mathcal{X}$ and $\mathcal{Y}$, respectively. Moreover, if $\mathcal{X} \in \mathbb{R}^{I_1}$ and $\mathcal{Y} \in \mathbb{R}^{I_1}$, then $\mathcal{X} \boxtimes^1 \mathcal{Y} = \mathcal{X}^T \mathcal{Y}$.

In the following lemma, some properties of tensor multiplications are given:

Lemma 2.2. *Let $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N \times m}$ be two $(N+1)$-mode tensors with $N$-mode column slices $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_m$ and $\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_m$, respectively, $U \in \mathbb{R}^{J_n \times I_n}$ and $t \in \mathbb{R}^{J_n}$. Then*

1) $(\mathcal{A} \times_n U) \bar{\times}_n t = \mathcal{A} \bar{\times}_n (U^T t)$ [7].
2) $\mathcal{X} \boxtimes^{(N+1)} (\mathcal{Y} \bar{\times}_{N+1} t) = (\mathcal{X} \boxtimes^{(N+1)} \mathcal{Y}) t$ [4].

## 3. The Adaptive Simpler GMRES_BTF Method

In this section, we propose the Ad-SGMRES method based on tensor format for solving the Sylvester tensor Eq. (1). By choosing an adaptive parameter $v \in [0, 1]$, the basis of the tensor Krylov subspace is constructed such that the condition number of the matrix corresponding to the basis is at an acceptable level. In the following, the numerical stable algorithm is elaborated.

Let $\mathcal{S}$ be the linear mapping defined as

$$\mathcal{S} : \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N} \longrightarrow \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$$

$$\mathcal{X} \longrightarrow \mathcal{S}(\mathcal{X}) := \sum_{n=1}^{N} \mathcal{X} \times_n A^{(n)}.$$

Thus, the Sylvester tensor Eq. (1) can be rewritten as

$$\mathcal{S}(\mathcal{X}) = \mathcal{D}.$$

Besides, suppose that $\mathcal{V}$ is any $N$-mode tensor in $\mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$, then the $m$-th tensor Krylov subspace associated to the pair $(\mathcal{S}, \mathcal{V})$ is defined by $\mathcal{K}_m(\mathcal{S}, \mathcal{V}) = \text{span}\left\{\mathcal{V}, \mathcal{S}(\mathcal{V}), \dots, \mathcal{S}^{m-1}(\mathcal{V})\right\}$, where $\mathcal{S}^i(\mathcal{V}) = \mathcal{S}(\mathcal{S}^{i-1}(\mathcal{V}))$ and $\mathcal{S}^0(\mathcal{V}) = \mathcal{V}$.

In the Adaptive simpler GMRES_BTF algorithm, the basis of the tensor Krylov subspace is selected as follows:

Let the $N$-mode tensors $\mathcal{Z}_j \in \mathbb{R}^{I_1 \times I_2 \times \dots \times I_N}$, for $j = 1, 2, \dots, m$ are a basis for the tensor Krylov subspace $\mathcal{K}_m(\mathcal{A}, \mathcal{R}_0)$, where $\mathcal{X}_0 \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ is a given initial guess and $\mathcal{R}_0 = \mathcal{D} - \mathcal{A}(\mathcal{X}_0)$ is its corresponding residual. The basis elements are chosen as follows:

• For $j = 1$, $\mathcal{Z}_1 = \frac{\mathcal{R}_0}{\|\mathcal{R}_0\|}$ and the case that the residual norm reduces to some sizes or in other words $\|\mathcal{R}_{j-1}\| \leq v\|\mathcal{R}_{j-2}\|$, then the tensor $\mathcal{Z}_j$ is picked as $\mathcal{Z}_j = \frac{\mathcal{R}_{j-1}}{\|\mathcal{R}_{j-1}\|}$, $j > 1$, wherein the residuals $\mathcal{R}_{j-2}$ and $\mathcal{R}_{j-1}$ are computed in the $j-2$ and $(j-1)$-th iterations.

• If the previous case does not occur, the same Arnoldi basis will be considered as the tensor $\mathcal{Z}_j$, namely $\mathcal{Z}_j = \mathcal{V}_{j-1}$.

Then Arnoldi_BTF's process [4] is applied to produce an orthonormal basis $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_m$ of the tensor Krylov subspace

$$\mathcal{A}\mathcal{K}_m(\mathcal{A}, \mathcal{R}_0) = \text{span}\left\{\mathcal{A}(\mathcal{R}_0), \mathcal{A}^2(\mathcal{R}_0), \dots, \mathcal{A}^{m-1}(\mathcal{R}_0)\right\}.$$

Suppose that $\widetilde{\mathcal{V}}_m$ is the $(N+1)$-mode tensor with the frontal slices $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m$ and $\widetilde{U}_{m-1}$ is the $m \times (m-1)$ upper Hessenberg matrix whose nonzero entries $u_{i,j}$ are computed by Arnoldi_BTF algorithm. Then the following relations hold

$$\mathcal{A}\widetilde{\mathcal{Z}}_{m-1} = \widetilde{\mathcal{V}}_m \times_{(N+1)} \widetilde{U}_{m-1}^T,$$

where $\mathcal{A}\widetilde{\mathcal{Z}}_{m-1}$ is the $(N+1)$-mode tensor with the column tensors $\mathcal{A}(\mathcal{Z}_j)$, for $j = 1, 2, \ldots, m-1$. Since the tensor Krylov subspace $\mathcal{K}_m(\mathcal{A}, \mathcal{R}_0)$ can be decomposed into:

$$\mathcal{K}_m(\mathcal{A}, \mathcal{R}_0) = span\{\mathcal{R}_0\} \bigoplus \mathcal{A}\mathcal{K}_{m-1}(\mathcal{A}, \mathcal{R}_0),$$

where $\bigoplus$ denotes the direct sum. Therefore, tensors $\mathcal{Z}_1 = \mathcal{R}_0/\|\mathcal{R}_0\|, \mathcal{Z}_2, \ldots, \mathcal{Z}_m$ form a basis for $\mathcal{K}_m(\mathcal{A}, \mathcal{R}_0)$. This implies that

$$\mathcal{A}\widetilde{\mathcal{Z}}_m = \widetilde{\mathcal{V}}_m \times_{(N+1)} F_m^T, \tag{2}$$

where $F_m = \begin{pmatrix} u_{1,1} & \\ 0_{(m-1)\times 1} & \widetilde{U}_{m-1} \end{pmatrix}$ and $\widetilde{\mathcal{Z}}_m$ is the $(N+1)$-mode tensor with the frontal slices $\mathcal{Z}_1 = \mathcal{R}_0/\|\mathcal{R}_0\|, \mathcal{Z}_2, \ldots, \mathcal{Z}_m$.
To describe the Ad-SGMRES_BTF for solving the Sylvester tensor Eq. (1), assume that $\mathcal{X}_0$ is an initial guess and $\mathcal{R}_0$ is its corresponding residual. Since the tensors $\mathcal{Z}_1, \mathcal{Z}_1, \ldots, \mathcal{Z}_m$ are a basis for the Krylov subspace $\mathcal{K}_m(\mathcal{A}, \mathcal{R}_0)$, which satisfies in property (2). Then the Ad-SGMRES_BTF method seeks an approximate solution

$$\mathcal{X}_m \in \mathcal{X}_0 + \mathcal{K}_m(\mathcal{A}, \mathcal{R}_0), \tag{3}$$

such that the corresponding residual tensor $\mathcal{R}_m = \mathcal{D} - \mathcal{A}(\mathcal{X}_m)$ satisfies the following orthogonal condition $\mathcal{R}_m \perp \mathcal{A}\mathcal{K}_m(\mathcal{A}, \mathcal{R}_0)$. It is clear that the relation (3) can be reformulated as

$$\mathcal{X}_m = \mathcal{X}_0 + \widetilde{\mathcal{Z}}_m \bar{\times}_{(N+1)} t_m,$$

in which $t_m \in \mathbb{R}^m$. Also, it follows from the first property of Lemma 2.2 and (3), that

$$\mathcal{R}_m = \mathcal{R}_0 - \mathcal{A}(\widetilde{\mathcal{Z}}_m \bar{\times}_{(N+1)} t_m) = \mathcal{R}_0 - \mathcal{A}\widetilde{\mathcal{Z}}_m \bar{\times}_{(N+1)} t_m = \mathcal{R}_0 - \widetilde{\mathcal{V}}_m \bar{\times}_{(N+1)} F_m t_m,$$

where $t_m \in \mathbb{R}^m$ and $\widetilde{\mathcal{V}}_m$ is the $(N+1)$-mode tensor with the column slices $\mathcal{V}_1, \ldots, \mathcal{V}_m$. According to orthogonal condition and $\widetilde{\mathcal{V}}_m \boxtimes^{(N+1)} \widetilde{\mathcal{V}}_m = I^{(m)}$, we have

$$0 = \widetilde{\mathcal{V}}_m \boxtimes^{(N+1)} \mathcal{R}_m = \widetilde{\mathcal{V}}_m \boxtimes^{(N+1)} \mathcal{R}_0 - F_m t_m.$$

As a result, $F_m t_m = \widetilde{\mathcal{V}}_m \boxtimes^{(N+1)} \mathcal{R}_0$. In addition,

$$\mathcal{R}_m = \mathcal{R}_0 - \mathcal{A}\widetilde{\mathcal{Z}}_m \bar{\times}_{(N+1)} t_m = \mathcal{R}_0 - \widetilde{\mathcal{V}}_m \bar{\times}_{(N+1)} (\widetilde{\mathcal{V}}_m \boxtimes^{(N+1)} \mathcal{R}_0) = \mathcal{R}_{m-1} - \alpha_m \mathcal{V}_m,$$

where $\alpha_m = \langle \mathcal{W}_m, \mathcal{R}_0 \rangle = \langle \mathcal{W}_m, \mathcal{R}_{m-1} \rangle$. Consequently, $F_m t_m = \widetilde{\mathcal{V}}_m \boxtimes^{(N+1)} \mathcal{R}_0$ can be written as

$$F_m t_m = [\alpha_1, \alpha_2, \ldots, \alpha_m]^T.$$

In fact, the above discussion is the description of the Adaptive simpler GMRES_BTF approach. In the following theorem, an upper bound for the condition number of the basis matrix is derived.

THEOREM 3.1. *Assume that* $\widetilde{\mathcal{Z}}_m$ *and* $\widehat{\mathcal{V}}_{p,l-1}$ *are the* $(N+1)$*-mode tensors with the column tensors* $\frac{\mathcal{R}_0}{\|\mathcal{R}_0\|}, \mathcal{V}_1, \ldots, \mathcal{V}_{q-1}, \frac{\mathcal{R}_{q-1}}{\|\mathcal{R}_{q-1}\|}, \ldots, \frac{\mathcal{R}_{m-1}}{\|\mathcal{R}_{m-1}\|}$ *and* $\mathcal{V}_p, \ldots, \mathcal{V}_{q-1}, \mathcal{V}_q, \ldots, \mathcal{V}_{l-1}, \frac{\mathcal{R}_{l-1}}{\|\mathcal{R}_{l-1}\|}$*, respectively, and* $1 < q < m$ *and* $q+1 \leq l \leq m$*. In addition, let* $B_m = diag(\widetilde{B}_{1,q}, I_{m-q})$*,* $C_m = diag(I_q, \widetilde{C}_{q,m})$ *and* $F_m = C_m B_m$*. If* $\|\mathcal{R}_{m-1}\| < \cdots < \|\mathcal{R}_{q-1}\|$*, then the following statements hold*

$$\widetilde{\mathcal{Z}}_m = \widehat{\mathcal{V}}_m \times_{(N+1)} F_m^T,$$

*and*

$$\kappa_2(Z_m) = \kappa_2(F_m) = \kappa_2(C_m B_m) \leqslant \kappa_2(C_m)\kappa_2(B_m),$$

*where* $Z_m = [\frac{vec(\mathcal{R}_0)}{\|\mathcal{R}_0\|}, vec(\mathcal{V}_1), \ldots, vec(\mathcal{V}_{q-1}), \frac{vec(\mathcal{R}_{q-1})}{\|\mathcal{R}_{q-1}\|}, \ldots, \frac{vec(\mathcal{R}_{m-1})}{\|\mathcal{R}_{m-1}\|})]$,

$$\kappa_2(C_m) \leq \sqrt{m}(q + \sum_{i=1}^{m-q} \frac{\beta_{q+i-2}^2 + \beta_{q+i-1}^2}{\beta_{q+i-2}^2 - \beta_{q+i-1}^2})^{\frac{1}{2}},$$

$$\kappa_2(B_m) = \kappa_2(\widetilde{B}_{1,q}) = \frac{\beta_0 + \sqrt{\beta_0^2 - \beta_{q-1}^2}}{\beta_{q-1}},$$

*with* $\beta_j = \frac{\mathcal{R}_j}{\|\mathcal{R}_j\|}$ *for* $j = 0, 1, \ldots, m-1$.

## 4. Numerical Example

In this section, the numerical behavior of the Ad-SGMRES_BTF method in comparison to the other methods based on tensor format has been investigated from four perspectives the number of iteration (refereed to iter.), run time (refereed to CPU), true residual norm and true error norm. The stopping criterion for all methods is $\frac{\|\mathcal{D}-\mathcal{S}(\mathcal{X}_k)\|}{\|\mathcal{D}\|} < 10^{-8}$ or the maximum number of iteration is 501.

EXAMPLE 4.1. In this example, we evaluate the efficiency of the proposed method against the other methods SGMRES_BTF, GMRES_BTF and FOM_BTF. Here, the matrices $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$ [9] are obtained by the following Matlab commands

$$A^{(1)} = gallery('poisson', n_0) \in \mathbb{R}^{n\times n},\ A^{(2)} = gallery('pei', n, \alpha) \in \mathbb{R}^{n\times n},$$
$$A^{(3)} = fdm\_2d\_matrix(n_0, \sin(xy), e^{xy}, y^2 - x^2) \in \mathbb{R}^{n\times n},$$

with $n = n_0^2$. In addition, the initial guess $\mathcal{X}_0$ is taken zero tensor, the right-hand side tensor $\mathcal{D}$ is selected such that tensor $\mathcal{X}^* = randn(n, n, n)$ is the exact solution of the Sylvester Eq. (1). Also, $m = 20$, $n = 64, 100$ and $v = 0.9$ are taken.

As observed in Table 1, the Ad-SGMRES_BTF method is superior to the other methods in terms of the CPU time.

TABLE 1. The obtained results of the Ad-SGMRES_BTF, SGMRES_BTF, GMRES_BTF and FOM_BTF methods.

| Grid | Method | iter. | CPU | $\|\mathcal{R}_k\|$ | $\|\mathcal{X}_k - \mathcal{X}^*\|$ |
|---|---|---|---|---|---|
| $64 \times 64 \times 64$ | Ad-SGMRES_BTF | 30 | 92.944 | 1.7934e-05 | 5.3811e-06 |
| | SGMRES_BTF | 30 | 103.68 | 1.7934e-05 | 5.3811e-06 |
| | GMRES_BTF | 30 | 93.143 | 1.5925e-05 | 4.3193e-06 |
| | FOM_BTF | † | † | † | † |
| $49 \times 49 \times 49$ | Ad-SGMRES_BTF | 35 | 41.688 | 9.1304e-06 | 2.3673e-06 |
| | SGMRES_BTF | 35 | 42.883 | 9.4081e-06 | 2.3162e-06 |
| | GMRES_BTF | 35 | 42.554 | 9.4081e-06 | 2.3162e-06 |
| | FOM_BTF | 61 | 86.587 | 3.1740e-06 | 1.1592e-07 |

## References

1. Z. Z. Bai, G. H. Golub and M. K. Ng, *Hermitian and skew-hermitian splitting methods for non-hermitian positive definite linear systems*, SIAM J. Matrix Anal. Appl. **24** (3) (2003) 603–626.
2. A. H. Bentbib, S. El-Halouy and El M. Sadek, *Krylov subspace projection method for Sylvester tensor equation with low rank right-hand side*, Numer. Algor. **84** (2020) 1411–1430.
3. L. Grasedyck, *Existence and computation of low Kronecker-rank approximations for large linear systems of tensor product structure*, Computing **72** (2004) 247–265.
4. M. Heyouni, F. Saberi-Movahed and A. Tajaddini, *A tensor format for the generalized Hessenberg method for solving Sylvester tensor equations*, J. Comput. Appl. Math **377** (2020) 112–878.
5. P. Jirá and M. Rozložnı́k, *Adaptive version of Simpler GMRES*, Numer. Algor. **53** (2010) 93–112.
6. B. N. Khoromskij, *Tensors-structured numerical methods in scientific computing: Survey on recent advances*, Chemom. Intell. Lab. Syst. **110** (1) (2012) 1–19.
7. T. G. Kolda and B. W. Bader, Tensor decompositions and applications, SIAM Review **51** (3) (2009) 455–500.
8. B. W. Li, Y. S. Sun and D. W. Zhang, *Chebyshev collocation spectral methods for coupled radiation and conduction in a concentric spherical participating medium*, ASME J. Heat Transfer **131** (6) (2009) 062701–062709.
9. T. Penzl, *A Matlab Toolbox for Large Lyapunov and Riccati Equations, Model Reduction Problems, and Linear Quadratic Optimal Control Problems*. Software available at `https://www.tu-chemnitz.de/sfb393/lyapack/` (2000).
10. H. Lu, K. N. Plataniotis and A. N. Venetsanopoulos, *A survey of multi-linear subspace learning for tensor data*, Pattern Recogn. **44** (7) (2011) 1540–1551.
11. H. F. Walker and L. Zhou, *A simpler GMRES*, Numer. Linear Algebra Appl. **1** (6) (1994) 571–581.

E-mail: alimin7.11@gmail.com
E-mail: atajadini@uk.ac.ir

# Certain Functors for Some $p$-Groups of Class Two with Elementary Abelian Derived Subgroup of Order $p^2$

Farangis Johari*
Departamento de Matemática, Instituto de Ciências Exatas, Universidade Federal de Minas Gerais, Av. Antônio Carlos 6627, Belo Horizonte, MG, Brazil

Abstract. Let $G$ be a finite $d$-generator $p$-group of class two such that $G/G'$ is elementary abelian and $G' \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$. The aim of this talk is to characterize the exact structure of some functors including the Schur multiplier, the non-abelian tensor square, and the non-abelian exterior square of $G$. We also give the corank of $G$.
**Keywords:** Schur multiplier, Non-abelian tensor square, Non-abelian exterior square, $p$-Groups.
**AMS Mathematical Subject Classification [2010]:** 20D15, 20C25.

## 1. Introduction and Preliminaries

For a given group $G$, the center, the derived subgroup, and the Frattini subgroup of $G$ are denoted by $Z(G)$, $G'$, and $\Phi(G)$, respectively. Let $p$ be a prime number. The subgroup $\langle x^p \mid x \in G\rangle$ of $G$ is denoted by $G^p$. Let $\exp(G)$ be used to denote the exponent of $G$. All $p$-groups of class two are considered finite throughout the paper.
The concept of the non-abelian tensor square $G\otimes G$ of a group $G$ is a special case of the non-abelian tensor product of two arbitrary groups that was introduced by Brown and Loday in [4]. It is easy to check that $\kappa : G \otimes G \to G'$ given by $g \otimes g' \to [g, g']$ for all $g, g' \in G$ is an epimorphism. Let $J_2(G)$ be the kernel of $\kappa$, and let $\nabla(G)$ be a subgroup of $G \otimes G$ generated by the set $\{g \otimes g \mid g \in G\}$. Clearly, $\nabla(G)$ is a central subgroup of $G \otimes G$. The non-abelian exterior square $G\wedge G$ is the quotient group $\dfrac{G \otimes G}{\nabla(G)}$. The element $(g\otimes g')\nabla(G)$ in $G\wedge G$ is denoted by $g \wedge g'$ for all $g, g' \in G$. The map $\kappa$ induces the epimorphism $\kappa' : G \wedge G \to G'$ given by $g \wedge g' \to [g, g']$ for all $g, g' \in G$. The concept of the Schur multiplier $\mathcal{M}(G)$ of a group $G$ was introduced by Schur while he was studying on projective representation of groups. The kernel of the map $\kappa'$ is isomorphic to the Schur multiplier of $G$ (for more information, see [4]).

The corank $t(G)$ for a group $G$ of order $p^n$ is defined a non-negative integer such that

$$t(G) = \frac{1}{2}n(n-1) - \log_p(|\mathcal{M}(G)|).$$

Many authors found the structure of the Schur multiplier, the non-abelian tensor square, and the non-abelian exterior square for some classes of groups such as finite abelian groups and extra-special $p$-groups (See [8, 9]).

*Speaker

Recall that a group $G$ is called capable if $G \cong E/Z(E)$ for some group $E$. Beyl, Felgner, and Schmid [2] introduced the epicenter $Z^*(G)$ of a group $G$. The epicenter of $G$ is the smallest central subgroup $K$ of $G$ such that $G/K$ is capable. In particular, $G$ is capable if and only if $Z^*(G) = 1$. A finite $p$-group $G$ is called special of rank $k$ if $G' = Z(G) = \Phi(G)$ and $Z(G)$ is an elementary abelian $p$-group of rank $k$. Special $p$-groups of rank one are extra-special $p$-groups. Capable extra-special $p$-groups were classified by Beyl, Felgner, and Schmid in [2]. It is shown [7] that if $G$ is a finite capable $p$-group of class two such that $\Phi(G) = G' \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$, then $p^5 \leq |G| \leq p^7$. Hatui [6] obtained the order of the Schur multiplier of special $p$-groups of rank two.

In the same motivation, the goal of this paper is to give a complete description of the structure of some functors, such as the Schur multiplier, the non-abelian tensor square, and the non-abelian exterior square for a $d$-generator $p$-group $G$ of class two such that $\Phi(G) = G' \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$.

We list some elementary observations that will be used in the next section.

Let $\mathbb{Z}_n^{(r)}$ denote the direct product of $r$-copies of the finite cyclic group of order $n$.

Theorem 1.1. *Let $G$ be a $d$-generator $p$-group of class two with $\Phi(G) = G' \cong \mathbb{Z}_p^{(2)}$. Then*

i) *$\mathcal{M}(G)$ is an elementary abelian $p$-group.*
ii) *$G \otimes G$ is an abelian $p$-group.*
iii) *Let $p \neq 2$. Then $|G \wedge G| = |\mathcal{M}(G)||G'|$, $G \otimes G \cong (G \wedge G) \oplus \mathbb{Z}_p^{(\frac{1}{2}d(d+1))}$, and $J_2(G) \cong \mathcal{M}(G) \oplus \mathbb{Z}_p^{(\frac{1}{2}d(d+1))}$.*
iv) *If $G^p = G'$ and $G$ is non-capable, then $Z^*(G) = G'$, $G \otimes G \cong G/G' \otimes G/G'$, $J_2(G) \cong \mathcal{M}(G) \oplus \mathbb{Z}_p^{(\frac{1}{2}d(d+1))}$, and $G \wedge G \cong \mathcal{M}(G) \oplus G'$.*
v) *If $\exp(G) = p$, then $exp(G \otimes G) = p$.*
vi) *If $G^p \cong \mathbb{Z}_p$, then $G$ is non-capable and $G \wedge G \cong \mathcal{M}(G) \oplus G'$.*
vii) *If $\exp(G) = p^2$ and $G$ is capable, then $exp(G \otimes G) = p^2$.*

Proof. i) [9, Corollary 3.2.4] implies that the sequence

$$1 \to \ker \beta \to G' \otimes (G/G') \xrightarrow{\beta} \mathcal{M}(G) \xrightarrow{\varepsilon} \mathcal{M}(G/G') \to G' \to 1,$$

is exact. It follows that $\mathcal{M}(G) \cong \ker \varepsilon \oplus \mathrm{Im}\ \varepsilon \cong \dfrac{G' \otimes (G/G')}{\ker \beta} \oplus \mathrm{Im}\ \varepsilon$. Since $G' \otimes (G/G')$ and $\mathcal{M}(G/G')$ are elementary abelian $p$-groups, we get $\mathcal{M}(G)$ is elementary abelian as well.

ii) The result follows [1, Proposition 3.1].

iii) Clearly, $|G \wedge G| = |\mathcal{M}(G)||G'|$. Using part (ii), we get $G \otimes G$ is abelian. Using [3, Lemma 1.2(i), Theorem 1.3(ii), and Corollary 1.4], we have $\nabla(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d+1))}$ and so $G \otimes G \cong (G \wedge G) \oplus \nabla(G) \cong (G \wedge G) \oplus \mathbb{Z}_p^{(\frac{1}{2}d(d+1))}$ and $J_2(G) \cong \mathcal{M}(G) \oplus \mathbb{Z}_p^{(\frac{1}{2}d(d+1))}$.

iv) If $p = 2$, then $G^2 = G'$. By a similar way used in the proof of [6, Theorem 1.1(a)], we get $Z^*(G) = G'$ for an arbitrary prime number $p$. [5, Proposition 16] implies that $G \otimes G \cong G/G' \otimes G/G'$, $\nabla(G) \cong \nabla(G/G')$, and $G \wedge G \cong G/G' \wedge G/G'$. By parts (i) and (ii), we have $\mathcal{M}(G)$ and $G \wedge G$

are elementary abelian $p$-groups. It follows that $J_2(G) \cong \mathcal{M}(G) \oplus \nabla(G) \cong \mathcal{M}(G) \oplus \mathbb{Z}_p^{(\frac{1}{2}d(d+1))}$ and $G \wedge G \cong \mathcal{M}(G) \oplus G'$.

v) The result follows from [1, Lemma 3.4].

vi) The result holds by a similar way used in the proof of [6, Theorem 1.3(a)] and part (iii).

vii) Assume that $G^p = \langle x^p \rangle \oplus \langle y^p \rangle$ for $x, y \in G$. Put $S = \langle x^p \wedge g, y^p \wedge g_1 \mid g, g_1 \in G \rangle$. By [5, Proposition 16], we have $(G \wedge G)/S \cong G/G' \wedge G/G'$ and $S \neq 1$. For some $g \in G$, we get $(x \wedge g)^p = (x^p \wedge g)(x \wedge [x, g])^{\frac{-1}{2}p(p-1)} \neq 1_{G \wedge G}$. We conclude that $exp(G \wedge G) = p^2$.

□

THEOREM 1.2. *Let $G$ be a $d$-generator $p$-group of class two such that $Z(G) \cong \mathbb{Z}_p^{(m)}$ and $\Phi(G) = G' \cong \mathbb{Z}_p^{(2)}$. Then $G \cong H \times \mathbb{Z}_p^{(m-2)}$, where $H$ is a special $p$-group of rank two. In particular, $G$ is capable if and only if $H$ is capable.*

PROOF. Clearly, $Z(G) = G' \times A$, where $A \cong \mathbb{Z}_{p^{(m-2)}}$. If $A = 1$, then $G = H$ and the proof is complete.

Let $A \neq 1$. Since $G/G'$ is elementary abelian, we have $\frac{G}{G'} = \frac{H}{G'} \times \frac{AG'}{G'}$, for a subgroup $H$ of $G$. Therefore, $G = HA$ and $G' = H \cap AG' = (H \cap A)G'$. Hence $H \cap A \subseteq G' \cap A = 1$ and so $G \cong H \times A$. Since $Z(H) \times A = Z(G) = G' \times A$ and $G' = H'$, we have $Z(H) = H'$ and so $H$ is a special $p$-group of rank two. Now, let $G$ be capable. Then $Z^*(H) \cap H' = 1$, by [10, Proposition 3.2]. Since $H/H'$ is elementary abelian, $H/H'$ is capable, by [2, Proposition 7.3]. Hence, $Z^*(H) \subseteq H'$ and so $Z^*(H) = 1$. The converse holds by [7, Remark (2) p. 247]. □

## 2. Main Results

This section is devoted to characterize the explicit structure of $G \wedge G, G \otimes G$, and $J_2(G)$ for a $d$-generator $p$-group $G$ of class two such that $\Phi(G) = G' \cong \mathbb{Z}_p^{(2)}$. We also give the corank of $G$.

The corank, the Schur multiplier, the non-abelian exterior square, and the non-abelian tensor square of a non-capable $p$-group $G$ of class two when $\Phi(G) = G' \cong \mathbb{Z}_p^{(2)}$ are given in Theorems 2.1 and 2.2.

THEOREM 2.1. *Let $G$ be a non-capable $d$-generator $p$-group of class two such that $G' \cong \mathbb{Z}_p^{(2)}$, $\exp(G) = p$, and $p \neq 2$. Then the following results hold:*

i) *$Z^*(G) \cong \mathbb{Z}_p$ if and only if $\mathcal{M}(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1))}$, $t(G) = 2d + 1$, $G \wedge G \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)+2)}$, $G \otimes G \cong \mathbb{Z}_p^{(d^2+2)}$, and $J_2(G) \cong \mathbb{Z}_p^{(d^2)}$.*

ii) *$Z^*(G) = G'$ if and only if $\mathcal{M}(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)-2)}$, $t(G) = 2d+3$, $G \wedge G \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1))}$, $G \otimes G \cong \mathbb{Z}_p^{(d^2)}$, and $J_2(G) \cong \mathbb{Z}_p^{(d^2-2)}$.*

PROOF. Using Theorem 1.1 (i), (ii) and (v), we obtain that $\mathcal{M}(G)$ and $G \wedge G$ are elementary abelian $p$-groups. Hence, $G \wedge G \cong \mathcal{M}(G) \oplus G'$. By a similar way used in the proof of [6, Theorem 1.4(a),(g), and (h)], we have

i) $Z^*(G) \cong \mathbb{Z}_p$ if and only if $\mathcal{M}(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1))}$ if and only if $t(G) = 2d + 1$.

ii) $Z^*(G) = G'$ if and only if $\mathcal{M}(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)-2)}$ if and only if $t(G) = 2d+3$.

By Theorem 1.1(iii), we determine the structure of $G \otimes G$, $G \wedge G$, and $J_2(G)$. □

Theorem 2.2. *Let $G$ be a non-capable $d$-generator $p$-group of class two such that $\Phi(G) = G' \cong \mathbb{Z}_p^{(2)}$ and $\exp(G) = p^2$. Then the following assertions hold:*

i) *Assume that $Z^*(G) = G^p \cong \mathbb{Z}_p$ for $p \neq 2$. Then $\mathcal{M}(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1))}$, $t(G) = 2d+1$, $G \wedge G \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)+2)}$, $G \otimes G \cong \mathbb{Z}_p^{(d^2+2)}$, and $J_2(G) \cong \mathbb{Z}_p^{(d^2)}$.*

ii) *Let $G^p = G'$ or $G^p \cong \mathbb{Z}_p$ and $Z^*(G) = G'$ for $p > 2$. Then $\mathcal{M}(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)-2)}$, $t(G) = 2d+3$, $G \wedge G \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1))}$, $G \otimes G \cong \mathbb{Z}_p^{(d^2)}$, and $J_2(G) \cong \mathbb{Z}_p^{(d^2-2)}$.*

Proof. The result holds by Theorem 1.1 and a similar way used in the proof of [6, Theorem 1.1(b) and Theorem 1.3(c) and (d)]. □

In what follows, we compute the corank, the Schur multiplier, the non-abelian exterior square, and the non-abelian tensor square of a capable $d$-generator $p$-group $G$ of class two when $\Phi(G) = G' \cong \mathbb{Z}_p^{(2)}$.

Theorem 2.3. *Let $G$ be a capable $d$-generator $p$-group of class two such that $G' \cong \mathbb{Z}_p^{(2)}$ and $\exp(G) = p$. Then one of the following cases holds:*

i) *$G \cong \Phi_4(1^5) \times \mathbb{Z}_p^{(d-3)}$, $\mathcal{M}(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)+3)}$, $t(G) = 2d+4$, $G \wedge G \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)+5)}$, $G \otimes G \cong \mathbb{Z}_p^{(d^2+5)}$, and $J_2(G) \cong \mathbb{Z}_p^{(d^2+3)}$.*

ii) *$G \cong H \times \mathbb{Z}_p^{(d-4)}$, $\mathcal{M}(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)+2)}$, $t(G) = 2d+3$, $G \wedge G \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)+4)}$, $G \otimes G \cong \mathbb{Z}_p^{(d^2+4)}$, and $J_2(G) \cong \mathbb{Z}_p^{(d^2+2)}$, where $H \cong \Phi_{12}(1^6)$, $H \cong \Phi_{13}(1^6)$, or $H \cong \Phi_{15}(1^6)$.*

iii) *$G \cong T \times \mathbb{Z}_p^{(d-5)}$, $\mathcal{M}(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)-1)}$, $t(G) = 2d+2$, $G \wedge G \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)+1)}$, $G \otimes G \cong \mathbb{Z}_p^{(d^2+1)}$, and $J_2(G) \cong \mathbb{Z}_p^{(d^2-1)}$.*

Proof. Theorem 1.2 implies that $G \cong H \times \mathbb{Z}_p^{(m-2)}$, where $H$ is a capable special $p$-group of rank two and exponent $p$. Using [6, Theorem 1.4(c)], let $H \cong \Phi_4(1^5)$. Then $G \cong \Phi_4(1^5) \times \mathbb{Z}_p^{(d-3)}$. By [6, Theorem 1.4(c)] and [9, Theorem 2.2.10 and Corollary 2.2.12], we get

$$\mathcal{M}(G) \cong \mathcal{M}(H) \oplus \mathcal{M}(\mathbb{Z}_p^{(d-3)}) \oplus (H/H' \otimes \mathbb{Z}_p^{(d-3)}) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)+3)}.$$

Hence, $t(G) = 2d+4$. Similarly, we can obtain the Schur multiplier of $G$ when $H$ is isomorphic to one of the $p$-groups $\Phi_{12}(1^6)$, $\Phi_{13}(1^6)$, $\Phi_{15}(1^6)$, or $T$. Using Theorem 1.1(iii), we may obtain the structure of $G \otimes G$, $G \wedge G$, and $J_2(G)$. □

Theorem 2.4. *Let $G$ be a capable $d$-generator $p$-group of class two with $G^p = G' \cong \mathbb{Z}_p^{(2)}$ and $\exp(G) = p^2$. Then*

i) *$\mathcal{M}(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)-1)}$ and $t(G) = 2d$.*

ii) *If $p \neq 2$, then either $G \wedge G \cong \mathbb{Z}_{p^2}^{(2)} \oplus \mathbb{Z}_p^{(\frac{1}{2}d(d-1)-3)}$, $G \otimes G \cong \mathbb{Z}_{p^2}^{(2)} \oplus \mathbb{Z}_p^{(d^2-3)}$, and $J_2(G) \cong \mathbb{Z}_p^{(d^2-1)}$ or $G \wedge G \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(\frac{1}{2}d(d-1)-1)}$, $G \otimes G \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(d^2-1)}$, and $J_2(G) \cong \mathbb{Z}_p^{(d^2-1)}$.*

iii) *If* $p = 2$, *then either* $G \wedge G \cong \mathbb{Z}_4^{(2)} \oplus \mathbb{Z}_2^{(\frac{1}{2}d(d-1)-3)}$, $(G \otimes G)/N \cong \mathbb{Z}_4^{(2)} \oplus \mathbb{Z}_2^{(d^2-3)}$, *and* $J_2(G)/N \cong \mathbb{Z}_2^{(d^2-1)}$ *or* $G \wedge G \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2^{(\frac{1}{2}d(d-1)-1)}$, $(G \otimes G)/N \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2^{(d^2-1)}$, *and* $J_2(G)/N \cong \mathbb{Z}_2^{(d^2-1)}$, *where* $N = \ker\big(\nabla(G) \to \nabla(G/G')\big)$.

Proof. Theorem 1.2 implies that $G \cong H \times \mathbb{Z}_p^{(m-2)}$, where $H$ is a capable special $p$-group of rank two and exponent $p^2$. Using Theorem 1.1(i), [6, Theorems 1.1(c) and 1.5], [9, Theorem 2.2.10, and Corollary 2.2.12], we get $\mathcal{M}(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)-1)}$ and $t(G) = 2d$. From Theorem 1.1(vii), we get $exp(G \wedge G) = p^2$. Since $(G \wedge G)/\mathcal{M}(G) \cong G'$, we have $(G \wedge G)^p \subseteq \mathcal{M}(G)$. It follows that $G \wedge G \cong \mathbb{Z}_{p^2}^{(2)} \oplus \mathbb{Z}_p^{(\frac{1}{2}d(d-1)-3)}$ or $G \wedge G \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(\frac{1}{2}d(d-1)-1)}$. Using Theorem 1.1(iii) and [3, Theorem 1.3(ii)], we may obtain the structure of $G \otimes G$ and $J_2(G)$. □

## Acknowledgement

The author was supported by the postdoctoral grant "CAPES/PRINT-Edital $n$°41/2017, Process number: 88887.511112/2020-00," at the Federal University of Minas Gerais.

## References

1. M. R. Bacon and L. C. Kappe, *The nonabelian tensor square of a* 2*-generator p-group of class* 2, Arch. Math. **61** (6) (1993) 508–516.
2. F. R. Beyl, U. Felgner and P. Schmid, *On groups occurring as center factor groups*, J. Algebra **61** (1) (1979) 161–177.
3. R. D. Blyth, F. Fumagalli and M. Morigi, *Some structural results on the non-abelian tensor square of groups*, J. Group Theory **13** (1) (2010) 83–94.
4. R. Brown, D. L. Johnson and E. F. Robertson, *Some computations of non-abelian tensor products of groups*, J. Algebra **111** (1) (1987) 177–202.
5. G. J. Ellis, *Tensor products and q-crossed modules*, J. London Math. Soc. **51** (2) (1995) 243–258.
6. S. Hatui, *Schur multipliers of special p-groups of rank* 2, J. Group Theory **23** (1) (2020) 85–95.
7. H. Heineken, *Nilpotent groups of class* 2 *that can appear as central quotient groups*, Rend. Sem. Mat. Univ. Padova **84** (1990) 241–248.
8. S. H. Jafari, F. Saeedi and E. Khamseh, *Characterization of finite p-groups by their non-abelian tensor square*, Comm. Algebra **41** (5) (2013) 1954–1963.
9. G. Karpilovsky, *The Schur Multiplier*, London Math. Soc. Monogr. (N. S.) 2, The Clarendon Press, Oxford University Press, New York, 1987.
10. S. Shahriari, *On normal subgroups of capable groups*, Arch. Math. **48** (3) (1987) 193–198.

E-mail: farangisjohari@ufmg.br; farangisjohari85@gmail.com

# On Schur Multipliers of Special $p$-Groups of Rank 3

Farangis Johari
Departamento de Matemática, Instituto de Ciências Exatas, Universidade Federal de Minas Gerais, Av. Antônio Carlos 6627, Belo Horizonte, MG, Brazil
and Azam Kaheni*
Department of Mathematics, University of Birjand, Birjand, Iran

ABSTRACT. Let $G$ be a special $p$-group of rank 3 and exponent $p$. In this talk, an explicit bound for the order of Schur multiplier of $G$ will be given.
**Keywords:** $p$-Group, Schur multiplier.
**AMS Mathematical Subject Classification [2010]:** 20J99, 20D15.

## 1. Introduction

Let $G$ be a group presented as the quotient $F/R$ of a free group $F$ by a normal subgroup $R$. The Schur multiplier of $G$ is defined as

$$\mathcal{M}(G) \cong \frac{R \cap \gamma_2(F)}{[R, F]}.$$

It is well known that the Schur multiplier of $G$ is abelian and independent of the choice of its free presentation. Also, the Schur multiplier of a direct product of two finite groups is isomorphic to the direct sum of the Schur multipliers of the direct factors and the tensor product of the two groups abelianized. Therefore, if $G \cong \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k}$, where $m_{i+1}|m_i$, for $1 \leq i \leq k-1$, then

$$\mathcal{M}(G) \cong \mathbb{Z}_{m_2} \oplus \mathbb{Z}_{m_3}^{(2)} \oplus \cdots \oplus \mathbb{Z}_{m_k}^{(k-1)},$$

where $\mathbb{Z}_m^{(n)}$ denotes the direct product of $n$ copies of the cyclic group $\mathbb{Z}_m$. In addition to abelian groups, the exact structures of Schur multipliers for some non abelian groups have been determined. Moreover, the problem of finding a sharp bound for the order of Schur multipliers was interested by some authors. For a $p$-group $G$ of order $p^n$, Green [4] proved that $|\mathcal{M}(G)| \leq p^{\frac{1}{2}n(n-1)}$. Niroomand [7] improved this bound for a non abelian group in terms of the order of its derived subgroup. The Schur multiplier of groups $G$ of nilpotency class 2 with elementary abelian $G/\gamma_2(G)$ are investigated by Evens and Blackburn [3]. They found the Schur multiplier of extra special $p$-groups. Rai [8] considered the other extreme of the special $p$-groups $G$ where $|\gamma_2(G)|$ is maximum and gave a sharp bound for the order of their Schur multipliers. The Schur multiplier of special $p$-groups of rank 2 was studied by Hatui [5]. Here, we would like to determine a sharp bound for the order of the Schur multipliers of special $p$-groups of rank 3.

By existing exact sequences, some bounds for the order of the Schur multipliers of groups are given. In the following, one of such exact sequences is stated.

*Speaker

THEOREM 1.1. [6, Theorem 2.5.6] *Let $Z$ be a central subgroup of a finite group $G$. Then the following sequence is exact.*

$$G/G' \otimes Z \xrightarrow{\lambda_Z} \mathcal{M}(G) \xrightarrow{\mu} \mathcal{M}(G/Z) \to G' \cap Z \to 1.$$

**Main Theorem**. *Let $G$ be a special $p$-group of rank 3, and exponent $p$. If $p$ is an odd prime and $d = d(G)$ is the minimal number of generators of $G$, then*

a) *$\mathcal{M}(G)$ is elementary abelian;*

b) *$p^{\frac{1}{2}d(d-1)-3} \leq |\mathcal{M}(G)| \leq p^{\frac{1}{2}d(d-1)+6}$,*

c) *$G \otimes G$ is an abelian $p$-group.*

d) *$|G \wedge G| \leq p^{\frac{1}{2}d(d-1)+9}, |G \otimes G| \leq p^{d^2+9}$, and $|J_2(G)| \leq p^{d^2+6}$.*

## 2. Main Results

Let $G$ be a special $p$-group of order $p^n$ and rank 3. Hence, the center, the Frattini subgroup and the derived subgroup of $G$ are coincide and isomorphic to $\oplus_1^3 \mathbb{Z}_p$. Consider three vector spaces $G', G/G'$ and $G/G' \otimes G'$ over $\mathbb{F}_p$. Let $x, y$, and $z$ be arbitrary elements in $G$. Following Rai [8], suppose that $(x, y)$ denotes the element $x\gamma_2(G) \otimes y^p + y\gamma_2(G) \otimes x^p$ and $(x, y, z)$ denotes the element $x\gamma_2(G) \otimes [y, z] + y\gamma_2(G) \otimes [z, x] + z\gamma_2(G) \otimes [x, y] \in G/\gamma_2(G) \otimes \gamma_2(G)$. Moreover, $X_2$ and $X_1$ are the spanned subspace by all elements $(x, x)$ and $(x, y, z)$ of $G/G' \otimes G'$, respectively. Let $X := X_1 + X_2$, and $d = d(G)$. Using Theorem 1.1, we will have

$$\frac{|\mathcal{M}(G)|}{|\mathrm{Im}\lambda_Z|} = \frac{|\mathcal{M}(G/Z)|}{|G' \cap Z|}.$$

Moreover, by [6, Corollary 3.2.4], $\mathrm{Ker}\lambda_{Z(G)} = X$. Therefore, $|\mathrm{Im}\lambda_{Z(G)}| = p^{3d}/|X|$, and

$$|\mathcal{M}(G)| = \frac{p^{\frac{1}{2}d(d-1)-3+3d}}{|X|}.$$

Hence, for finding a suitable bound for the order of Schur multiplier of $G$, it is enough to characterize the set $X$.

Now, following the method used by Hatui [5], we can prove the main result.

**Proof of Main Theorem**. $(a)$ Consider, the homomorphism $\sigma : G/G' \wedge G/G' \to (G/G' \otimes G')/X$ given by $\sigma(\overline{x} \wedge \overline{y}) = (\overline{x} \otimes y^p + \binom{p}{2}\overline{y} \otimes [x, y]) + X$. Evens and Blackburn [3, Theorem 3.1] showed that, there exists an abelian group $M$ with a subgroup $N$ isomorphic to $(G/G' \otimes G')/X$, such that

$$1 \to N \to M^* \xrightarrow{\xi} G/G' \wedge G/G' \to 1,$$

is exact and $\sigma\xi(m) = m^p$ for all $m \in M^*$. Also, they considered the epimorphism $\rho : G/G' \wedge G/G' \to G'$ given by $\rho(\overline{x} \wedge \overline{y}) = [x, y]$ and proved that $\mathcal{M}(G) \cong M$, in which $M$ is the subgroup of $M^*$ containing $N$ such that $M/N \cong Ker\rho$. Since $p$ is odd and $G^p = 1$, the homomorphism $\sigma$ is the trivial map, and therefore $\sigma\xi(x) = x^p = 1$. Thus $\mathcal{M}(G)$ is elementary abelian.

$(b)$ Let $z_1, z_2$, and $z_3$ be the generators of $G'$, and let $x_1, x_2, \ldots, x_d$ be the generators of $G$ such that $[x_1, x_2] \in \langle z_1 \rangle$ is non trivial. Then the set $A_1 := \{(x_1, x_2, x_i) \mid 3 \leq i \leq d\}$ consists of $d - 2$ linearly independent elements of $X_1$. Now if, for some $k, 3 \leq k \leq d$ $[x_1, x_k] \in \langle z_2 \rangle$ is non trivial, then the set $A_2 := \{(x_1, x_k, x_i) \mid 3 \leq i \leq d, i \neq k\}$ consists of $d - 3$ linearly independent elements of $X_1$. Moreover, let for

some $j, 3 \leq j \leq d$ $[x_1, x_j] \in \langle z_3 \rangle$ is non trivial, then the set $A_3 := \{(x_1, x_j, x_i) \mid 3 \leq i \leq d, i \neq k, j\}$ consists of $d-4$ linearly independent elements of $X_1$. Clearly, $A_t$'s for $t = 1, 2, 3$ are three disjoint sets and $A_1 \cup A_2 \cup A_3$ is the smallest linearly independent set of elements in $X_1$. Therefore, $p^{3d-9} \leq |X_1|$. Since $G^p$ is trivial, we will have $|X| = |X_1|$. Hence $p^{\frac{1}{2}d(d-1)-3} \leq |\mathcal{M}(G)| \leq p^{\frac{1}{2}d(d-1)+6}$, as desirable.
$(c)$ The result follows [1, Proposition 3.1].
$(d)$ Let $\nabla(G) = \langle \{g \otimes g \mid g \in G\} \rangle$, and $J_2(G)$ be the kernel of $\kappa : G \otimes G \to G'$ given by $g_1 \otimes g_2 \to [g_1, g_2]$ for all $g_1, g_2 \in G$. Clearly, $|G \wedge G| = |\mathcal{M}(G)||G'|$. Using part $(c)$, we get $G \otimes G$ is abelian. By [2, Lemma 1.2$(i)$, Theorem 1.3$(ii)$, and Corollary 1.4], we will have $\nabla(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d+1))}$. Thus $G \otimes G \cong (G \wedge G) \oplus \nabla(G) \cong (G \wedge G) \oplus \mathbb{Z}_p^{(\frac{1}{2}d(d+1))}$ and $J_2(G) \cong \mathcal{M}(G) \oplus \mathbb{Z}^{(\frac{1}{2}d(d+1))}$. Now, one can obtain the result by part $(b)$. □

## Acknowledgement

F. Johari was supported by the postdoctoral grant "CAPES/PRINT-Edital $n°41/$ 2017, Process number: 88887.511112/2020-00," at the Federal University of Minas Gerais.

## References

1. M. R. Bacon and L. C. Kappe, *The nonabelian tensor square of a* 2*-generator p-group of class* 2, Arch. Math. (Basel) **61** (6) (1993) 508–516.
2. D. Blyth, F. Fumagalli and M. Morigi, *Some structural results on the non-abelian tensor square of groups*, J. Group Theory **13** (1) (2010) 83–94.
3. L. Evens and N. Blackburn, *Schur multipliers of p-groups*, J. Reine Angew. Math. **309** (1979) 100–113.
4. J. A. Green, *On the number of automorphisms of a finite p-group*, Proc. Royal Soc. A **237** (1956) 574–581.
5. S. Hatui, *Schur multipliers of special p-groups of rank* 2, J. Group Theory **23** (1) (2020) 85–95.
6. G. Karpilovsky, *Schur Multipliers*, London Math. Soc. Monographs, (N.S.) 2, Oxford University Press, Oxford, New York, 1987.
7. P. Niroomand, *On the order of Schur multiplier of non-abelian p-groups*, J. Algebra **322** (12) (2009) 4479–4482.
8. P. K. Rai, *On the Schur multiplier of special p-groups*, J. Pure Appl. Algebra **222** (2) (2018) 316–322.

E-mail: farangisjohari@ufmg.br; farangisjohari85@gmail.com
E-mail: azamkaheni@birjand.ac.ir

# On Trivial Extensions of Morphic Rings

Niloufar Keshavarz*
Faculty of Intelligent Systems Engineering and Data Science, Persian Gulf University, Bushehr, Iran
and Najmeh Dehghani
Faculty of Intelligent Systems Engineering and Data Science, Persian Gulf University, Bushehr, Iran

ABSTRACT. The aim of this work is to study (quasi-)morphic property for the trivial extension $R \propto M$ of a bimodule $M$ over a ring $R$. For instance, we show that if $R$ is a commutative domain and $\mathrm{ann}_R(x) = 0$ for some $x \in M$, then $R \propto M$ is (quasi-)morphic if and only if $R$ is a field and $M \simeq R$. Moreover, examples which illustrate our results will be provided.
**Keywords:** Bimodule, Morphic ring, Quasi-morphic ring, Trivial extension.
**AMS Mathematical Subject Classification [2010]:** 16S70, 16D20,16U10.

## 1. Introduction

Throughout this paper, we assume that $R$ is a ring (not necessarily commutative) with a nonzero identity and $M$ is an $R-R$ bimodule. The notions $\mathrm{r.ann}_R(X)$ and $\mathrm{l.ann}_R(X)$ mean the right annihilator and left annihilator of $X$ in $R$, respectively, where $X$ is a nonempty subset of $M$. If $R$ is a commutative ring then the annihilator of $X$ in $R$ is denoted by $\mathrm{ann}_R(X)$.

A ring $R$ is called *left quasi-morphic* if for any $a \in R$, there exist elements $b, c \in R$ such that $\mathrm{l.ann}_R(a) = Rb$ and $Ra = \mathrm{l.ann}_R(c)$. The ring $R$ is called *left morphic* provided that the elements $b$ and $c$ can be chosen equal. Right (quasi-)morphic rings are defined analogously. A left and right (quasi-)morphic ring $R$ is called (quasi-)morphic. These rings were first introduced by Nicholson, Campos and Camillo in [2, 8] and were discussed in great detail in [1, 3, 4, 6] and [7]. Clearly left morphic rings are left quasi-morphic however the converse does not hold true in general. It is proved that for a commutative ring $R$, the notions morphic and quasi-morphic coincide [2, Corollary 4]. Unit-regular rings are examples of morphic rings [8, Example 4] and also every von-Neumann regular ring is quasi-morphic [2]. Moreover, it is proved that unit-regular rings are precisely von-Neumann regular and morphic rings [8, Proposition 5]. Besides, extensions of (quasi-)morphic rings has been of focus by a number of researchers, for example see [1, 4] and [7]. It has been proved that a ring $R$ is unit-regular if and only if $R[x]/(x^{n+1})$ is morphic, where $n \geq 1$. Moreover, (quasi-)morphic property for the ring $R[x, \sigma]/(x^{n+1})$ ($n \geq 1$) where $\sigma$ is a ring homomorphism over $R$, has been also investigated [7, 5]. Quasi-morphic property of the trivial extension $R \propto M$ has also been studied,

*Speaker

where $R$ is a principal ideal domain and $M$ is an $R$-module. For example, it has been shown that $\mathbb{Z} \propto M$ is morphic if and only if $M \simeq \mathbb{Q}/\mathbb{Z}$ where $\mathbb{Q}$ is the set of rational numbers [4, Theorem 14].

These motivated us to investigate when the trivial extension $R \propto M$ is a left (quasi-)morphic ring. We give some examples showing that $R \propto M$ is (quasi-)morphic does not imply that $R$ is (quasi-)morphic and vice versa. Among other results, we will show that if $R$ is a commutative domain, $M$ is an $R$-module and $0 \neq x \in M$ such that $\mathrm{ann}_R(x) = 0$ then $R \propto M$ is (quasi-)morphic if and only if $R$ is a field and $M \simeq R$. As an application of our results, we obtain Corollary 2.7, which is also proved in [5, Proposition 11].

## 2. Main Results

We remind that in whole of the paper $R$ is a ring and $M$ is an $R-R$ bimodule. The trivial extension of $R$ and $M$ is denoted by $R \propto M$ and defined by $\{(a, m) \mid a \in R,\ m \in M\}$. The addition is defined componentwise and multiplication is defined by

$$(a_1, m_1)(a_2, m_2) = (a_1a_2, a_1m_2 + m_1a_2).$$

We note that it is easy to see that $R \propto M$ is isomorphic to the subring $\{\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \mid r \in R, m \in M\}$ of upper triangular matrix ring $\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$. We are interested to investigate when the trivial extension $R \propto M$ is (quasi-)morphic. We begin with the following two examples which show that the condition "$R \propto M$ is (quasi-)morphic" does not imply that "$R$ has the property" and vice versa.

Example 2.1. We show that if $R$ is left (quasi-)morphic then $R \propto M$ does not have the property in general. Note that if $S$ is a commutative domain and $M$ is a $S$-module then $R \propto R$ is never left quasi-morphic where $R = S \propto M$ [1, Proposition 2.4].
Now let $F$ be a field and $R = F \propto F$. Thus by Theorem 2.7, $R$ is a commutative morphic ring however by the above note $R \propto R$ is not even quasi-morphic.

Example 2.2. If $R \propto M$ is left (quasi-)morphic then $R$ is not necessarily left (quasi-)morphic. To see it, consider the trivial extension $S = \mathbb{Z} \propto \mathbb{Q}/\mathbb{Z}$. By [4, Theorem 14], $S$ is a morphic ring. While $\mathbb{Z}$ is not quasi-morphic by the fact that left quasi-morphic domains are exactly division rings [2, Lemma 1].

In the following we proceed with the study of quasi-morphic property for the ring $R \propto M$. First, we prove the following lemma for latter uses.

Lemma 2.3. *Let $R \propto M$ be left morphic. If $0 \neq x \in M$ and $\mathrm{r.ann}_R(x) = 0$ then $\mathrm{l.ann}_R(x) = 0$.*

Proof. Let $S := R \propto M$ be left morphic and $0 \neq x \in M$ with $\mathrm{r.ann}_R(x) = 0$. There exists an element $(s, y) \in S$ such that $S(0, x) = \mathrm{l.ann}_S((s, y))$ and $S(s, y) = \mathrm{l.ann}_S((0, x))$. Therefore $(0, x)(s, y) = 0$. Since $\mathrm{r.ann}_R(x) = 0$, $s = 0$. Now let $r \in \mathrm{l.ann}_R(x)$. Therefore $(r, 0) \in \mathrm{l.ann}_S((0, x)$ and so $(r, 0) \in S(0, y)$. Thus there exists an element $(t, m) \in S$ such that $(r, 0) = (t, m)(0, y) = (0, ty)$. Hence $r = 0$. □

Theorem 2.4. *Let $R \propto M$ be left quasi-morphic. If there exists a nonzero element $x \in M$ such that either* $\text{r.ann}_R(x) = 0$ *or* $\text{l.ann}_R(x) = 0$, *then ${}_RM$ is cyclic. Moreover, if $R \propto M$ is left morphic then $M \simeq R$ as left $R$-module.*

Proof. Let $S := R \propto M$. We note that it is routine to check that

$$\text{l.ann}_S((0, y)) = \text{l.ann}_R(y) \propto M,$$

and $S(0, y) = 0 \propto Ry$, where $y \in M$. Suppose that $S$ is a left quasi-morphic ring, $0 \neq x \in M$ and $a := (0, x) \in S$. Therefore there exist elements $(r, m), (s, n) \in S$ such that $\text{l.ann}_S((r, m)) = Sa$ and $\text{l.ann}_S(a) = S(s, n)$. We consider the case $\text{r.ann}_R(x) = 0$. Since $a(r, m) = 0$, $(0, xr) = 0$ and so $r = 0$. By the above note $Sa = 0 \propto Rx$ and $\text{l.ann}_S((0, m)) = \text{l.ann}_R(m) \propto M$. Therefore $0 \propto Rx = \text{l.ann}_R(m) \propto M$. Thus $M = Rx$ and we are done. Now in case $\text{l.ann}_R(x) = 0$, since $\text{l.ann}_S(a) = S(s, n)$, $(0, sx) = 0$ and so $s \in \text{l.ann}_R(x) = 0$. We remind that

$$0 \propto M = \text{l.ann}_R(x) \propto M = \text{l.ann}_S(a) = S(0, n) = 0 \propto Rn.$$

Therefore $M = Rn$ as desired. In particular, assume that $S$ is left morphic. By Lemma 2.3, it is enough to prove the case $\text{l.ann}_R(x) = 0$. By the previous part, we know that $M = Rn$ where $n \in M$ and $\text{l.ann}_S(a) = S(0, n)$. Since $S$ is left morphic, $Sa = \text{l.ann}_S((0, n))$. Therefore $0 \propto Rx = \text{l.ann}_R(n) \propto M$ and so $\text{l.ann}_R(n) = 0$. Therefore $M = Rn \simeq R$ as left $R$-module. The proof is complete. ☐

Corollary 2.5. *Let $R$ be a commutative ring, $M$ be an $R$-module and $0 \neq x \in M$ such that* $\text{ann}_R(x) = 0$. *If $R \propto M$ is quasi-morphic then $M \simeq R$ and $R$ is also quasi-morphic.*

Proof. Let $R \propto M$ be quasi-morphic. We remind that every commutative quasi-morphic ring is morphic [2, Corollary 4]. Therefore by Theorem 2.4, $M \simeq R$ as $R$-module and so the trivial extension $R \propto M$ is isomorphic to $R \propto R$. Therefore $R$ must be quasi-morphic [1, Corollary 2.3]. ☐

Theorem 2.6. *Let $R$ be a commutative domain and $x$ be a nonzero element of $M$ such that* $\text{ann}_R(x) = 0$. *Then the following statements are equivalent.*

a) *$R \propto M$ is a morphic ring;*
b) *$R \propto M$ is a quasi-morphic ring;*
c) *$R$ is a field and $M \simeq R$.*

Proof. (a) $\Rightarrow$ (b). It is clear.
(b) $\Rightarrow$ (c). It follows from Corollary 2.5 and the fact that quasi-morphic domains are exactly division rings [2, Lemma 1].
(c) $\Rightarrow$ (a). Let $M \simeq R$ and $R$ be a field. Therefore $R \propto M \simeq R \propto R$. Let $(a, x)$ be any nonzero arbitrary element in $S$ where $S = R \propto R$. If $a = 0$ then it is easy to see that $S(0, x) = \text{ann}_S((0, x))$. If $a \neq 0$ then it is also routine to check that $S(a, x) = S$ and $\text{ann}_S((a, x)) = 0$. Therefore $R \propto R$ is a morphic ring and so is $R \propto M$. ☐

As an application of Theorem 2.6, we can deduce the following corollary which is proved in [6, Proposition 11].

Corollary 2.7. *Let $D$ be a field and $V$ be a bimodule over $D$. Then $D \propto V$ is (quasi-)morphic if and only if* $\dim({}_DV) \leq 1$.

Proof. (⇒). If $V$ is a nonzero $D$-module and $D \propto V$ is (quasi-)morphic, then by Theorem 2.6, $V \simeq D$ and so $\dim({}_D V) = 1$.
(⇐). If $V = 0$ then clearly $D \propto V \simeq D$ is (quasi-)morphic. Otherwise, $V \simeq D$ and then by the above theorem, $D \propto V$ is (quasi-)morphic. □

We end the paper with the following corollary showing that $R \propto Q$ is not quasi-morphic when $R$ is a commutative domain and $Q$ is the quotient field of $R$ such that $R \neq Q$.

Corollary 2.8. *If $R$ is an integral domain which is not division ring then $R \propto Q$ is not quasi-morphic where $Q$ is the quotient field of $R$.*

Proof. It is an application of Theorem 2.6. □

## References

1. L. V. An, T. G. Nam and N. S. Tung, *On quasi-morphic rings and related problems*, Southeast Asian Bull. Math. **40** (1) (2016) 23–34.
2. V. Camillo and W. K. Nicholson, *Quasi-morphic rings*, J. Algebra Appl. **6** (5) (2007) 789–799.
3. V. Camillo, W. K. Nicholson and Z. Wang, *Left quasi-morphic rings*, J. Algebra Appl. **7** (6) (2008) 725–733.
4. J. Chen and Y. Zhou, *Morphic rings as trival extensions*, Glasgow Math. J. **47** (2005) 139–148.
5. T. K. Lee and Y. Zhou, *A theorem on unit-regular rings*, Canad. Math. Bull. **53** (2) (2010) 321–326.
6. T. K. Lee and Y. Zhou, *Morphic rings and unit regular rings*, J. Pure Appl. Algebra **210** (2) (2007) 501–510.
7. T. K. Lee and Y. Zhou, *Regularity and morphic property of rings*, J. Algebra **322** (4) (2009) 1072–1085.
8. W. K. Nicholson and E. S. Campos, *Rings with the dual of the isomorphism theorem*, J. Algebra **271** (2004) (1) 391–406.

E-mail: n.dehghani@pgu.ac.ir
E-mail: Niloufar_072@yahoo.com

# Characterization of Finite Groups by the Number of Elements of Prime Order

Alireza Khalili Asboei*
Department of Mathematics, Farhangian University, Tehran, Iran

Abstract. Let $S$ be a non-abelian simple group non-isomorphic to $L_2(q)$, where $q$ is a Mersenne prime and $p$ be the greatest prime divisor of $|S|$. In [6, Conjecture E] A. Moreto conjectured that if every finite group $G$ that is generated by elements of order $p$ and has the same number of elements of order $p$ as $S$, then $G/Z(G) \cong S$. In this paper, we verify the conjecture for the sporadic simple groups.
**Keywords:** Element orders, Simple groups, Normal Sylow number.
**AMS Mathematical Subject Classification [2010]:** 20D99, 20D06.

## 1. Introduction

Let $G$ be a finite group. We denote by $n_p(G)$ the number of Sylow $p$-subgroup of $G$, that is, $n_p(G) = |\mathrm{Syl}_p(G)|$. The number of elements of order $i$ of $G$ is denoted by $m_i(G)$. Given a positive integer $n$ and a prime $r$, we write $n_r$ to denote the full $r$-part of $n$, so we can factor $n = n_r m$, where $m$ is not divisible by $r$. Now fix a prime $p$. We say that a positive integer $n$ is a *normal Sylow number* for $p$ if for every prime $q$, the full $q$-part $n_q$ of $n$ satisfies $n_q \equiv 1 \pmod p$. Note that if $n$ is a normal Sylow number for $p$, then $n \equiv 1 \pmod p$, and thus $n$ is not divisible by $p$. Note also that the set of normal Sylow numbers for $p$ is closed under multiplication. We called the set $\omega(G)$ of its element orders, the spectrum of a group $G$. The $\omega(G)$ together with its order retains a substantial part of the information on the structure of a finite group $G$. But, as demonstrated by the example of the dihedral group $D_8$ of order 8 and the quaternion group $Q_8$, does not necessarily determine $G$ uniquely. There is a long bibliography of papers on element orders of finite groups, with special emphasis on element orders of finite simple groups. However, most of the literature has been devoted to proving that certain simple groups are determined by the set of element orders (See [7] or [5] and their references) or to proving that certain simple groups $S$ are determined by the set of multiplicities of element orders and order of $S$ (See [1] and its references). The assumption on the order of the group is very great, so A. Moreto posed the below conjecture that is more interesting (See [6, Conjecture E]).

**Conjecture 1.1.** *Let $S$ be a non-abelian simple group that $S \ncong L_2(q)$, where $q$ is a Mersenne prime. Also, let $p$ be the greatest prime divisor of $|S|$. If $G$ is generated by elements of order $p$ and $m_p(G) = m_p(S)$, then $G/Z(G) \cong S$.*

A. Moreto [6] is proved that the above conjecture is true for $A_p$, the alternating group of degree $p$, where $p$ be every prime number that is not a Wilson prime or

*Speaker

a near Wilson prime of order 2 and $L_2(p)$, where $p$ be every prime number that is not a Mersenne prime. W. J. Shi [8], provided some counterexamples for the above conjecture. He showed that $A_8$, $L_3(4)$, $O_7(3)$, and $S_6(3)$ are counterexamples. In this paper as the main result we give positive answer to the above conjecture for the sporadic simple groups. Our main theorem is the following.

THEOREM 1.2. *Let $p$ be the greatest prime divisor of the order of the finite group $G$. Assume that $G$ is generated by elements of order $p$ and $m_p(G) = m_p(S)$, where $S$ is the sporadic simple group. Then $G/Z(G) \cong S$.*

We have proved the main theorem of this paper in [2].

## 2. Preliminary Results

In this section, we bring some preliminary results.

LEMMA 2.1. [3] *Let $G$ be a finite group without cyclic Sylow $p$-subgroups. Then $m_p(G)$, is congruent to $-1$ modulo $p^2$.*

The following lemma is elementary (See [6, Lemma 2.3]).

LEMMA 2.2. *Let $G$ be a finite group such that a cyclic Sylow $p$-subgroups of $G$ has order $p^n$, with $n \geq 1$. Then the number of subgroups of order $p$ of $G$ is congruent to 1 modulo $p^n$.*

LEMMA 2.3. *Let $G$ be a finite group such that $|G| = p^\alpha \cdot n$, where $(p^\alpha, n) = 1$. Let $P$ be a $p$-subgroup that acts on a $p'$-subgroup $N$, and let $C = C_N(P)$. Then $|N : C|$ is a normal Sylow number for $p$.*

For example, if $p = 11$, we cannot have $|N : C| = 12$ because 12 is not a normal Sylow number for 11.

LEMMA 2.4. *Let $G$ be a $p$-solvable group. Then $n_p(G)$ is a normal Sylow number for $p$.*

## 3. Proof of Theorem 1.2

Now, we will prove the main theorem of this paper.
**Proof of Main Theorem**. First, we will show that $|P| = p$, where $P \in \mathrm{Syl}_p(G)$. By [4, Table 1 and 2], we can compute $n_p(S)$ for every sporadic simple group $S$. Since $p$ is the greatest prime divisor of $|S|$, we have $p^2 \nmid |S|$, so $m_p(S) = (p-1) \times n_p(S)$. Now, we can easily compute $m_p(S)$. Also, it is easy to check that $m_p(G) = m_p(S) \not\equiv -1 \pmod{p^2}$. By Lemma 2.1, the group $G$ has a cyclic Sylow $p$-subgroup $P$. Also, $G$ has $n_p(S)$ subgroups of order $p$. It is easy to check that $n_p(S) \not\equiv -1 \pmod{p^2}$. By Lemma 2.2, we have $|P| = p$, as desired.

Now, we will show that $G$ is not a $p$-solvable group. Let $G$ be $p$-solvable. Then by Lemma 2.4, $n_p(G) = n_p(S) = q_1^{\beta_1} q_2^{\beta_2} \ldots\ q_s^{\beta_s}$ is a normal Sylow number for $p$. For every simple sporadic group $S$, it is easy to check that there exists some $i$ $(1 \leq i \leq s)$ such that $q_i^{\beta_i} \not\equiv 1 \pmod{p}$, which is a contradiction.

We can prove that $G$ has a normal series $N \trianglelefteq K \trianglelefteq G$ such that $K/N$ is a simple group. Since $p$ divides $|K/N|$, $|G|_p = p$, $G$ is not $p$-solvable and $G$ is generated by elements of order $p$, we have $K = G$ (note that $K \trianglelefteq G$ and

$n_p(K) = n_p(G)$, so $m_p(G) = m_p(K)$) and $G/N$ is simple non-abelian such that it's Sylow $p$-subgroups has order $p$.

In order to complete the proof, we must show that $G/N \cong S$. For instance, we consider two cases $S = Co_1$ and $M$ (the Monster group) will show that $G/N \cong S$. The other cases are similar.

**Case a:** $S = Co_1$. In this case $p = 23$ and $n_{23}(S) = 2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 13$. By Table 1 in [9], the all possibilities for $G/N$ are: $L_2(23)$, $U_3(23)$, $M_{23}$, $M_{24}$, $Co_3$, $Co_2$, $Co_1$, $F_{i_{23}}$, $A_n$, where $n \in \{23, 24, 25, 26, 27, 28\}$.

If $G/N = L_2(23)$, then $n_{23}(G/N) = |G : N_G(P)|/|N : N_N(P)| = n_{23}(G)/|N : N_N(P)| = n_{23}(S)/|N : N_N(P)| = 24$. It follows that $|N : N_N(P)| = |N : C_N(P)| = 2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 13$. By Lemma 2.3, $13 \equiv 1 \pmod{23}$, a contradiction.

Similarly, if $G/N = U_3(23)$, $M_{23}$, $M_{24}$, $Co_3$, $Co_2$, then we get a contradiction.

If $G/N = F_{i_{23}}$, then $n_{23}(G/N) = 2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$. Since $n_{23}(G/N) \mid n_{23}(G)$, we get a contradiction.

Similarly, if $G/N = A_n$, where $n \in \{23, 24, 25, 26, 27, 28\}$, then we get a contradiction. Therefore, $G/N = Co_1$.

**Case b:** $S = M$. In this case $p = 71$ and $n_{71}(S) = 2^{46} \cdot 3^{20} \cdot 5^8 \cdot 7^5 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59$. By Table 1 in [9], the all possibilities for $G/N$ are: $L_2(71)$, $L_5(5)$, $L_6(5)$, $M$, $A_{71}$, $A_{72}$.

If $G/N = L_2(71)$, then $n_{23}(G/N) = |G : N_G(P)|/|N : N_N(P)| = 72$. It follows that $|N : N_N(P)| = |N : C_N(P)| = 2^{43} \cdot 3^{18} \cdot 5^8 \cdot 7^5 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59$. By Lemma 2.3, $17 \equiv 1 \pmod{71}$, a contradiction.

Similarly, if $G/N = L_5(5)$, $L_6(5)$, then we get a contradiction.

If $G/N = A_{71}$, then $n_{71}(G/N) = (71 - 2)!$. Since $n_{71}(G/N) \mid n_{71}(G)$, we get a contradiction.

If $G/N = A_{72}$, then $n_{71}(G/N) = 72!/4970$. Since $n_{71}(G/N) \mid n_{71}(G)$, we get a contradiction. Therefore, $G/N = M$.

Now, we show that $N$ is central in $G$. Let $P$ be a Sylow $p$-subgroup of $G$. Since $n_p(G/N) = n_p(G)$ and $n_p(G/N) = |G : N_G(P)|/|N : N_N(P)| = n_p(G)/|N : N_N(P)|$, we have $|N : N_N(P)| = 1$. It follows that $N = N_N(P)$ and so $N \leq N_G(P)$. Therefore, $[P, N] \leq P \cap N = 1$, so $N$ commutes with any Sylow $p$-subgroup of $G$. Since $G$ is generated by elements of order $p$, we conclude that $N \subseteq Z(G)$. On the other hand, $Z(G)/N \trianglelefteq G/N$. Since $G/N$ is a non-abelian simple group, we have $N = Z(G)$, as desired. This completes the proof of the main theorem.

## References


[1] A. Kh. Asboei, S. S. Salehi, A. Iranmanesh and A. Tehranian, *A characterization of sporadic simple groups by NSE and order*, J. Algebra Appl. **12** (2) (2013) 1250158.

[2] A. Kh. Asboei and S. S. Salehi, *The influence of the number of elements of prime order on the structure of finite groups*, submitted.

[3] P. Hall, *On a theorem of Frobenius*, Proc. London Math. Soc. **40** (6) (1936) 468–501.

[4] A. Khosravi and B. Khosaravi, *Two new characterization of sporadic simple groups*, Pure Math. Appl. **16** (2005) 287–293.

[5] A. S. Mamontov and E. Yabara, *Recognition of the group $L_3(4)$ by the set of element orders in the class of all groups*, Algebra Log. **54** (2015) 279–282.

[6] A. Moreto, *The number of elements of prime order*, Monatsh. Math. **186** (1) (2018) 189–195.

[7] W. Shi, *Arithmetical Properties of Finite Groups*, Groups St. Andrews 2005. Vol. 2, 646–653, London Math. Soc. Lecture Note Ser., 340, Cambridge Univ. Press, Cambridge, 2007.

[8] W. Shi, *A counterexample for the conjecture of finite simple groups*, (2018). arXiv:1810.03786

[9] A. V. Zavarnitsine, *Finite simple groups with narrow prime spectrum*, Sib. Élektron. Mat. Izv **6** (2009) 1–12.

E-mail: khaliliasbo@yahoo.com

# Finite Groups with the Kappe Property

Hassan Khosravi*
Faculty of Basic Sciences and Engineering, Gonbad Kavous University, Gonbad, Iran

ABSTRACT. Let $m$ and $n$ be positive integer numbers. In this note we study all finite groups that for every finite subsets $M$ and $N$ containing $m$ and $n$ elements, respectively, there exist $x \in M$ and $y \in N$ such that $\langle x, y \rangle$ is $r$-Kappe (call this condition $\mathcal{K}_r(m,n)$). In fact we fined some bounds for $m$ and $n$ such that $G \in \mathcal{K}_r(m,n)$ implies that $G$ is Kappe and we find a bound for order of $G$ when $G$ is not Kappe group in $\mathcal{K}_r(m,n)$ and $r = 2, 3$. Also we study all finite groups such that every two subsets $M$ and $N$ of $G$, containing $m$ and $n$ elements, there exist $x \in M$ and $y \in N$, such that $\langle x \rangle$ is subnormal in $\langle x, y \rangle$, (call this condition $\mathfrak{S}(m,n)$), and we will fine some bounds for $m$ and $n$ such that all finite groups in this class are nilpotent. Also we find a bound for order of $G$ when $G$ is a non-nilpotent finite $\mathfrak{S}(m,n)$-group.
**Keywords:** Finite group, Fitting subgroup, Kappe group.
**AMS Mathematical Subject Classification [2010]:** 20B05, 20D15.

## 1. Introduction

In [6], M. Zarrin defined the class $\mathcal{X}(m,n)$ as follow: Let $\mathcal{X}$ be a class. Then a finite group $G$ is in the class $\mathcal{X}(m,n)$ for some positive integer numbers $m$ and $n$, if for all subsets $M$ and $N$ of $G$ such that $|M| = m$ and $|N| = n$ there exist $x \in M$ and $y \in N$ that $\langle x, y \rangle \in \mathcal{X}$. This definition is motivated by B. H. Neumann [4] when $\mathcal{X} = \mathfrak{U}$ is the class of abelian groups (he called this condition $Comm(m,n)$).

By a result of Neumann [5], Abdollahi et al. [1] have shown that if $G$ is an infinite group satisfying in the condition $Comm(m,n)$, for some $m$ and $n$, then $G$ is abelian. They also proved that if $G$ is a nonabelian group in $Comm(m,n)$, then $|G|$ is bounded by a function of $m$ and $n$. Bryce in [2], defined the class $\mathfrak{Y}^{[n]}$ with respect to the class $\mathfrak{Y}$ and positive integer $n$ as fallow. A group $G$ is in $\mathfrak{Y}^{[n]}$, if, whenever $X$ and $Y$ are subsets of cardinality $n$ in $G$ there exist $x \in X$ and $y \in Y$ for which $\langle x, y \rangle \in \mathfrak{Y}$. In [2], Bryce introduce a class of groups that he called star groups which containing the class of abelian groups, nilpotent groups and supersoluble groups and find a bound for order of groups in $\mathfrak{Y}^{[n]}$ where $\mathfrak{Y}$ is a class of star groups. Zarrin [6], studied the class $\mathfrak{N}(m,n)$ when $\mathfrak{N}$ is the class of all weakly nilpotent groups and find a bound for order of finite non–nilpotent groups in $\mathfrak{N}(m,n)$. Although Bryce [2] fined a bound for the cardinality of non–nilpotent finite groups in $\mathfrak{Y}(n,n)$, the bound given in [6] is more accurate than the Bryce's bound. In fact he has shown that, among other things, if $G$ is a non–soluble finite $\mathfrak{N}(m,n)$-group, then

$$|G| \leq \max\{m,n\} \times c^{2\max\{m,n\}^2}[\log_{60}^{\max\{m,n\}}]!,$$

*Speaker

where $c \leq \max\{m, n\}$ is a constant. Now let $G$ be a finite group and let $m$ and $n$ be two positive integer numbers. Then we say $G$ is a $Sn(m, n)$-group if for all subsets $M$ and $N$ of $G$ such that $|M| = m$ and $|N| = n$ there exist $x \in M$ and $y \in N$ such that $\langle x \rangle$ is subnormal in $\langle x, y \rangle$. It is clear that if $m = 1$ and $n = 1$ then for all $x, y \in G$, $\langle x \rangle$ is subnormal in $\langle x, y \rangle$ and therefore $[y,_k x] = 1$ for some positive integer k. Thus $G$ is an Engel group and $G$ is nilpotent by a result of Zorn [7]. It is not difficult to see that $S_3 \notin Sn(4, 1) \setminus Sn(1, 4)$ and therefore $Sn(4, 1) \neq Sn(1, 4)$. Thus we define $\mathfrak{S}(m, n) = Sn(m, n) \bigcap Sn(n, m)$ for symmetry. Then it is clear that $\mathfrak{S}(m, n) = \mathfrak{S}(n, m)$ for all positive integer numbers $m$ and $n$. We recall that a group $G$ is said to be an $n$-Kappe group if $[x^n, y, y] = 1$, for all $x, y \in G$. In fact $G$ is $n$-Kappe if $\frac{G}{R_2(G)}$ is a group of exponent $n$ where $R_2(G) = \{x \in G |\ [x, y, y] = 1,$ for all $y \in G\}$ is the set of all right 2-Engel elements of $G$. Primoz Moravec [3] study $n$-Kappe groups and characterize 2 -Kappe, 3-Kappe and metabelian $p$-Kappe groups. In fact he has shown that if $p$ is a prime number, then $G$ is a metabelian $p$-Kappe group if and only if $G$ is nilpotent of class$\leq p + 1$. Also it is shown that $G$ is a 2-Kappe or 3-Kappe if and only if $G$ is a 2-Engel or 3-Engel group, respectively. In this talk we study finite groups in $\mathcal{K}_r(m, n)$ and find some bounds for $m$ and $n$ such that every group in $\mathcal{K}_r(m, n)$ is $r$-Kappe. Also we will use the result of [3] and find some bound for order of non–Kappe finite $\mathcal{K}_r(m, n)$-groups where $r = 2, 3$. Also we study finite groups $G \in \mathfrak{S}(m, n)$ and find some bounds for $m$ and $n$ such that every $\mathfrak{S}(m, n)$-group is nilpotent. Also we find a bound for order of $G$ when $G$ is a non–nilpotent finite $\mathfrak{S}(m, n)$-group.

## 2. Main Results

In this section we study finite groups in $\mathcal{K}_r(m, n)$ and $\mathfrak{S}(m, n)$ for positive integer numbers $m$ and $n$. We will use the following theorem and find some bounds for $m$ and $n$ such that $G \in \mathcal{K}_r(m, n)$ implies that $G$ is $r$-Kappe.

THEOREM 2.1. *Let $G$ be a $\mathcal{K}_r(m, n)$-group and let $N$ be a normal subgroup of $G$ such that $\frac{G}{N}$ is not a $r$-Kappe geoup. Then $|N| < \max\{m, n\}$.*

THEOREM 2.2. *Let $G$ be a finite group in class $\mathcal{K}_r(m, n)$ and let $q$ be the least prime number dividing $|G|$. Also let $N$ be a normal subgroup of $G$ such that $(q - 1)|N| > \max\{m, n\}$ then $\frac{G}{N}$ is a $r$-Kappe group.*

COROLLARY 2.3. *Let $G$ be a finite group in $\mathcal{K}_r(m, n)$, $r \in \{2, 3\}$ and let $(q - 1)\big|Z^*(G)\big| < \max\{m, n\}$. Then $G$ is nilpotent.*

REMARK 2.4. Let $G \in \mathcal{K}_r(m, n)$. Then if $m \leqslant m'$ and $n \leqslant n'$ Then $G \in k_r\,(m', n')$. In special $k_r(m, n) \subseteq k_r\,(m', n')$.

THEOREM 2.5. *Let $G$ be a finite group in $\mathcal{K}_r(m, n)$ such that $r \in \{2, 3\}$. Also let $a \in G$ be an element that $\varphi\,(|a|) \geqslant \max\{m, n\}$. Then $G$ is nilpotent.*

THEOREM 2.6. *Let $G \in \mathcal{K}_r(m, n)$ such that $m + n \leqslant 5$. Then $G$ is a $r$-Kappe group.*

PROOF. By Remark 2.4 it is enough to consider only the cases $G \in \mathcal{K}_r(1, 4)$ and $G \in \mathcal{K}_r(2, 3)$. If $G \in k_r(1, 4)$ and $x$ and $y$ are two arbitrary elements of $G$, then put: $N = \{y, xy, yx, y^x\}$ and $M = \{x\}$. If $y = xy$ or $yx$ then $x = 1$. If

$y = x^y$ then $y = x$ and $\langle x, y\rangle = \langle x\rangle$ is cyclic. If $xy = x^y = y^{-1}xy$ then $y = 1$ and if $yx = x^y$ then $y = x^y x^{-1} = y^{-1}xyx^{-1} = \left[y, x^{-1}\right]$ and therefore $[x^{-1}, y, y] = 1$. Thus suppose that $N$ have four distinct elements. then

$$\langle x, y\rangle = \langle x, xy\rangle = \langle x, yx\rangle = \langle x, x^{-1}yx\rangle,$$

and $G \in \mathcal{K}_r(1, 4)$ implies that $\langle x, y\rangle$ is a $r$-Kappe group. Now if $G \in \mathcal{K}_r(2, 3)$ and $o(x) \neq 2$ then $N = \{y, xy, yx\}$ and $M = \left\{x, x^{-1}\right\}$. I this case $\langle x, y\rangle$ is a $r$-Kappe group and if $o(x) = o(y) = 2$ then we put $N = \{y, yx, xy\}$ and $M = \{x, x^y\}$. In this case $\langle x, y\rangle = \langle x, xy\rangle = \langle x, yx\rangle$ or $\langle x^y, y\rangle = \langle x, y\rangle$ or $\langle x^y, yx\rangle = \langle x^y, xy\rangle = \langle x, y\rangle$ is a $r$-Kappe and since $x^y = y^{-1}xy = yxy$, $G$ is a $r$-Kappe group. □

Corollary 2.7. *Let $G \in \mathcal{K}_r(m, n)$, where $m + n \leqslant 5$ and $r \in \{2, 3\}$. Then $G$ is nilpotent.*

Theorem 2.8. *Let $G \in \mathcal{K}_r(m, n)$ be a finite group that is not $r$-Kappe, where $r \in \{2, 3\}$. Then $|G|$ is bounded by a function of $m$ and $n$.*

In the following we find some similar results above for finite groups in the class $\mathfrak{S}(m, n)$.

Theorem 2.9. *Let $G$ be a non–nilpotent finite group in $\mathfrak{S}(m, n)$ and let $N$ be a normal subgroup of $G$. Also let $q$ be the least prime number dividing $|G|$. Then*

1) *if $m \leq q(q-1)|N|$ and $n \leq (q-1)|N|$, then $\frac{G}{N}$ is nilpotent,*
2) *if $m \leq 2(q-1)|N|$ and $n \leq 2|N|$, then $\frac{G}{N}$ is nilpotent.*

Corollary 2.10. *Let $G$ be a finite $\mathfrak{S}(m, n)$-group. Then*

1) *if $1 \leq m, n \leq 2$, then $G$ is nilpotent, and*
2) *if $1 \leq m, n \leq 4$ and $Z(G) \neq 1$, then $G$ is nilpotent.*

We will extend this corollary for finite $\mathfrak{S}(m, n)$-groups when $m + n \leq 5$.

Corollary 2.11. *Let $G$ be a finite $\mathfrak{S}(m, n)$-group and let $Z^*(G)$ be the hypercenter of $G$. Also let $q$ be the least prime number dividing $|G|$, then*

1) *if $\max\{m, n\} \leq (q-1)|Z^*(G)|$, then $G$ is nilpotent, and*
2) *if $m \leq 2(q-1)|Z^*(G)|$, $n \leq 2|Z^*(G)|$, then $G$ is nilpotent.*

Proof. It is clear that if $G$ is not nilpotent, then $\frac{G}{Z^*(G)}$ is not nilpotent. Now the assertions are clear by applying Theorem 2.9. □

The first main result is the following theorem.

Theorem 2.12. *Let $G$ be a finite $\mathfrak{S}(m, n)$-group and let $q$ be the least prime number dividing $|G|$. If $a \in G$ is a non–trivial element and $u = \varphi(|a|)$, where $\varphi$ is Euller $\varphi$-function, then*

1) *if $m \leq (u+1)(q-1)$, $n \leq u$ or*
2) *if $m \leq qu$, $n \leq q-1$,*

*then $G$ is nilpotent.*

Corollary 2.13. *Let $G$ be a finite group in $\mathfrak{S}(m, n)$. Then $G$ is nilpotent if $m + n \leq 5$.*

Proof. It is clear that the smallest distinct prime numbers that may divide $|G|$ is $q = 2$ and $p = 3$. By Cauchy's theorem $G$ must have an element of order 3. Now since $u = \varphi(3) = 2$ if $m \leq (q-1)(u+1) = 3$ and $n \leq u(q-1) = 2$, then $G$ is nilpotent by Theorem 2.12 (1). Also if $m \leq qu = 4$ and $n \leq q - 1 = 1$, then $G$ is nilpotent by Theorem 2.12 (2). $\square$

The second main result of this talk says,

Theorem 2.14. *Let $G$ be a non-nilpotent finite group in $\mathfrak{S}(m,n)$. Then*

$$|G| \leq \frac{1}{2}\max\{m,n\} \times \max\{c^{2\max\{m,n\}^2}[\log_{60}^{\max\{m,n\}}]!, (m+n)^{113\sqrt{m+n}+2}\},$$

*where $c$ is a constant.*

## References

1. A. Abdollahi, A. Azad, A. Mohammadi Hassanabadi and M. Zarrin, *B. H. Neumann's question on ensuring commutativity of finite groups*, Bull. Austral. Math. Soc. **47** (2) (2006) 121–132.
2. R. A. Bryce, *Ensuring a finite group is supersoluble*, Bull. Austral. Math. Soc. **74** (2) (2006) 219–226.
3. P. Moravec *Some groups with n-central normal closures*, Publ. Math. Debrecen. **16** (2004) 1–17.
4. B. H. Neumann, *Ensuring commutativity of finite groups*, J. Aust. Math. Soc. **71** (2) (2001) 233–234.
5. B. H. Neumann, *A problem of Paul Erdos on groups*, J. Austral. Math. Soc. Ser. A **21** (1976) 467–472.
6. M. Zarrin, *Ensuring a finite group is weekly nilpotent*, Comm. Algebra **40** (2012) 4739–4752
7. M. Zorn, *Nilpotency of finite groups*, Bull. Amer. Math. Soc. **42** (1936) 485–486.

E-mail: khosravi@gonbad.ac.ir

# Some Applications of Tridiagonal Matrices in P-Polynomial Table Algebras

Masoumeh Koohestani*
Faculty of Mathematics, K. N. Toosi University of Technology, Tehran, Iran
Amirhossein Amiraslani
School of STEM, Department of Mathematics, Capilano University, North Vancouver, BC, Canada
Faculty of Mathematics, K. N. Toosi University of Technology, Tehran, Iran
and Amir Rahnamai Barghi
Faculty of Mathematics, K. N. Toosi University of Technology, Tehran, Iran

Abstract. Here, we study the characters of two classes of P-polynomial table algebras. To obtain the characters of these table algebras, we use some tridiagonal matrices and linear algebra methods.
**Keywords:** Character, P-Polynomial table algebra, Tridiagonal matrix.
**AMS Mathematical Subject Classification [2010]:** 05C50, 15A18, 15A23.

## 1. Introduction

Tridiagonal matrices and their applications have been studied in many papers such as [4, 5] and [9]. Moreover, tridiagonal matrices are used in P-polynomial table algebras. More precisely, the first intersection matrix of a P-polynomial table algebra is a tridiagonal matrix whose eigenvalues can give all characters of the P-polynomial table algebra, see [1, Remark 3.1]. Additionally, the Bose-Mesner algebra of any association scheme is a table algebra and hence, the characters of table algebras can be applied in studying the properties of association schemes, see [6]. However, calculating the characters of table algebras explicitly is sometimes hard or impossible.

Here, we intend to calculate the characters of two classes of P-polynomial table algebras which are studied in [7] and their first intersection matrices are as follows:

(1)
$$C = \begin{pmatrix} 0 & 1 & & & & \\ 2\alpha^2 & 0 & \alpha & & & \\ & \alpha & 0 & \alpha & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha & 0 & \alpha \\ & & & & \alpha & \alpha \end{pmatrix}_{(d+1)\times(d+1)},$$

*Speaker

$$D = \begin{pmatrix} 0 & 1 & & & & \\ 2\alpha\gamma & 0 & \gamma & & & \\ & \alpha & 0 & \gamma & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha & 0 & \gamma \\ & & & & 2\alpha & 0 \end{pmatrix}_{(d+1)\times(d+1)},$$

for $\alpha, \gamma \in \mathbb{R}^+$. To this end, we apply some linear algebra methods and the tridiagonal matrices in the forms of

$$(2) \qquad P_n = \begin{pmatrix} 0 & 1 & & & & \\ c & 0 & 1 & & & \\ & c & 0 & 1 & & \\ & & c & 0 & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & c & 0 \end{pmatrix}_{n\times n},$$

$$Q_n = \begin{pmatrix} a & b & & & & \\ 2c & a & b & & & \\ & c & a & b & & \\ & & c & a & \ddots & \\ & & & \ddots & \ddots & b \\ & & & & c & a \end{pmatrix}_{n\times n}, a, b, c \in \mathbb{C},\ bc \neq 0.$$

Also, we can calculate the characteristic polynomial of $P_n$ and $Q_n$ from the results in [2] and [3] as follows

$$(3)\quad |xI_n - P_n| = \left(\sqrt{c}\right)^n U_n\left(\frac{x}{2\sqrt{c}}\right),\ |xI_n - Q_n| = 2(\sqrt{bc})^n T_n\left(\frac{x-a}{2\sqrt{bc}}\right),$$

where $U_n$ and $T_n$ are the $n$-th degree Chebyshev polynomial of the second and first kind, respectively.

## 2. P-Polynomial Table Algebras

In this section, we review some important concepts from table algebras and P-Polynomial table algebras; see [1, 8] for more details.

Let $A$ be an associative commutative algebra with finite-dimension and a basis $\mathbf{B} = \{x_0, x_1, \dots, x_d\}$, where $x_0 = 1_A$. Then $(A, \mathbf{B})$ is called a table algebra if the following conditions hold:

i) $x_i x_j = \sum_{m=0}^{d} \beta_{ijm} x_m$ with $\beta_{ijm} \in \mathbb{R}^+ \cup \{0\}$, for all $i$, $j$;
ii) there is an algebra automorphism of $A$ (denoted by $^-$), whose order divides 2, such that if $x_i \in \mathbf{B}$, then $\overline{x}_i \in \mathbf{B}$ and $\overline{i}$ is defined by $x_{\overline{i}} = \overline{x}_i$;
iii) for all $i$, $j$, we have $\beta_{ij0} \neq 0$ if and only if $j = \overline{i}$; moreover, $\beta_{i\overline{i}0} > 0$.

$(A, \mathbf{B})$ is called a real table algebra, if $i = \bar{i}$, for $0 \le i \le d$. The $i$-th intersection matrix of $(A, \mathbf{B})$ is as

$$B_i = \begin{pmatrix} \beta_{i00} & \beta_{i01} & \dots & \beta_{i0d} \\ \beta_{i10} & \beta_{i11} & \dots & \beta_{i1d} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{id0} & \beta_{id1} & \dots & \beta_{idd} \end{pmatrix}_{(d+1)\times(d+1)},$$

where $x_i x_j = \sum_{m=0}^{d} \beta_{ijm} x_m$, for all $i, j, k$.

For any table algebra $(A, \mathbf{B})$ with $\mathbf{B} = \{x_0 = 1_A, x_1, \cdots, x_d\}$, there exists a unique algebra homomorphism $f : A \to \mathbb{C}$ such that $f(x_i) = f(x_{\bar{i}}) \in \mathbb{R}^+$, for $0 \le i \le d$, see [8]. If $f(x_i) = \beta_{i\bar{i}0}$ for all $i$, then $(A, \mathbf{B})$ is called standard. A real standard table algebra is called P-polynomial if for each $i$, $2 \le i \le d$, there exists a complex coefficient polynomial $\nu_i(x)$ of degree $i$ such that $x_i = \nu_i(x_1)$. If $(A, \mathbf{B})$ is a P-polynomial table algebra, then for all $i$, there exist $b_{i-1}, a_i, c_{i+1} \in \mathbb{R}$ such that

$$x_1 x_i = b_{i-1} x_{i-1} + a_i x_i + c_{i+1} x_{i+1}, \tag{4}$$

with $b_i \neq 0$, $(0 \le i \le d-1)$, $c_i \neq 0$, $(1 \le i \le d)$, and $b_{-1} = c_{d+1} = 0$. Hence, the first intersection matrix of a P-polynomial table algebra is as follows.

$$B_1 = \begin{pmatrix} a_0 & c_1 & & & \\ b_0 & a_1 & c_2 & & \\ & b_1 & a_2 & \ddots & \\ & & \ddots & \ddots & c_d \\ & & & b_{d-1} & a_d \end{pmatrix}_{(d+1)\times(d+1)}.$$

Let $(A, \mathbf{B})$ be a table algebra. Since $A$ is semisimple, the primitive idempotents of $A$ form another basis for $A$, see [8]. Consequently, if $\{e_0, e_1, \dots, e_d\}$ is the set of the primitive idempotents of $A$, then we have $x_i = \sum_{j=0}^{d} p_i(j) e_j$, where $p_i(j) \in \mathbb{C}$, for $0 \le i, j \le d$. The numbers $p_i(j)$ are the characters of the table algebra. Let $(A, \mathbf{B})$ be a P-polynomial table algebra. Then the $p_1(j)$ are equal to the eigenvalues of its first intersection matrix and for $2 \le i \le d$, we have

$$p_i(j) = \nu_i(p_1(j)), \tag{5}$$

where $\nu_i(x)$ is a complex coefficient polynomial such that $x_i = \nu_i(x_1)$.

## 3. Main Results

We now study the characters of two classes of P-polynomial table algebras whose first intersection matrices are given in (1).

THEOREM 3.1. *Let $(A, \mathbf{B})$ be a P-polynomial table algebra with $\mathbf{B} = \{x_0 = 1_A, x_1, \dots, x_d\}$ and its first intersection matrix $B_1$ is equal to the matrix $C$ in (1). Then the characters of $(A, \mathbf{B})$ are*

$$p_0(j) = 1, \ p_1(j) = \lambda_j = 2\alpha \cos\left(\frac{2j\pi}{2d+1}\right),$$
$$p_i(j) = (\sqrt{\alpha})^{i-4}\left((\lambda_j^2 - 2\alpha^2) U_{i-2}\left(\frac{\lambda_j}{2\sqrt{\alpha}}\right) - \alpha\sqrt{\alpha}\lambda_j U_{i-3}\left(\frac{\lambda_j}{2\sqrt{\alpha}}\right)\right),$$

*for $2 \le i \le d$ and $0 \le j \le d$.*

PROOF. For each $i$, $0 \leq i \leq d$, the $p_i(j)$, $0 \leq j \leq d$, are equal to the eigenvalues of the $i$-th intersection matrix $B_i$. So, it is obvious that $p_0(j) = 1$ for all $j$. Let $R_{d+1}(x) = |xI_{d+1} - B_1|$ and $M_n$ be a tridiagonal matrix in the form of

$$M_n = \begin{pmatrix} 0 & \alpha & & & & \\ 2\alpha & 0 & \alpha & & & \\ & \alpha & 0 & \alpha & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha & 0 & \alpha \\ & & & & \alpha & \alpha \end{pmatrix}_{n\times n}.$$

Set $K_n(x) = |xI_n - M_n|$. We can see that $R_{d+1}(x) = K_{d+1}(x)$. By Laplace expansion and using the characteristic polynomial of $Q_n$ in (3), we get

$$K_{d+1}(x) = \quad 2\alpha^{d+1}\left(T_{d+1}\left(\tfrac{x}{2\alpha}\right) - T_d\left(\tfrac{x}{2\alpha}\right)\right). \tag{6}$$

So, the $p_1(j)$ can be obtained from (6). To calculate the $p_i(j)$, $2 \leq i \leq d$, we obtain the polynomial $\nu_i(x)$, where $x_i = \nu_i(x_1)$. Obviously, $\nu_1(x) = x$, and from (4), we get

$$\nu_2(x) = \frac{1}{\alpha}\left(x^2 - 2\alpha^2\right), \ \nu_3(x) = \frac{1}{\alpha}\left(x\nu_2(x) - \alpha\nu_1(x)\right), \ldots,$$

$$\nu_d(x) = \frac{1}{\alpha}\left(x\nu_{d-1}(x) - \alpha\nu_{d-2}(x)\right).$$

Let the recursive function $\varphi_n(x) = x\varphi_{n-1}(x) - \alpha\varphi_{n-2},(x)$ with $\varphi_1(x) = \alpha x$ and $\varphi_2(x) = x^2 - 2\alpha^2$. Hence, $\varphi_n(x)$ can be obtained by the following determinant and equation

$$\begin{vmatrix} \alpha x & 1 & & & & \\ 2\alpha^2 & x/\alpha & 1 & & & \\ & \alpha & x & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha & x & 1 \\ & & & & \alpha & x \end{vmatrix}_{n\times n}, \tag{7}$$

$$\varphi_n(x) = (x^2 - 2\alpha^2)H_{n-2}(x) - \alpha^2 x H_{n-3}(x),$$

where $H_n(x)$ is the characteristic polynomial of the matrix $P_n$ in (2) with $c = \alpha$. Finally from (3), (5) and (7), the proof is completed. □

THEOREM 3.2. *Let* $(A, \mathbf{B})$ *be a P-polynomial table algebra with* $\mathbf{B} = \{x_0 = 1_A, x_1, \ldots, x_d\}$ *and its first intersection matrix is equal to* $D$ *in* (1). *Then the characters of* $(A, \mathbf{B})$ *are*

$$p_0(j) = 1, \ p_1(j) = \lambda_j = 2\sqrt{\alpha\gamma}\cos\left(\tfrac{j\pi}{d}\right),$$
$$p_i(j) = \frac{\left(\sqrt{\alpha}\right)^{i-2}}{\gamma}\left((\lambda_j^2 - 2\alpha\gamma)U_{i-2}\left(\tfrac{\lambda_j}{2\sqrt{\alpha}}\right) - \sqrt{\alpha}\gamma\lambda_j U_{i-3}\left(\tfrac{\lambda_j}{2\sqrt{\alpha}}\right)\right),$$

*for* $2 \leq i \leq d$ *and* $0 \leq j \leq d$.

PROOF. Obviously, $p_0(j) = 1$ for all $j$. Set $R_{d+1}(x) = |xI_{d+1} - B_1|$. Let $N_n$ be the tridiagonal matrix as follows

$$N_n = \begin{pmatrix} 0 & \gamma & & & & \\ 2\alpha & 0 & \gamma & & & \\ & \alpha & 0 & \gamma & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha & 0 & \gamma \\ & & & & 2\alpha & 0 \end{pmatrix}_{n\times n},$$

and $K_n(x) = |xI_n - N_n|$. We have $R_{d+1}(x) = K_{d+1}(x)$. By Laplace expansion and using the characteristic polynomial of $Q_n$ in (3), we have

$$K_{d+1}(x) = 2(\sqrt{\alpha\gamma})^{d+1}\left(T_{d+1}\left(\frac{x}{2\sqrt{\alpha\gamma}}\right) - T_{d-1}\left(\frac{x}{2\sqrt{\alpha\gamma}}\right)\right). \tag{8}$$

So, the $p_1(j)$ are obtained from (8). To calculate the $p_i(j)$, $2 \leq i \leq d$ by the argument as given in Theorem 3.1, we consider the recursive function $\varphi_n(x) = x\varphi_{n-1}(x) - \alpha\varphi_{n-2},(x)$ with $\varphi_1(x) = \gamma x$ and $\varphi_2(x) = x^2 - 2\alpha\gamma$. So, $\varphi_n(x)$ can be obtained by the following determinant and equation

$$\begin{vmatrix} \gamma x & 1 & & & & \\ 2\alpha\gamma & x/\gamma & 1 & & & \\ & \alpha & x & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha & x & 1 \\ & & & & \alpha & x \end{vmatrix}_{n\times n}, \tag{9}$$

$$\varphi_n(x) = (x^2 - 2\alpha\gamma)H_{n-2}(x) - \alpha\gamma x H_{n-3}(x),$$

where $H_n(x)$ is the characteristic polynomial of the matrix $P_n$ in (2). So from (3), (5) and (9), the proof is complete. □

## References


1. H. I. Blau and R. J. Hein, *A Class of P-polynomial table algebras with and without integer multiplicities*. Comm. Algebra **42** (12) (2014) 5387–5424.
2. C. M. da Fonseca, S. Kouachi, D. A. Mazilu and I. Mazilu, *A multi-temperature kinetic Ising model and the eigenvalues of some perturbed Jacobi matrices*, Appl. Math. Comput. **259** (2015) 205–211.
3. C. M. da Fonseca and V. Kowalenko, *Eigenpairs of a family of tridiagonal matrices: Three decades later*, Acta Math. Hungar. **160** (2020) 376–389.
4. L. Losonczi, *Eigenvalues and eigenvectors of some tridiagonal matrices*, Acta Math. Hungar. **60** (1992) 309–332.
5. J. Rimas, *On computing of arbitrary positive integer powers for one type of tridiagonal matrices*, App. Math. Comput. **161** (3) (2005) 1037–1040.
6. B. Xu, *Characters of table algebras and applications to association schemes*, J. Combin. Theory Ser. A **115** (8) (2008) 1358–1373.
7. B. Xu, *On perfect P-polynomial table algebras*, Comm. Algebra **37** (1) (2009) 120–153.
8. B. Xu, *On P-polynomial table algebras and applications to association schemes*. Comm. Algebra **40** (6) (2012) 2171–2183.

9. A. R. Willms, *Analytic results for the eigenvalues of certain tridiagonal matrices*, SIAM J. Matrix Anal. Appl. **30** (2) (2008) 639–656.

E-mail: m.kuhestani@email.kntu.ac.ir
E-mail: amirhosseinamiraslan@capilanou.ca
E-mail: rahnama@kntu.ac.ir

# Group Rings which are Right Gr-Ring

Maryam Masoudi Arani*
Department of Fundamental Sciences, Faculty of Fadak, Kashan Branch,
Technical and Vocational University (TVU), Isfahan, Iran
and Reza Jahani-Nezhad
Department of Mathematics, University of Kashan, Kashan, Iran

Abstract. A ring $R$ is called a reversible ring, if $ab = 0$ implies that $ba = 0$, for every $a, b \in R$. Many studies have been conducted on reversible group rings in recent years. The aim of this paper is to generalize some of the previous results about reversible group rings to more general cases. For this purpose, we introduce a generalization of reversible rings as right gr-ring, where a right gr-ring is a ring in which $ab \in I$ implies $ba \in I$, for every right ideal $I$ of $R$ and $a, b \in R$. We will study conditions under which a group ring $R[G]$ becomes a right gr-ring. We show that the group ring $K[Q_8]$ of a group of quaternions $Q_8$ over field $K$ is a right gr-ring if and only if char(K)=0 and the equation $x^2 + y^2 + 1 = 0$ has no solution in $K$.
**Keywords:** Reversible, Group ring, Right duo.
**AMS Mathematical Subject Classification [2010]:** 16P99, 16S34, 16D25.

## 1. Introduction

The rings in this paper are associative with nonzero identity and $char(R)$ is the characteristic of $R$. A ring $R$ is called a *right duo* ring, if every right ideal of $R$ is an ideal. The notion of reversible ring was introduced by Cohn in [3]. He called a ring $R$ *reversible*, if $ab = 0$ implies $ba = 0$, for all $a, b \in R$. Kim and Lee in [6], continued the study of reversible rings. They showed that polynomial rings over reversible rings need not be reversible and sequentially argue about the reversibility of some kinds of polynomial rings. Gutan and Kisielewicz in [5] characterized reversible group ring $K[G]$ of torsion group $G$ over field $K$.

In this paper, we introduce the noation of right gr-ring as a generalization of reversible rings which has a close relationship with reversible, symmetric and right duo rings, where *symmetric ring* $R$ is a ring which for all $a, b, c \in R$, if $abc = 0$ then $bac = 0$. A ring $R$ is called a *right gr-ring*, if for every right ideal $I$ of $R$ and $a, b \in R$, $ab \in I$ implies that $ba \in I$. We will study conditions under which a group ring $R[G]$ of a group $G$ over a ring $R$ becomes a right gr-ring. We show that the group ring $K[Q_8]$ of a group of quaternions $Q_8$ over field $K$ is a right gr-ring if and only if $char(K) = 0$ and the equation $x^2 + y^2 + 1 = 0$ has no solution in $K$. Using the results, we can give an example of a right duo ring which is not a right gr-ring. Also, if $M$ is a maximal ideal of a commutative ring $R$ such that $\frac{R}{M}[Q_8]$ is a right gr-ring, then $char(R) = 0$ and for every prime number $p \in \mathbb{N}$, we have $p.1 \notin M$.

*Speaker

**1.1. Introduce the Noation of Right Gr-Ring.** In this section, we present the noation of right gr-ring and study some properties of it which we need in the main results.

DEFINITION 1.1. A ring $R$ is said to be a *right gr-ring*, if for every right ideal $I$ of $R$ and $a, b \in R$, $ab \in I$ implies that $ba \in I$.

It is obvious that every finite direct product of division rings are right gr-rings. Also $\mathbb{Z} \times D$, where $D$ is a division ring, is a right gr-ring. In the following Example, we give another example of right gr-ring.

EXAMPLE 1.2. Let $F$ be a field and $F(x)$ be the quotient field of the polynomial ring $F[x]$. Let $\varphi : F(x) \longrightarrow F(x^2)$ be a map satisfying

$$\varphi(\frac{f(x)}{g(x)}) = \frac{f(x^2)}{g(x^2)}.$$

We see at once that $\varphi$ is a ring homomorphism. Now, let

$$R = \left\{ \begin{pmatrix} a & 0 \\ b & \varphi(a) \end{pmatrix} \; ; \; a, b \in F(x) \right\}.$$

It is easy to check that $R$ is a subring of $M_2(F(x))$. If

$$H = \begin{pmatrix} 0 & 0 \\ F(x) & 0 \end{pmatrix},$$

it is easily seen that $H$ is the unique nonzero proper right ideal of $R$ and $R$ is a right gr-ring.

Before stating the next proposition, let us first recall that a ring $R$ is called a *right duo* ring, if every right ideal of $R$ is an ideal.

PROPOSITION 1.3. *Let $R$ be a right gr-ring. Then $R$ is a right duo ring.*

In the next section, we will give an example which shows that in general every right duo ring is not a right gr-ring.

Recall that a ring $R$ is called a symmetric ring, if $abc = 0$ implies that $bac = 0$, for all $a, b, c \in R$.

PROPOSITION 1.4. *Let $R$ be a right gr-ring. Then $R$ is a symmetric ring.*

DEFINITION 1.5. Let $R$ be a ring. If $R$ is a right (left) injective $R$-module, then $R$ is said to be a *right (left) self injective* ring.

THEOREM 1.6. *For a left self injective ring $R$, the following conditions are equivalent:*

1) *$R$ is a right gr-ring.*
2) *$R$ is a symmetric ring.*

## 2. Main Results

In this section, we study the group ring $R[G]$ of a group $G$ over a ring $R$ which is a right gr-ring.

DEFINITION 2.1. A non abelian group $G$ is called a *Hamiltonian group*, if every subgroup of $G$ is a normal subgroup of $G$.

Recall that a torsion group is a group in which each element has finite order. It is well known that if $G$ is a torsion group and $R[G]$ is a reversible group ring, then $G$ is an abelian or is a Hamiltonian group, see [2]. The following Proposition gives this result for the group ring $R[G]$ of a torsion group $G$ over a ring $R$ which is a right gr-ring.

PROPOSITION 2.2. *Let $R$ be a ring and $G$ a group. If the group ring $R[G]$ is a right gr-ring, then the following statements hold:*

1) *$R$ is a right gr-ring.*
2) *If $G$ is a torsion group, then $G$ is an abelian or a Hamiltonian group.*

THEOREM 2.3. *Let $R$ be a ring and the group ring $R[Q_8]$ be a right gr-ring. Then $char(R) = 0$.*

PROOF. Let $char(R) = n \neq 0$. This gives $\mathbb{Z}_n[Q_8] \subseteq R[Q_8]$. Since $R[Q_8]$ is a right gr-ring, $R[Q_8]$ is a reversible ring. Thus $\mathbb{Z}_n[Q_8]$ is also a reversible ring. From this, we have $n = 2$, by [8, Theorem 2.5]. On the other hand, [5, Corollary 4.3] shows that $\mathbb{Z}_2[Q_8]$ is not a symmetric ring. Hence $R[Q_8]$ is not also a symmetric ring and so is not a right gr-ring, by Proposition 1.4, which contradicts the assumption. □

For the general case, the converse of Theorem 2.3 is false. For example $char(\mathbb{Z}) = 0$ but the group ring $\mathbb{Z}[Q_8]$ is not a right gr-ring, because [1, Example 1.2] shows that it is not a right duo ring.

COROLLARY 2.4. *For every natural number $n \neq 1$, the group ring $\mathbb{Z}_n[Q_8]$ is not a right gr-ring.*

REMARK 2.5. Marks showed that $\mathbb{Z}_2[Q_8]$ is a right duo ring, see [9, Example 7]. Thus $\mathbb{Z}_2[Q_8]$ is a right duo ring, but not a right gr-ring, by Corollary 2.4.

COROLLARY 2.6. *Let $R$ be a ring and $G$ a nonabelian torsion group. If the group ring $R[G]$ is a right gr-ring, then $char(R) = 0$.*

PROOF. Proposition 2.2 implies that $G$ is a Hamiltonian group. So $G \cong Q_8 \times A \times B$, where $A$ is an abelian group of exponent 2 and $B$ is an abelian group all of whose elements are of odd order. Since $R[G] \cong (R[Q_8])[A \times B]$ and $R[G]$ is a right gr-ring, $R[Q_8]$ is also a right gr-ring, by Proposition 2.2. From this we have $char(R) = 0$, by Theorem 2.3. □

COROLLARY 2.7. *Let $G$ be a nonabelian finite group and $K$ a field. Then the following statements are equivalent:*

1) *The group ring $K[G]$ is a right gr-ring.*
2) *The group ring $K[G]$ is a finite direct product of division rings.*

THEOREM 2.8. *If $K$is a field, then the following sets are equivalent:*

1) *The group ring $K[Q_8]$ over field $K$ is a right gr-ring.*
2) *$char(K) = 0$ and the equation $x^2 + y^2 + 1 = 0$ has no solution in $K$.*

PROOF. 1 $\Rightarrow$ 2. If the group ring $K[Q_8]$ is a right gr-ring, then $K[Q_8]$ is a reversible ring and Theorem 2.3 implies $char(K) = 0$. From this the equation $x^2 + y^2 + 1 = 0$ has no solution in $K$, by [1, Theorem 2.1].

$2 \Rightarrow 1$. Since $char(K) = 0$ and $x^2+y^2+1 = 0$ has no solution in $K$, $K[Q_8]$ is a reversible right duo ring, by [1, Theorem 2.1]. Furthermore, [5, Corollary 3.3] tells us the group ring $K[Q_8]$ is a symmetric ring. Therefore $K[Q_8]$ is a right duo symmetric ring. On the other hand, $K[Q_8]$ is a semisimple ring, by [7, Theorem 6.1], which implies that $K[Q_8]$ is a left self injective ring, by [4, Exercise 4H]. From these we conclude $K[Q_8]$ is a right gr-ring, by Theorem 1.6. □

Remark 2.9. Theorem 2.8 shows that $\mathbb{R}[Q_8]$ and $\mathbb{Q}[Q_8]$ are right gr-rings but $\mathbb{C}[Q_8]$ is not a right gr-ring.

Corollary 2.10. *Let $K$ be a field of zero characteristic. Then the following statements are equivalent:*

1) *$K[Q_8]$ is a right gr-ring.*
2) *$K[Q_8]$ is a reversible ring.*
3) *The equation $1 + x^2 + y^2 = 0$ has no solutions in $K$.*
4) *$K[Q_8]$ is a finite direct product of division rings.*

Corollary 2.11. *Let $R$ be a commutative ring and $M$ a maximal ideal of $R$. If $\frac{R}{M}[Q_8]$ is a right gr-ring, then*

1) *$char(\frac{R}{M}) = 0$ and therefore $char(R) = 0$.*
2) *For every prime number $p \in \mathbb{N}$, we have $p.1 \notin M$.*

## References


1. H. E. Bell and Y. Li, *Duo group rings*, J. Pure Appl. Algebra **209** (2007) 833–838.
2. H. E. Bell and Y. Li, *Reversible and Duo Group Rings*, In: D. Van Huynh, S. R. López-Permouth (eds) Advances in Ring Theory, Trends in Mathematics, Birkhäuser Basel, 2010. DOI:10.1007/978-3-0346-0286-0_3
3. P. M. Cohn, *Reversible rings*, Bull. London Math. Soc. **31** (1999) 641–648.
4. K. R. Goodearl and R .B. Warfield, Jr., *An Introduction to Noncommutative Noetherian Rings*, London Math. Soc. Stud. Text Ser. Vol. 16, Cambridge Univ. Press, Cambridge, 1989.
5. M. Gutan and A. Kisielewicz, *Reversible group rings*, J. Algebra **279** (2004) 280–291.
6. N. K. Kim and Y. Lee, *Extensions of reversible rings*, J. Pure Appl. Algebra **185** (2003) 207–223.
7. Y. Lam, *A First Course in Noncommutative Ring*, Graduate Texts in Mathematics, Vol. 131, Springer-Verlag, New York, 1991.
8. Y. Li and M. M. Parmenter, *Reversible group rings over commutative rings,* Comm. Algebra **35** (2007) 4096–4104.
9. G. Marks, *Reversible and symmetric rings*, J. Pure Appl. Algebra **174** (2002) 311–318.


E-mail: masoudiar@gmail.com
E-mail: jahanian@kashanu.ac.ir

# On the Generalized Telephone Numbers of Some Groups of Nilpotency Class 2

Elahe Mehraban*
Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran
and Mansour Hashemi
Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran

Abstract. In this paper, we study the generalized telephone numbers modulo $m$ and define the generalized telephone numbers on a finite group. Also, by considering some special groups of nilpotency class 2, we examine the lengths of the period of the generalized telephone numbers.
**Keywords:** Period, The generalized telephone numbers.
**AMS Mathematical Subject Classification [2010]:**
20F05,11B39, 20D60.

## 1. Introduction

Definition 1.1. The classical telephone numbers are given by the following recurrence relation

$$T(n) = T(n-1) + (n-1)T(n-2),$$

for $n \geqslant 2$, and with initial conditions $T(0) = T(1) = 1$ (See [1, 3]).

A sequence of elements is periodic, if after a certain point, it consists only of repetitions of a fixed subsequence. For example, the sequence $1, 0, 2, 3, 5, 7, 3, 5, 7, \ldots$ is periodic and has the period 3. A sequence of elements is simply periodic with period $l$ if the first $l$ elements in the sequence form a repeating subsequence. For example, the sequence $1, 2, 3, 8, 1, 2, 3, 8, \ldots$ is simply periodic with the period 4. First, we state a lemma without proof that establishes some properties of groups of nilpotency class 2.

Lemma 1.2. *If $G$ is a group and $G' \subseteq Z(G)$, then the following propositions hold for every integer $k$ and $u, v, w \in G$:*

i) $[uv, w] = [u, w]\,[v, w]$ *and* $[u, vw] = [u, v]\,[u, w]$.
ii) $\left[u^k, v\right] = \left[u, v^k\right] = [u, v]^k$.
iii) $(uv)^k = u^k v^k\,[v, u]^{\frac{k(k-1)}{2}}$.
iv) *If* $G = \langle a, b\rangle$ *then* $G' = \langle [a, b]\rangle$.

For integer $m$, we consider the finitely presented groups $G_m$:

$$G_m = \langle a, b \mid a^m = b^m = 1,\ [a, b]^a = [a, b],\ [a, b]^b = [a, b]\rangle, \quad m \geqslant 2.$$

*Speaker

LEMMA 1.3. [2] *Every element of $G_m$ may be uniquely presented by $a^r b^s [a, b]^t$, where $0 \leqslant r, s, t \leqslant m-1$. Also $|G_m| = m^3$.*

## 2. Main Results

In this section, first by using the definition of the generalized telephone numbers, we give some results that will be used later. Then, we introduce the generalized telephone numbers in a finite group. Lastly, we study the generalized telephone numbers of $G_m$ with respect to $X = \{a, b\}$.

DEFINITION 2.1. The generalized telephone numbers $T_n^k$ defined for integers $n \geqslant 1$ and $k \geqslant 1$ by the following formula

$$T_n^k = kT_{n-1}^k + (n-1)T_{n-2}^k, \quad n \geqslant 4,$$

with initial conditions $T_1^k = 0, T_2^k = 1$, and $T_3^K = k$.

THEOREM 2.2. *For $k = 2^\alpha, \alpha \in \mathbb{N}$, $\{T_n^k\}_{n=1}^{\infty}$ is a periodic sequence.*

PROOF. Suppose $W = \{(x_1, x_2) \mid 0 \leqslant x_1, x_2 \leqslant m-1\}$. Then $|W| = m^2$ is finite. For $i \geqslant 1$, $a \geqslant 0$ and $b \geqslant a$, we have

$$T_{a+i}^k \equiv T_{b+i}^k \pmod m,$$
$$T_{a+i+1}^k \equiv T_{b+i+1}^k \pmod m.$$

By using Definition 2.1 (definition of the generalized telephone numbers), we have

$$T_i^k \equiv T_{b-a+i}^k \pmod m,$$
$$T_{i+1}^k \equiv T_{b-a+i+1}^k \pmod m.$$

It results that $\{T_n^k\}_{n=1}^{\infty}$ is a periodic sequence. □

The smallest period of $T_m^k$, denoted by $hT_m^k$, is called the period of the generalized telephone numbers modulo $m$.

EXAMPLE 2.3. By Definition 2.1, we have $\{T_3^2\} = \{0, 1, 2, 1, 1, 1, 2, 2, 2, 1, 1, \ldots\}$. Therefore, $hT_3^2 = 6$.

THEOREM 2.4. *If $m = \prod_{i=1}^{t} p_i^{e_i}, t \geqslant 1$, where $p_i, 1 \leqslant i \leqslant t$, are distinct prime, then*

$$hT^k_{\prod_{i=1}^{t} p_i^{e_i}} = l.c.m[hT^k_{p_1^{e_1}}, hT^k_{p_2^{e_2}}, \ldots, hT^k_{p_t^{e_t}}].$$

PROOF. By using elementary number theory, we can get easy the proof. □

By using the period of the generalized telephone numbers, we have the following lemma.

LEMMA 2.5. *For integers $k = 2^\alpha, n \geqslant 2, t \geqslant 1$, and $i \geqslant 3$, we have*

i) $T^k_{hT_m^k + i} \equiv T_i^k \pmod m$,
ii) $T^k_{t \times (hT_m^k) + i} \equiv T_i^k \pmod m$.

Definition 2.6. For $k \geqslant 1$, a generalized telephone numbers in a finite group is a sequence of group elements $x_1, x_2, \ldots, x_n, \ldots$, for which, given an intial (seed) set in $X = \{a_1, \ldots, a_j\}$, each element is definted by:

$$x_n = \begin{cases} a_n, & \text{for } n \leqslant j, \\ x_{n-2}^{n-1}\, x_{n-1}^k, & \text{for } n > j. \end{cases}$$

We denote the generalized telephone numbers of the group $G = \langle X \rangle$ by $Q_T^k(G; X)$ and the period of the sequence $Q_T^k(G; X)$ by $LQ_T^k(G; X)$.

Here, we consider $G_m = \langle a, b | a^m = b^m = 1, [a,b]^a = [a,b], [a,b]^b = [a,b] \rangle$, $m \geqslant 2$. In this section, we study the generalized telephone numbers of $G_m$ with respect to $X = \{a, b\}$ and find the period of $Q_T^k(G_m; X)$ for $k = 2^\alpha, \alpha \in N$. For this we define the sequence $\{h_n\}_1^\infty$ and $\{g_n\}_1^\infty$ of numbers as follows:

$$\begin{aligned} &h_1 = 1, h_2 = 0, \\ &h_n = (n-1)h_{n-2} + kh_{n-1}, \quad n \geqslant 3, \\ &g_1 = g_2 = g_3 = 0, \\ &g_n = kg_{n-1} + (n-1)g_{n-2} + \frac{(n-1)(n-2)}{2} T_{n-2}^k h_{n-2} + k(n-1)T_{n-2}^k \\ &\quad + \frac{k(k-1)}{2} T_{n-1}^k h_{n-1}, \quad n \geqslant 4. \end{aligned}$$

Now, we find a standard form of the generalized telephone numbers $x_4, x_5, \ldots$, of $G_m, n \geqslant 4$.

Lemma 2.7. *For $k = 2^\alpha, \alpha \in \mathbb{N}$, every element of $Q_T^k(G_m; X)$ may be presented by*

$$x_n = a^{h_n} b^{T_n^k} [a,b]^{g_n}, \quad n \geqslant 4.$$

Proof. Let $k = 2$. For $n = 4$, we have $x_4 = a^4 b^7 [a,b]^{22} = a^{h_4} b^{T_4^2} [a,b]^{g_4}$. Then, by induction method on $n$, we get

$$\begin{aligned} x_n &= x_{n-2}^{n-1} x_{n-1}^2 = (a^{h_{n-2}} b^{T_{n-2}} [a,b]^{g_{n-2}})^{n-1} (a^{h_{n-1}} b^{T_{n-1}^2} [a,b]^{g_{n-1}})^2 \\ &= a^{h_{n-2}} b^{T_{n-2}^2} [a,b]^{g_{n-2}} \ldots\ a^{h_{n-2}} b^{T_{n-2}^2} [a,b]^{g_{n-2}} (a^{h_{n-1}} b^{T_{n-1}^2} [a,b]^{g_{n-1}})^2 \\ &= a^{(n-1)h_{n-2}} b^{(n-1)T_{n-2}^2} [a,b]^{(n-1)g_{n-2} + \frac{(n-1)(n-2)}{2} T_{n-2}^2 h_{n-2}} (a^{h_{n-1}} b^{T_{n-1}^2} [a,b]^{g_{n-1}})^2 \\ &= a^{h_n} b^{T_n^k} [a,b]^{kg_{n-1} + (n-1)g_{n-2} + (\frac{(n-1)(n-2)}{2}) T_{n-2}^2 h_{n-2} + (2(n-1)T_{n-2}^k + \frac{2(2-1)}{2} T_{n-1}^2 h_{n-1}} \\ &= a^{h_n} b^{T_n^2} [a,b]^{g_n}. \end{aligned}$$

Other cases are similar to the proof for $k = 2$, thus they are omitted. □

Lemma 2.8. *For $t \in \mathbb{Z}$, we have*

i) *The elements $hT_{m+1}^k$−th and $hT_{m+2}^k$−th of the generalized telephone numbers $Q_T^k(G_m; X)$ are as*

$$x_{hT_{m+1}^k} \equiv a^{i_1} b^{j_1} [a,b]^{q_1}, (\text{mod } m), \quad x_{hT_{m+2}^k} \equiv a^{i_2} b^{j_2} [a,b]^{q_2}, (\text{mod } m).$$

ii) *The elements* $t \times hT^k_{m+1}-th$ *and* $t \times hT^k_{m+2}-th$ *of the generalization telephone numbers* $Q^k_T(G_m;X)$ *are as*

$$x_{t\times hT^k_{m+1}} \equiv a^{i_1}b^{j_1}[a,b]^{q_1}, (\text{mod } m), \quad x_{t\times hT^k_{m+2}} \equiv a^{i_2}b^{j_2}[a,b]^{q_2}, (\text{mod } m).$$

PROOF. (i) For $k = 2^\alpha, \alpha \in \mathbb{N}$, by using Lemma 2.7, we have
$x_1 = a, x_2 = b, x_3 = ab, \ldots, x_{hT^k_m} = a^{h_{hT^k_m}} b^{T^k_{hT^k_m}} [a,b]^{g_{hT^k_m}}$,
$x_{hT^k_{m+1}} = a^{h_{hT^k_{m+1}}} b^{T^k_{hT^k_{m+1}}} [a,b]^{g_{hT^k_{m+1}}}$,
$x_{hT^k_{m+2}} = a^{h_{hT^k_{m+2}}} b^{T^k_{hT^k_{m+2}}} [a,b]^{g_{hT^k_{m+2}}}, \ldots,$
$x_{t.hT^k_{m+1}} = a^{h_{t.hT^k_{m+1}}} b^{T^k_{t.hT^k_{m+1}}} [a,b]^{g_{t.hT^k_{m+1}}}$,
$x_{t.hT^k_{m+2}} = a^{h_{t.hT^k_{m+2}}} b^{T^k_{t.hT^k_{m+2}}} [a,b]^{g_{t.hT^k_{m+2}}}, \ldots.$
So, we get the elements $hT^k_{m+1}-$th and $hT^k_{m+2}-$th of the generalization telephone numbers $Q^k_T(G_m;X)$ are

$$x_{hT^k_{m+1}} \equiv a^{i_1}b^{j_1}[a,b]^{q_1}, (\text{mod } m), \quad x_{hT^k_{m+2}} \equiv a^{i_2}b^{j_2}[a,b]^{q_2}, (\text{mod } m).$$

The proof (ii) is similar to (i), so it's omitted. □

By Lemma 2.8, we can obtain the following corollary.

COROLLARY 2.9. *For* $k = 2^\alpha, \alpha \in \mathbb{N}$, *we have*

$$hT^k_m \Big| LQ^k_T(G_m;X).$$

EXAMPLE 2.10. For $m = 5$ and $k = 2$, we have
$x_1 = a, x_2 = b, x_3 = ab, x_4 = a^4b^2[a,b]^1, x_5 = a^1b^2[a,b]^3, x_6 = a^2b^4[a,b]^3$,
$x_7 = a^0b^0[a,b]^0 = e, \ldots, x_{24} = a^4b^3[a,b]^4, x_{25} = a^1b^2[a,b]^3, x_{26} = a^2b^4[a,b]^3, \ldots.$
We have $x_5 = x_{25}$ and $x_6 = x_{26}$. Therefore, $LQ^2_T(G_5;X) = 20$ and $hT^2_5 | LQ^2_T(G_5;X)$.

In Table 1, by using the software Maple 18, we calculate some the period of generalization telephone numbers $Q^k_T(G_m;X)$.

TABLE 1. The period of generalization telephone numbers $Q^k_T(G_m;X)$.

| $m$ | $LQ^2_T(G_m;X)$ | $hT^2_m$ | $LQ^4_T(G_m;X)$ | $hT^4_m$ | $LQ^8_T(G_m;X)$ | $hT^8_m$ | $LQ^16_T(G_m;X)$ | $hT^16_m$ |
|---|---|---|---|---|---|---|---|---|
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| 4 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| 5 | 20 | 20 | 10 | 10 | 20 | 20 | 5 | 5 |
| 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| 7 | 21 | 21 | 21 | 21 | 7 | 7 | 21 | 21 |
| 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| 9 | 18 | 18 | 9 | 9 | 18 | 18 | 9 | 9 |
| 10 | 20 | 20 | 10 | 10 | 20 | 20 | 20 | 20 |

We finish this section with an open question as follows:
Prove or disprove, for every $k = 2^\alpha, \alpha \in \mathbb{N}$,

$$LQ^k_T(G_m;X) = hT^k_m.$$

### References


1. U. Bednarz and M. Wolowiec-Musial, *On a new generalization of telephone numbers*, Turk. J. Math. **43** (2019) 1595–1603.
2. M. Hashemi and E. Mehraban, *On the generalized order* 2*-Pell sequence of some classes of groups*, Commun. Algebra **46** (9) (2018) 4104–4119.
3. A. Wloch and M. Wolowiec-Musial, *On generalized telephone number, their interpretations and matrix generators*, Utilitas Math. **105** (2017) 31–39.


E-mail: e.mehraban.math@gmail.com
E-mail: m_hashemi@guilan.ac.ir

# Power Graphs Based on the Order of Their Groups

Mahsa Mirzargar*
Faculty of Science, Mahallat Institute of Higher Education, Mahallat, I. R. Iran

Abstract. The power graph $P(G)$ of a group $G$ is a graph with vertex set $G$, where two vertices $u$ and $v$ are adjacent if and only if $u \neq v$ and $u^m = v$ or $v^m = u$ for some positive integer $m$. The present paper aims to classify power graphs based on group orders, which can be a new look at the power graphs classification. We raise and study the following question: For which natural numbers $n$ every two groups of order $n$ with isomorphic power graphs are isomorphic? We denote the set of all such numbers by $\bar{S}$ and consider the elements of $\bar{S}$. Moreover, we show that if two finite groups have isomorphic power graphs and one of them is nilpotent or has a normal Hall subgroup, the same is true with the other one.
**Keywords:** Power graph, Conformal groups, Nilpotent group.
**AMS Mathematical Subject Classification [2010]:** 05C12, 91A43, 05C69.

## 1. Introduction

There are many different ways to associate a graph to the given group, including the commuting graphs, prime graphs, and of course Cayley graphs, which have a long history and applications. Graphs associated with groups and other algebraic structures have been actively investigated since they have valuable applications and specially are related to automata theory [6, 7]. The rigorous development of the mathematical theory of complexity via algebraic automata theory reveals deep and unexpected connections between algebra (semigroups) and areas of science and engineering.

Let $G$ be a finite group. The undirected power graph $P(G)$ is the undirected graph with vertex set $G$, where two vertices $a, b \in G$ are adjacent if and only if $a \neq b$ and $a^m = b$ or $b^m = a$ for some positive integer $m$. Likewise, the directed power graph $\overrightarrow{P}(G)$ is the directed graph with vertex set $G$, where for two vertices $u, v \in G$ there is an arc from $a$ to $b$ if and only if $a \neq b$ and $b = a^m$ for some positive integer $m$. In [1] you can see a survey of results and open questions on power graphs, also it is explained that the definition given in [5] covers all undirected graphs as well. This means that the undirected power graphs were also defined in [5] for the first time and used only the brief term power graph, even though they covered both directed and undirected power graphs. Cameron proved in [3], if $G_1$ and $G_2$ are finite groups whose undirected power graphs are isomorphic, then their directed power graphs are also isomorphic. Clearly, the converse is also true. Clearly $G \cong H$ implies $P(G) \cong P(H)$. The converse is false for finite groups in general. For example, if $p$ is an odd prime and $m > 2$, besides the elementary abelian group $H$ of order $p^m$, there are non-abelian groups $G$ of

*Speaker

order $p^m$ and exponent $p$, so $H$ and $G$ are non-isomorphic but have isomorphic power graphs. On the other hand, it is shown in [2, 9] that if both $G$ and $H$ are abelian then $P(G) \cong P(H)$ implies $G \cong H$. Also in [9], it is proved that if $G$ is one of the following finite groups:

1) a simple group,
2) a cyclic group,
3) a symmetric group,
4) a dihedral group,
5) a generalized quaternion group,

and $H$ is a finite group such that $P(G) \cong P(H)$ then $G \cong H$.

Following [8, 10], two finite groups $G$ and $H$ are said to be conformal if and only if they have the same number of elements of each order. Such groups need not be isomorphic (See the above example of groups of exponent $p$). The relevance of this concept to power graphs is due to the fact that, as proved by Cameron [3], two finite groups with isomorphic undirected power graphs are conformal. Note that the converse is not true. For example, two groups of order 16 with the same numbers of elements of each order, e.g. $C_4 \times C_4$ and $C_2 \times Q_8$ are SmallGroup$(16, 2)$ and SmallGroup$(16, 4)$ in GAP respectively [4]. Their power graphs are not isomorphic. In fact, in the group $C_4 \times C_4$, each element of order 2 has four square roots, but in $C_2 \times Q_8$, the involution in $Q_8$ has twelve square roots and the other two have none. In [8], an algorithm is described to find the number of elements of a given order in abelian groups, so if $G$ and $H$ are finite conformal abelian groups, then $G \cong H$.

In [10], the following question was investigated:

**Question:** For which natural numbers $n$ every two conformal groups of order $n$ are isomorphic?

In [10], the set of all such numbers was denoted by $S$ and odd and square-free elements of $S$ were characterized.

In this paper we raise another question along the same lines:

**Question:** For which natural numbers $n$, every two groups of order $n$ with isomorphic power graphs are isomorphic?

Let us denote the set of all such numbers by $\bar{S}$. Since two finite groups with isomorphic power graphs are conformal, it is easy to see that $S \subseteq \bar{S}$.

There is not a one to one function between groups and power graphs. Therefore, the power graphs do not always determine the groups. An interesting study would be to find out for which groups $G$ and $H$, $P(G) \cong P(H)$ implies $G \cong H$. The present paper aims to classify power graphs based on group orders, which can be a new look at the power graphs classification. Moreover, the concept of conformal groups and the order of the elements of a group play an important role in the results of this paper and guide us to classify power graphs of nilpotent groups and groups which have a normal Hall subgroup. The authors believe that it is possible to classify power graphs based on the order of their groups. This topic can continue and leads many open questions motivated by classification problems for future work.

## 2. Main Results

In this section, we study the set $\bar{S}$, often exploiting methods and results already used for $S$.

In [10], Lemma 1, it is proved that if $p$ and $q$ are prime and $q|(p-1)$, then $p^2q \in S$ if and only if $q = 2$. Since $S \subseteq \bar{S}$, the following result is straightforward.

PROPOSITION 2.1. *If $p$ is an odd prime number, then $2p^2 \in \bar{S}$.*

Note that $8 \in S$, because the two non-abelian groups of order 8 are either the dihedral group $D_8$ or the quaternion group $Q_8$, and the number of elements of order 4 in these groups is 2 and 6, respectively. There are three abelian groups of order 8, which are pair-wise non-conformal and non-conformal to $D_8$ or $Q_8$. Therefore $8 \in S$ and $8 \in \bar{S}$.

The following result shows that $\bar{S}$ contains natural numbers with an arbitrary number of prime factors.

THEOREM 2.2. *If $n \notin \bar{S}$ and $(n, k) = 1$, then $nk \notin \bar{S}$.*

LEMMA 2.3. *Let $G$ be a 2-group and $A$ be an elementary abelian 2-group. Two vertices $(a, x)$, $(b, y)$ of the graph $P(G \times A)$ are adjacent if and only if one of the following holds:*

1) *$x = y = 1$ and $b$ is a power of $a$,*
2) *$x = y \neq 1$ and $b$ is an odd power of $a$,*
3) *$x \neq 1$, $y = 1$ and $b$ is an even power of $a$,*
4) *$x = 1$, $y \neq 1$ and $a$ is an even power of $b$.*

THEOREM 2.4. *Let $n = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ $(r \geq 0)$. If $\alpha_0 \geq 4$ or there exists $i \neq 0$ such that $\alpha_i \geq 3$, then $n \notin \bar{S}$.*

COROLLARY 2.5. *Every odd element of $\bar{S}$ is cube-free.*

As mentioned above, we have $S \subseteq \bar{S}$. On the other hand, when we look closely at computer programming, we notice that many small numbers belong to both $S$ and $\bar{S}$ or to neither. It is then natural to ask whether this inclusion is indeed strict.

THEOREM 2.6. *The set $\bar{S} \setminus S$ is non-empty. Its smallest element is 72.*

Again exploiting the necessary condition of conformality, we are going to show here some situations where a property of a group $G$ is inherited by all groups with the same power graph.

THEOREM 2.7. *If $G$ and $H$ are conformal and $H$ is nilpotent, then also $G$ is nilpotent.*

COROLLARY 2.8. *If $P(G) \cong P(H)$ and $H$ is nilpotent, then also $G$ is nilpotent.*

A subgroup of a finite group is said to be a Hall subgroup if its order and index are relatively prime.

THEOREM 2.9. *Let $G$ and $H$ be conformal groups. If $H$ has a normal Hall subgroup of order $m$ and $G$ is solvable, then also $G$ has a normal Hall subgroup of order $m$.*

COROLLARY 2.10. *If $P(G) \cong P(H)$, $H$ has a normal Hall subgroup of order $m$, and $G$ is solvable, then also $G$ has a normal Hall subgroup of order $m$.*

## References


1. J. Abawajy, A. Kelarev and M. Chowdhury, *Power graphs: A survey*, Electron. J. Graph Theory Appl. **1** (2) (2013) 125–147.
2. P .J. Cameron and S. Ghosh, *The power graph of a finite group*, Discrete Math. **311** (13) (2011) 1220–1222.
3. P. J. Cameron, *The power graph of a finite group II*, J. Group Theory **13** (6) (2010) 779–783.
4. The GAP group, GAP-Groups, Algorithms, and Programming, Version 4.11.0, 2020. http://www.gap-system.org
5. A. V. Kelarev and S. J. Quinn, *A combinatorial property and power graphs of groups*, The Vienna Conference, Contrib. General Algebra **12** (58) (2000) 229–235.
6. A. V. Kelarev, *Graph Algebras and Automata*, Monographs and Textbooks in Pure and Applied Mathematics, 257. Marcel Dekker, Inc., New York, 2003.
7. A. V. Kelarev, *Labelled Cayley graphs and minimal automata*, Australas. J. Combin. **30** (2004) 95–101.
8. G. A. Miller, H. F. Blichfeldt and L. E. Dickson, *Theory and applications of finite group*, Dover Publications, Inc., New York, 1961.
9. M. Mirzargar, A. R. Ashrafi and M. J. Nadjafi-Arani, *On the power graph of a finite group*, Filomat **26** (6) (2012) 1201–1208.
10. R. Scapellato, *Finite groups with the same number of elements of each order*, Rend. Mat. Appl. VII, **8** (3) (1988) 339–344.


E-mail: M. Mirzargar@gmail.com

# Hyperring-Based Graph

Mohammad Hamidi
Department of Mathematics, Payame Noor University, Tehran, Iran
and Hoda Mohammadi*
Department of Mathematics, Payame Noor University, Tehran, Iran

Abstract. In this paper, we study a concept of graph based on hyperideals of a hyperring and investigate some graph property such connectedness, completeness and etc. In particular, we obtain some necessary and sufficient conditions such that mentioned graph is complete.
**Keywords:** Hyperring, Hyperideals, Intersection graph.
**AMS Mathematical Subject Classification [2010]:** 20N20.

## 1. Introduction

For the last few decades several mathematicians studied such graphs on various algebraic structures. The first step in this direction was taken by Bosak in 1964 [3]. Then Csakany and Pollak studied the graphs of subgroups of a finite group [4]. Zelinka continued the work on intersection graphs of nontrivial subgroups of finite abelian groups [10]. Various constructions of graphs related to the ring structure are found in [1, 2, 5]. The theory of hyperstructures has been introduced by Marty in 1934 during the $8^{th}$ Congress of the Scandinavian Mathematicians [9]. Marty introduced hypergroups as a generalization of groups and hyperring is structure generalizing that of a ring, but where the addition is a composition, but a hypercomposition, i.e, the sum and the product of two elements is not an element but a subset. The notation of hyperring was introduced by Krasner [8], who used it as a technical tool in a study of his on the approximation of valued fields. Further materials regarding intersection graphs, ring and multirings are available in the literature too [5, 6, 7].

The purpose of this paper is the study of intersection graphs of hyperideals of hyperrings, as a generalization of intersection graphs of classical rings. In this regards, the notation of absorbing elements with respect are introduced and the intersection graphs of hyperideals of hyperrings and investigates their properties.

## 2. Preliminaries

A map $\varrho : G^n \to P^*(G)$ is an $n$-ary hyperoperation with *arity* $n$, where for $n = 0$ (*nullary hyperoperation*) is an element of $P^*(G)$ and $(G, \{\varrho_i\}_{i\in \mathbf{I}})$ is a *hyperalgebra* (for $|\mathbf{I}| = 1$ is called *hypergroupoid*) of type $\varphi : \mathbf{I} \to \mathbb{N}^*$, where two hyperalgebras of the same type are called *similar* hyperalgebras. A $\emptyset \neq W \subseteq G$ is said to be a *subhyperalgebra* of $G$ if $\forall\ (b_1, \ldots, b_{n_i}) \in W^{n_i}$, $\varrho_i(b_1, \ldots, b_{n_i}) \subseteq W$. For similar hyperalgebras $(G, \{\varrho_i\}_{i\in\mathbf{I}}), (G', \{\varrho'_i\}_{i\in\mathbf{I}})$, a map $g : G \to G'$ is called a *homomorphism* if $\forall\ i \in \mathbf{I}, \forall\ (b_1, \ldots, b_{n_i}) \in G^{n_i}$ we have $g(\varrho_i((b_1, \ldots, b_{n_i})) \subseteq$

*Speaker

${\varrho'}_i(g(b_1),\ldots,g(b_{n_i}))$ and a *good homomorphism* if $\forall\ i \in \mathbf{I}, \forall\ (b_1,\ldots,b_{n_i}) \in G^{n_i}$, $g(\varrho_i((b_1,\ldots,b_{n_i})) = {\varrho'}_i(g(b_1),\ldots,g(b_{n_i}))$. A *hypergroupoid* $(G,\varrho)$ together with an associative binary hyperoperation is said a *semihypergroup* and a semihypergroup $(G,\varrho)$ is called a *hypergroup* if $\forall\ y \in G, \varrho(y,G) = \varrho(G,y) = G$(*reproduction axiom*). A hypergroup $(G,\varrho)$ is said to be a *canonical* , if always $(i)$ $\varrho(x,y) = \varrho(y,x)$ $(ii)$ $\exists!\ e \in G, \forall\ x \in G,$ in a way $\varrho(e,x) = \varrho(x,e) = \{x\}$(*neutral element*), $(iii)$ $x \in \varrho(y,z)$ concludes that $y \in \varrho(x,\eta(z))$ and $z \in \varrho(\eta(y),x)$, where $\eta$ is an unitary operation on $G\big(\forall\ x \in G,\ \exists!\ \eta(x) \in G$ i.e $e \in (\varrho(x,\eta(x)) \cap (\varrho(\eta(x),x)), \eta(e) = e, \eta(\eta(x)) = x\big)$ and is denoted by $(G,\varrho,e,\eta)$ or $(G,+,0,-)$. A *Krasner hyperring* is a hyperstructure $(K,+,.)$, where $(i)$ $(K,+)$ is a canonical hypergroup, $(ii)$ $(K,.)$ is a semigroup, $(iii)$ $\forall k,s,t \in K : k(s+t) = ks+kt$ and $(s+t)k = sk+tk$, $(iv)$ $\forall k \in K : k.0 = 0.k = 0$, i.e. $\exists\ 0 \in K$ is an absorbing element.

## 3. Graphs Derived from Hyperrings

In this section, we introduce graph based on hyperideals and seek to some conditions on hyperideals in hyperring such that obtain especial graphs.

Definition 3.1. Let $(K,+,\cdot)$ be a hyperring. We say that

i) $0 \in K$ is a $(+)$-absorbing element of $K$, if for all $k \in K, k \in (0+k) \cap (k+0)$,
ii) $0 \in K$ is a $(\cdot)$-absorbing element of $K$, if for all $k \in K, 0 \in (k \cdot 0 \cap 0 \cdot k)$,
iii) $0 \in K$ is an absorbing element of $K$, if it is both $(+)$-absorbing element and $(\cdot)$-absorbing element of $K$.

From now on, we consider the set of all $(+)$-absorbing elements of $K$ by $\mathcal{O}_K^+$, all $(\cdot)$-absorbing elements of $K$ by $\mathcal{O}_K^{\cdot}$ and absorbing elements of hyperring $K$ by $\mathcal{O}_K$. It is clear that $\mathcal{O}_K = \mathcal{O}_K^+ \cap \mathcal{O}_K^{\cdot}$.

Definition 3.2. Let $(K,+,\cdot)$ be a hyperring and $\emptyset \neq \mathbf{I} \subseteq K$. Then $\mathbf{I}$ is a hyperideal of $K$ if and only if satisfies in the following conditions:

i) for all $y \in \mathbf{I}, y + \mathbf{I} = \mathbf{I} + y = \mathbf{I}$,
ii) for all $k \in K$ and $y \in \mathbf{I}$, we have $(k \cdot y) \cup (y \cdot k) \subseteq \mathbf{I}$.

Let $(K,+,\cdot)$ be a hyperring. Then we will denote the set of all hyperideals of $K$ by $\mathcal{I}(K)$. Clearly, $K \in \mathcal{I}(K) \neq \emptyset$ and will call $K$ as a non-proper hyperideal of any hyperring.

Definition 3.3. Let $K$ be a hyperring. The intersection graph of $\mathcal{I}(K)$ is the undirected simple graph (without loops and multiple edges) whose vertices are in a one-to-one correspondence with all nontrivial hyperideals of $K$ and two distinct vertices are joined by an edge if and only if the corresponding hyperideals of $K$ have intersection(if $\mathcal{O}_K \neq \emptyset$, then this intersection must be non-absorbing element). We will denote an intersection graph of $\mathcal{I}(K)$ by $\Gamma(K) = (\mathcal{I}(K), E)$.

In the following, we present an examples for clarifying the definition of intersection graph of hyperrings.

Example 3.4. Let $K = \{a_1,a_2,a_3,a_4\}$. Then $(K,+',\cdot')$ is a hyperring as follows:

| $+'$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|
| $a_1$ | $a_1$ | $\{a_1, a_2\}$ | $\{a_1, a_3\}$ | $\{a_1, a_4\}$ |
| $a_2$ | $\{a_1, a_2\}$ | $a_2$ | $\{a_3, a_2\}$ | $\{a_4, a_2\}$ |
| $a_3$ | $\{a_1, a_2, a_3\}$ | $\{a_2, a_3\}$ | $\{a_2, a_3\}$ | $\{a_2, a_3, a_4\}$ |
| $a_4$ | $\{a_1, a_2, a_4\}$ | $\{a_2, a_4\}$ | $\{a_2, a_3, a_4\}$ | $\{a_2, a_4\}$ |

*and*

| $\cdot'$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|
| $a_1$ | $a_2$ | $a_2$ | $a_2$ | $a_2$ |
| $a_2$ | $a_2$ | $a_2$ | $a_2$ | $a_2$ |
| $a_3$ | $a_2$ | $a_2$ | $a_2$ | $a_2$ |
| $a_4$ | $a_2$ | $a_2$ | $a_2$ | $a_2$ |

.

Clearly $\mathcal{O}_K^{+} = K, \mathcal{O}_K^{\cdot} = \{a_2\}$ and so $\mathcal{O}_K = \{a_2\}$. Also

$$\mathcal{I}(K) = \big\{\mathbf{I}_1 = \{a_2\}, \mathbf{I}_2 = \{a_1, a_2\}, \mathbf{I}_3 = \{a_2, a_3\}, \mathbf{I}_4 = \{a_4, a_2\},$$
$$\mathbf{I}_5 = \{a_1, a_2, a_3\}, \mathbf{I}_6 = \{a_1, a_2, a_4\}, \mathbf{I}_7 = \{a_2, a_3, a_4\}, \mathbf{I}_8 = K\big\},$$

where $\{a_2\}$ and $K$ are trivial hyperideals of $K$. So we obtain the intersection graph $\Gamma(K) = (\mathcal{I}(K), E)$ in Figure 1.

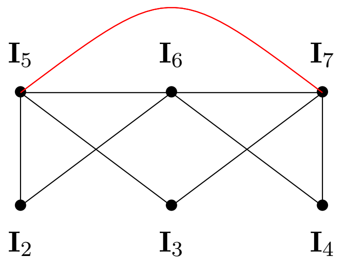



FIGURE 1. Intersection graph $\mathcal{I}(K)$.

THEOREM 3.5. *Let $q$ be an odd prime. Then $\mathcal{I}((\mathbb{Z}_q, +_q, \cdot_q) = \{\{\overline{0}\}, \mathbb{Z}_q\}$.*

THEOREM 3.6. *Assume $n \in \mathbb{N}$ is an even integer. Then there exist binary hyperoperations $\boxplus$ and $\boxtimes$", suchthat*

$$\overline{x} \boxplus \overline{y} = \overline{x} +_{\overline{b}} \overline{y} = \{\overline{x+y}, \overline{x+y+b}\}.$$

*and*

$$\overline{x} \boxtimes \overline{y} = \overline{x} \cdot_{\overline{b}} \overline{y} = \{\overline{xy}, \overline{xy+b}\}.$$

*Then $(\mathbb{Z}_n, \boxplus, \boxtimes)$ is a hyperring.*

Let $K = (\mathbb{Z}_n, \boxplus, \boxtimes)$ be the hyperring in Theorem 3.6 and $\overline{y} \in K$. Define $\langle \overline{y} \rangle = \bigcup_{r \in \mathbb{N}} r\overline{y}$. The next result immediately follows.

THEOREM 3.7. *Let $2 \leq n \in \mathbb{N}$ be even, $\overline{b} \in K$ and $\overline{y} \in K$. If $\overline{2b} = \overline{0}$, then*

i) *$\langle \overline{y} \rangle \in \mathcal{I}(\mathbb{Z}_n, \boxplus, \boxtimes)$,*
ii) *$\langle \overline{0} \rangle = \langle \overline{b} \rangle$,*
iii) *if $y \neq b$ and $\gcd(y, b) = d$, we have $\langle \overline{y} \rangle = \langle \overline{d} \rangle$,*
iv) *$\boldsymbol{I} \in \mathcal{I}(\mathbb{Z}_n, \boxplus, \boxtimes)$ if and only if there exists $\overline{y} \in K$, such that $\boldsymbol{I} = \langle \overline{y} \rangle$.*

THEOREM 3.8. *Let $2 \leq n \in \mathbb{N}$ and $\overline{b} \in K$. If $\overline{2b} = \overline{0}$, then*

i) *$|\mathcal{I}(\mathbb{Z}_n, \boxplus, \boxtimes)| = |Div(b)| + 1$,*
ii) *if for any $\overline{y} \in K, y \mid b$, then $\overline{b} \in \langle \overline{y} \rangle$.*

Corollary 3.9. *Let* $2 = q_1, q_2, \ldots, q_r$ *be primes,* $r, \beta_1, \beta_2, \ldots, \beta_r \in \mathbb{N}$ *and* $n = \prod_{i=1}^{r} q_i^{\beta_i}$. *Then*

$$\mathcal{I}(\mathbb{Z}_n, \boxplus, \boxtimes) = \{\overline{0}\} \cup \{\langle \overline{q_1^{s_1} q_2^{s_2} \ldots q_j^{s_j}} \rangle \mid 0 \leq s_1 \leq \beta_1 - 1, \text{ and for all } j \neq 1,\ 0 \leq s_j \leq \beta_i\}\rangle.$$

Theorem 3.10. *Let* $n \in \mathbb{N}$ *be an even. Then* $\Gamma(\mathbb{Z}_n, \boxplus, \boxtimes) = (\mathcal{I}(\mathbb{Z}_n), E)$ *is a disconnected graph if and only if for some distinct primes* $p, q$ *we have* $n = pq$.

Theorem 3.11. *Let* $n \in \mathbb{N}$ *be an even,* $I, J \in (\mathcal{I}(\mathbb{Z}_n, \boxplus, \boxtimes), E)$. *Then* $I \cap J = \langle \overline{lcm(d, d')} \rangle$, *where* $I = \langle \overline{d} \rangle$, $J = \langle \overline{d'} \rangle$ *and* $d, d' \in Div(n/2)$.

Theorem 3.12. *Let* $n = q^m$ *be an even, where* $q$ *is a prime. Then* $m \geqslant 3$ *if and only if* $\Gamma(\mathbb{Z}_n, \boxplus, \boxtimes) = (\mathcal{I}(\mathbb{Z}_n), E)$ *is a complete graph.*

## References


1. M. Afkhami and K. Khashyarmanesh, *The intersection graph of ideals of a lattice*, Note Mat. **34** (2) (2014) 135–143.
2. M. Berg, S. Kisfaludi-Baka and G. Woeginger, *The complexity of dominating set in geometric intersection graphs*, Theoret. Comput. Sci. **769** (2019) 18–31.
3. J. Bosák, *The graphs of semigroups*, Proc. Sympos. Smolenice, (1963) pp. 119–125.
4. B. Csakany and G. Pollak, *The graph of subgroups of a finite group*, Czechoslovak Math. J. **19** (2) (1969) 241–247.
5. A. Delfan, H. Rasouli and A. Tehranian, *Intersection graphs associated with semigroup acts*, Categ. Gen. Algebr. Struct. Appl. **11** (1) (2019) 131–148 (in Persian).
6. E. Galby, M. P. Mazzoleni and B. Ries, *Proper circular arc graphs as intersection graphs of paths on a grid*, Discrete Appl. Math. **262** (2019) 195–202.
7. M. Hamidi and A. Borumand Saeid, *Zero-divisor graphs of (non-distributive) multirings*, submitted.
8. M. Krasner, *Approximation Des Corps Values Complète de Characteristique* $p \neq 0$ *par Ceux de Characteristique 0*, Colloque dAlgebra Superieure, C.B.R.M., Bruxelles, 1956.
9. F. Marty, *Sur une generalization de la notion de groupe*, 8th congress Math., Scandinaves, Stockholm, (1934) pp. 45–49.
10. B. Zelinka, *Intersection graphs of finite abelian groups*, Czechoslovak Math. J. **25** (2) (1975) 171–174.


E-mail: m.hamidi@pnu.ac.ir
E-mail: hodamohamadi0@gmail.com

# Some Results on Divisibility Graph in Some Classes of Finite Groups

Danial Khoshnevis
School of Mathematics, Iran University of Science & Technology, Tehran, Iran
and Zohreh Mostaghim*
School of Mathematics, Iran University of Science & Technology, Tehran, Iran

Abstract. A finite group $G$ is called an $F-$group, if for every $x, y \in G \setminus Z(G)$, $C_G(x) \leq C_G(y)$ implies that $C_G(x) = C_G(y)$. The graph $D(G)$ is called the divisibility graph of $G$ if its vertex set is the non-central conjugacy class sizes of $G$ and there is an edge between vertices $a$ and $b$ if and only if $a|b$ or $b|a$. We determine the number of connected components of the divisibility graph $D(G)$ where $G$ is an $F-$group.
**Keywords:** Divisibility graph, F-Group, Conjugacy class.
**AMS Mathematical Subject Classification [2010]:** 20E45, 05C25.

## 1. Introduction

There are some graphs related to finite groups and this graphs have been widely studied; see, for example [4, 5, 6, 7].

In [8] A. R. Camina and R. D. Camina introduced a graph. This graph is called divisibility graph $\overrightarrow{D}(X)$ for a set of positive integers $X$. Its vertex set is $V(\overrightarrow{D}(X)) = X^*$ and the edge set is $E(\overrightarrow{D}(X)) = \{(x, y); x, y \in X^*, x|y\}$. Throughout the paper, $G$ denotes a finite non-abelian group and $x$ an element of $G$. $x^G$ denotes the $G$-conjugacy class containing $x$, $|x^G|$ denotes the size of $x^G$ and $cs(G) = \{|x^G|; x \in G\}$ denotes the set of $G$-conjugacy class sizes and $cs^*(G) = cs(G) \setminus \{1\}$. $Z(G)$ and $C_G(x)$ denote the center of $G$ and the centralizer of $x$ in $G$, respectively. We consider $D(G)$ instead of $D(cs(G))$. The number of connected components of the divisibility graph $D(G)$ is denoted by $n(D(G))$.
In [8], the authors posed a question about the number of components of $D(G)$. To answer this question the authors in [1] showed that the divisibility graph $D(G)$ has at most two or three connected components where $G$ is the symmetric or alternating group, respectively. Also they found the number of connected components of the divisibility graph $D(G)$ where $G$ is a simple Zassenhaus group or an sporadic simple group in [2]. The authors in [3] proved that if $G$ is a finite group of Lie type in odd characteristic, then the divisibility graph $D(G)$ has at most one connected component which is not a single vertex.

In this paper, we investigate the structure of the divisibility graph $D(G)$ where $G$ is an $F-$group. We obtain the number of connected components of the divisibility graph $D(G)$ where $G$ is an $F-$group. A finite group $G$ is called an $F-$group,

*Speaker

if for every $x, y \in G \setminus Z(G)$, $C_G(x) \leq C_G(y)$ implies that $C_G(x) = C_G(y)$.

## 2. Preliminaries and Main Results

In [9], the structure of non-abelian $F-$groups is given by J. Rebmann that we show this complete list below:

Theorem 2.1. [9] *Let $G$ be a non-abelian group. Then $G$ is an $F-$group if and only if it is one of the following types:*

(1) *$G$ has an abelian normal subgroup of prime index.*
(2) *$G/Z(G)$ is a Frobenius group with Frobenius kernel $L/Z(G)$ and Frobenius complement $K/Z(G)$, where $L$ and $K$ are abelian.*
(3) *$G/Z(G)$ is a Frobenius group with Frobenius kernel $L/Z(G)$ and Frobenius complement $K/Z(G)$ with $K$ abelian, $Z(L) = Z(G)$, $L/Z(G)$ has prime power order and $L$ is an $F-$group.*
(4) *$G/Z(G) \cong S_4$ and if $V/Z(G)$ is the Klein four-group in $G/Z(G)$, then $V$ is non-abelian.*
(5) *$G \cong A \times P$ where $P$ is an $F-$group of prime power order and $A$ is abelian.*
(6) *$G/Z(G) \cong PSL(2, p^n)$ or $PGL(2, p^n)$, $G' \cong SL(2, p^n)$, where $p$ is a prime and $p^n > 3$.*
(7) *$G/Z(G) \cong PSL(2, 9)$ or $PGL(2, 9)$ and $G'$ is isomorphic to the Schur cover of $PSL(2, 9)$.*

Lemma 2.2. [10] *Let $N$ be a normal subgroup of $G$ and $B = b^G$, $C = c^G$ with $(|B|, |C|) = 1$ that $b, c \in N$. Then*

i) *$G = C_G(b) \cdot C_G(c)$.*
ii) *$BC = CB$ be a conjugacy class of $N$ and $|BC| \,|\, |B| \cdot |C|$.*

A graph is a star if at least one of its vertices is adjacent to all the remaining vertices. A complete graph is certainly a star. We investigate the structure of the divisibility graph $D(G)$ where $G$ is a non-abelian $F-$group.

Theorem 2.3. *Let $G$ be a non-abelian $F-$group. Then the divisibility graph $D(G)$ is either disconnected or a star.*

Proposition 2.4. *Let $G$ be a non-abelian $F-$group and for $k \geqslant 1$ the divisibility graph $D(G)$ be a $k-$regular graph. Then the divisibility graph $D(G)$ is a connected graph.*

Proof. If the divisibility graph $D(G)$ is not connected, then by Theorem 2.3, it contains at least one isolated vertex and so it is not a $k-$regular graph for $k \geqslant 1$. $\square$

In the following theorem we investigate the number of connected components of the divisibility graph $D(G)$, whenever $G$ is a non-abelian $F-$group.

Theorem 2.5. *Let $G$ be a non-abelian $F-$group. Then the divisibility graph $D(G)$ has at most three connected components.*

Proof. Let $G$ be a non-abelian $F-$group. Therefore $G$ satisfies one of the condition $(1) - (7)$ of Theorem 2.1, We will investigate each case one by one.

(1) Suppose that (1) holds. So by Theorem 2.3, the divisibility graph $D(G) \cong K_2$ or $2K_1$. Thus $n(D(G)) \leqslant 2$.

(2) For case (2), according to Theorem 2.3, the divisibility graph $D(G) \cong K_2$. So $n(D(G)) = 1$.

(3) Let (3) holds. Since by Theorem 2.3, the divisibility graph $D(G)$ is a complete graph, so $n(D(G)) = 1$.

(4) According to Theorem 2.3, the divisibility graph $D(G) \cong K_2 + K_1$ and therefore $n(D(G)) = 2$.

(5) Suppose that (5) holds. By Theorem 2.3, we have that the divisibility graph $D(G)$ is a complete graph, so $n(D(G)) = 1$.

(6), (7) Due to Theorem 2.3, when $G/Z(G) \cong PSL(2, p^n)$, $p$ is a prime and $p^n > 3$, the divisibility graph $D(G) \cong K_2 + 2K_1$ or $3K_1$ and whenever $G/Z(G) \cong PGL(2, p^n)$, where $p$ is a prime and $p^n > 3$, the divisibility graph $D(G) \cong 2K_2 + K_1$ or $3K_1$. So in two cases, we have $n(D(G)) = 3$.

Due to above proof, if $G$ is a non-abelian $F-$group, then the divisibility graph $D(G)$ has at most three connected components. □

## References


1. A. Abdolghafourian and M. A. Iranmanesh, *Divisibility graph for symmetric and alternating groups*, Comm. Algebra **43** (7) (2015) 2852–2862.
2. A. Abdolghafourian and M. A. Iranmanesh, *On the number of connected components of divisibility graph for certain simple groups*, Trans. Comb. **5** (2) (2016) 33–40.
3. A. Abdolghafourian and M. A. Iranmanesh, *The divisibility graph of finite groups of Lie type*, J. Pure Appl. Algebra **221** (10) (2017) 2482–2493.
4. A. Abdollahi, S. Akbari and H. R. Maimani, *Non-commuting graph of a group*, J. Algebra **298** (2) (2006) 468-492.
5. A. Beltran, M. J. Felipe and C. Melchor, *Triangles in the graph of conjugacy classes of normal subgroups*, Monatsh. Math. **182** (1) (2017) 5–21.
6. E. A. Bertram, *Some applications of graph theory to finite groups*, Discrete Math. **44** (1) (1983) 31–43.
7. E. A. Bertram, M. Herzog and A. Mann, *On a graph related to conjugacy classes of groups*, Bull. London Math. Soc. **22** (6) (1990) 569–575.
8. A. R. Camina and R. D. Camina, *The influence of conjugacy class sizes on the structure of finite groups: A survey*, Asian-Eur. J. Math. **4** (4) (2011) 559–588.
9. J. Rebmann, *F-Gruppen*, Arch. Math. **22** (1971) 225–230.
10. X. Zhao, G. Chen and J. Shi, *On the normal subgroup with coprime G-conjugacy class sizes*, Proc. Indian Acad. Sci. Math. Sci. **121** (4) (2011) 397–404.


E-mail: danial.khoshnevis1364@gmail.com
E-mail: mostaghim@iust.ac.ir

# $\mathcal{NAC}$-Groups

Hamid Mousavi*
Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran

Abstract. A finite non-Dedekind group $G$ is called an $\mathcal{NAC}$-group if all non-normal abelian subgroups are cyclic. In this paper, we classify all finite $\mathcal{NAC}$-groups. We show that the center of such groups is cyclic. If $G$ has a non-abelian non-normal Sylow subgroup of odd order, then other Sylow subgroups of $G$ are cyclic or of Quaternion type.
**Keywords:** $\mathcal{NAC}$-group, Abelian non-normal subgroup.
**AMS Mathematical Subject Classification [2010]:** 20D99, 20E45.

## 1. Introduction

Let all Sylow subgroups of $G$ be cyclic, then by [1, Theorem 5.16] we can write $G = G'Y$ where $G'$ is a cyclic Hall subgroup and $Y$ is cyclic too. If $S \leqslant G$ is nilpotent, then $S = LK$ where $K \leqslant G'$ and $L \leqslant Y$. As $S$ is nilpotent and $G'$ is a Hall subgroup, thus $(|L|, |K|) = 1$, so $[K, L] = 1$. Therefore $S$ is cyclic. So every non-normal nilpotent (in particular abelian) subgroup of $G$ will be cyclic. But the converse does not hold, that is, if all non-normal nilpotent (in particular abelian) subgroups of $G$ are cyclic, necessarily Sylow subgroups are not cyclic.

A finite non-Dedekind group $G$ is called an $\mathcal{NAC}$-group ($\mathcal{NNC}$-group) if all of whose non-normal abelian (nilpotent) subgroups are cyclic.

The authors in [2], provide the complete characterization of finite non-nilpotent $\mathcal{NNC}$-groups. In [3], Zhang and Zhang, gave the classification of $\mathcal{NAC}$-$p$-groups.

The purpose of this paper is to investigate the structure of finite non-Dedekindian $\mathcal{NAC}$-groups such that containing at least a non-cyclic Sylow subgroup.

In this paper we use $Q_{2^n}$, $D_{2^n}$ and $\mathbb{Z}_{p^n}$ to denote the generalized quaternion group of order $2^n$, the dihedral group of order $2^n$ and the cyclic group of order $p^n$, respectively. Our notations are standard and can be found in [1].

Throughout this paper we used the following notations for the minimal non-abelian $p$-groups which are not isomorphic to $Q_8$.

$$M_p(m, n) = \langle a, b \,|\, a^{p^m} = b^{p^n} = 1, a^b = a^{1+p^{m-1}} \rangle,$$

where $m \geq 2$.

$$M_p(m, n, 1) = \langle a, b, c \,|\, a^{p^m} = b^{p^n} = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle,$$

where $m \geq n$, and if $p = 2$, then $m + n \geq 3$.

In the following theorem Zhang and Zhang, give the structure of non-abelian $\mathcal{NAC}$-$p$-group of odd order.

*Speaker

Theorem 1.1. [3, Theorem 3.3] *Assume $G$ is a finite non-Dedekindian $p$-group and $p$ is an odd prime. Then all non-normal abelian subgroups of $G$ are cyclic if and only if $G$ is one of the following groups.*

i) $M_p(m, n)$, *where* $m \geq 2$.
ii) $M_p(1, 1, 1) * C_{p^n}$.
iii) $P_{81} = \langle a, b \,|\, a^9 = c^3 = 1, b^3 = a^3, [a, b] = c, [c, a] = a^3, [c, b] = 1\rangle$.

The group $P_{81}$ is a 3-group of maximal class of order 81.

## 2. $\mathcal{NAC}$-Groups with an Abelian Sylow Subgroup

In this section we show that the center of an $\mathcal{NAC}$-group is cyclic and then we characterize the structure of $\mathcal{NAC}$-groups with an abelian Sylow subgroup.

Theorem 2.1. *The center of any non-nilpotent $\mathcal{NAC}$-group is cyclic.*

Theorem 2.2. *Let $G$ be a non-Dedekindian nilpotent group. Then $G$ is $\mathcal{NAC}$-group if and only if $G$ is isomorphic to one of the following groups.*

i) $Q \times C$, *where* $Q \ncong Q_8$ *is non-abelian $\mathcal{NAC}$-2-group.*
ii) $Q \times P \times C$, *where $P$ is non-abelian $\mathcal{NAC}$-$p$-group of odd order and $Q$ is cyclic or* $Q \cong Q_8$.

*Where, $C$ is cyclic Hall subgroup of odd order.*

Theorem 2.3. *Let $G$ be a non-nilpotent group with a non-cyclic abelian Sylow 2-subgroup. Then $G$ is an $\mathcal{NAC}$-group if and only if* $G \cong ((\mathbb{Z}_2 \times \mathbb{Z}_2) \times C) \rtimes \mathbb{Z}_{3^m}$ *where $C$ is a cyclic $\{2, 3\}'$-Hall subgroup.*

We observed that in Theorem 2.3, if $Q$, the Sylow 2-subgroup of $G$ is non-cyclic abelian, then it is of type $(2, 2)$. Actually because the center of a non-nilpotent $\mathcal{NAC}$-group is cyclic, so $Q \cap Z(G) = 1$, by Mashke's theorem. Therefore no subgroup of $Q$ is normal in $G$. We now extend this problem to the abelian Sylow $p$-subgroups of odd order.

Theorem 2.4. *Let non-nilpotent group $G$ with a non-cyclic abelian Sylow subgroup $P$ of odd order. Then $G$ is $\mathcal{NAC}$-group if and only if $G$ is isomorphic to one of the following groups.*

i) *If $P$ has a subgroup which is non-normal in $G$, then $G$ has one of the following structures.*
   (i-1) $G \cong (P \times C) \rtimes H$, *where any Sylow subgroup of $H$ is cyclic or generalized Quaternion.*
   (i-2) $G \cong Q \times (P \times C) \rtimes H$, *where $H$ is cyclic Hall subgroup and* $Q \in \mathcal{S}y\ell_2(G)$ *is cyclic or* $Q \cong Q_8$.
   *In all cases $C$ is cyclic normal Hall subgroup of odd order, $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$ is the only non-cyclic abelian Sylow subgroup of $G$ and $H$ acts irreducibly on $P$.*
ii) *If any subgroup of $P$ is normal in $G$, then $G \cong N \rtimes H$, where $N$ is Dedekindian Hall subgroup of $G$ and any Sylow subgroup of $H$ is cyclic or generalized Quaternion. We can assume that $p$ is the smallest prime factor of $|G|$ such that $G$ has a subgroup of type $(p, p)$. Also any prime factor of $|H|$ is a divisor of $p - 1$.*

COROLLARY 2.5. *Let $G$ be a non-nilpotent $\mathcal{NAC}$-group such that all Sylow subgroups of $G$ are abelian. Then $G$ has one of the following structures.*

i) *$G$ is non-abelian meta-cyclic group such that $G'$ is cyclic Hall-subgroup.*
ii) *$G \cong ((\mathbb{Z}_2 \times \mathbb{Z}_2) \times C) \rtimes \mathbb{Z}_{3^m}$ where $C$ is a cyclic $\{2,3\}'$-Hall subgroup.*
iii) *$G \cong ((\mathbb{Z}_p \times \mathbb{Z}_p) \times C) \rtimes H$ where $p$ is odd, $C$ and $H$ are cyclic Hall subgroups and $H$ acts irreducibly on $\mathbb{Z}_p \times \mathbb{Z}_p$.*
iv) *$G \cong (P \times C) \rtimes H$ where $P$ is non-cyclic abelian Sylow $p$-subgroup of odd order, $C$ is abelian and $H$ is cyclic Hall subgroup. Also every subgroup of $P$ is $H$-invariant.*

## 3. $\mathcal{NAC}$-Groups with Non-Abelian Sylow Subgroup

Section 2, is shown that if $\mathcal{NAC}$-group contains a subgroup of type $(p,p)$, then for any $2 < q \neq p$, Sylow $q$-subgroup is abelian. Therefore, if an $\mathcal{NAC}$-group contains one non-abelian Sylow subgroup of odd order, then other Sylow subgroups are cyclic or Quaternion (ordinary or generalized). Hence $G$ can only contain one non-abelian Sylow subgroup of odd order.

In this section we characterized the $\mathcal{NAC}$-group $G$ with non-abelian Sylow subgroup. By Theorems 2.3 and 2.4, in the following we can assume that $G$ is not contain a non-cyclic abelian Sylow subgroup. First we assume that a non-abelian Sylow subgroup is of odd order, next that all Sylow subgroups of odd order are cyclic.

THEOREM 3.1. *Assume that the group $G$ contains a non-abelian non-normal Sylow subgroup of odd order, $P$ say, and $Q \in \mathcal{S}y\ell_2(G)$. Then $G$ is $\mathcal{NAC}$-group if and if $G \cong Q \times C \rtimes P$, where $C$ is the normal cyclic $\{2,p\}'$-Hall subgroup of $G$, $Q$ is either cyclic or $Q \cong Q_8$ and $P$ is one of the following groups.*

i) *$M_p(m,1) \cong \mathbb{Z}_{p^m} \rtimes \mathbb{Z}_p$, where $m \geq 2$.*
ii) *$P_{81} = \langle a,b,c \,|\, a^9 = c^3 = 1, a^3 = b^3, [a,b] = c, [c,a] = a^3, [c,b] = 1\rangle$.*

*Furthermore $\mathcal{C}_P(C) = T$ where $T = \langle a^p, b\rangle$ if $P \cong M_p(m,1)$ and $T = \langle b,c\rangle$ if $P \cong P_{81}$.*

THEOREM 3.2. *Assume that the group $G$ contains a non-abelian normal Sylow subgroup of odd order, $P$ say, and $Q \in \mathcal{S}y\ell_2(G)$. Then $G$ is $\mathcal{NAC}$-group if and if $G$ is one of the following groups.*

i) *$G \cong Q \times (P \times C) \rtimes H$, where $Q$ is cyclic or $Q \cong Q_8$.*
ii) *$G \cong (P \times C) \rtimes H$, if $Q \ntrianglelefteq G$.*

*Where $C$ is the cyclic normal Hall subgroup of $G$, any Sylow subgroup of $H$ is either cyclic or of Quaternion type and $P$ is one of the groups listed in Theorem 1.1. Also all non-cyclic abelian subgroups of $P$ are $H$-invariant and any prime factor of $|H|$ is a divisor of $p-1$.*

*Furthermore, let $L \leqslant P$ be of type $(p,p)$, then $\mathcal{C}_Q(L) = \mathrm{Core}_G(Q) \trianglelefteq G$ is Dedekindian and $\mathcal{C}_H(L) = \mathrm{Core}_G(H) \trianglelefteq G$ is cyclic. Also for any $K \leqslant H$ which is non-normal in $G$, $\mathcal{C}_P(K)$ is cyclic.*

Finally we assume that $G$ does not contain any non-cyclic Sylow subgroup of odd order.

Theorem 3.3. *Let $G$ be a non-nilpotent group such that whose odd order Sylow subgroups are cyclic. Assume that $Q$ is a non-abelian non-normal Sylow 2-subgroup of $G$. Then $G$ is $\mathcal{NAC}$-group if and only if $G$ is isomorphic to one of the following groups.*

i) *$G \cong N \rtimes Q$, where $N$ is cyclic of odd order and $Q$ is one of the following group, that acts by inverse on $N$.*
   (i-1) $\langle a, b, c \mid a^8, b^2a^4, c^2, [a, b]c, [c, a]a^4, [c, b]\rangle$.
   (i-2) *$M_{2^{\ell+2}}$ the modular 2-group of order $2^{\ell+2}$, where $\ell \geq 2$.*
   (i-3) *$\langle a, c \mid a^{2^\ell}, a^{2^{\ell-1}}c^4, [a, c]a^2\rangle$, where $\ell \geq 2$.*

ii) *$G \cong N \rtimes Q_{2^n}$, where $N$ is meta-cyclic subgroup of odd order.*

iii) *$G \cong N \rtimes (QR)$, where $N$ is meta-cyclic $\{2, 3\}'$-Hall subgroup of $G$, $Q \cong Q_8$ or $Q_{16}$ and $R \cong \mathbb{Z}_{3^n}$ for some $n$. Also for any $K \leqslant QR$, if $K \ntrianglelefteq G$, $\mathcal{C}_N(K)$ is cyclic. If $Q \cong Q_8$ then $QR \cong (Q_8 \rtimes R)$ otherwise $QR$ contains a subgroup $K$ of index 2, such that $K \cong Q_8 \rtimes R$.*

iv) *$G \cong N \rtimes H$, where $N$ is odd order cyclic subgroup and $H$ contains a subgroup $H_1$ such that $|H : H_1| \leq 2$ and $H_1 \cong Z \times \mathrm{SL}(2, q)$ for some prime number $q$ and all Sylow subgroups of $Z$ are cyclic.*

Theorem 3.4. *Let $G$ be a non-nilpotent group such that whose Sylow subgroups of odd order are cyclic and whose Sylow 2-subgroup is non-abelian and normal. Then $G$ is $\mathcal{NAC}$-group if and only if $G \cong (Q_{2^n} \times C) \rtimes H$, where $H$ is Hall subgroup with cyclic Sylow subgroups, $C$ is a cyclic Hall subgroup.*

## References


1. I. M. Isaacs, *Finite Group Theory*, Graduate Studies in Mathematics, Vol. 92, AMS Providence, RI, 2008.
2. H. Mousavi and G. Tiemouri, *The structure of finite groups with trait of non-normal subgroups*, Commun. Algebra **47** (3) (2019) 1023–1028.
3. L. Zhang and J. Zhang, *Finite p-groups all of whose non-normal abelian subgroups are cyclic*, J. Algebra Appl. **12** (8) (2013) 1350052.


E-mail: hmousavi@tabrizu.ac.ir

# Essentially Retractable Acts over Monoids

Roghaieh Khosravi
Department of Mathematics, Faculty of Sciences, Fasa University, Fasa, Iran
and Mohammad Ali Naghipoor*
Department of Mathematics, Faculty of Sciences, Jahrom University, Jahrom, Iran

Abstract. In this paper we introduce a class of right $S$-acts called essentially retractable $S$-acts which are right $S$-acts with homomorphisms into their essential subacts. We also give some classifications of monoids and acts by essentially retractable $S$-acts.
**Keywords:** Essential subact, Retractable act, $S$-Act.
**AMS Mathematical Subject Classification [2010]:** 20M30.

## 1. Introduction

Throughout this paper $S$ will denote a monoid. A right $S$-act $A$ is a non-empty set on which $S$ acts unitarily. To simplify, by an $S$-act we mean a right $S$-act. Recall from [1] that a monomorphism $f : B \longrightarrow A$ of $S$-acts is said to be *essential* if for each homomorphism $g : A \longrightarrow C$ which $gf$ is a monomorphism, then $g$ is so. If $f$ is an inclusion map, then $A$ is said to be an *essential extension* of $B$ or $B$ is called *essential (large)* in $A$. We denote this situation by $B \subseteq' A$. It is shown that $B \subseteq' A$ if and only if for every non trivial congruence $\theta$ on $A$, $\theta \cap (B \times B) \neq \Delta_B$. Also, a subact $B$ of a right $S$-act $A$ is called *intersection large* if $B \cap C \neq \emptyset$ for each subact $C$ of $A$. The reader is referred to [2] for basic results and definitions relating to semigroups, acts and other properties which are used here.

Khuri in [4] introduced the notion of retractable modules, and then some excellent papers have been appeared investigating this subject. Also some weaker and stronger classes of retractable modules are considered. For instance in [6], essentially retractable modules are studied. In the category of $S$-acts, first in [3] retractable $S$-acts are introduced. In [3], a right $S$-act $A$ is called retractable if for any subact $B$ of $A$, $\hom(A, B) \neq \emptyset$. In [5] a slightly different definition of retractable acts over semigroups with zeros are investigated and the authors introduced some smaller classes of retractable acts, i.e., strong retractable, epi-retractable, mono-retractable and largely mono-retractable acts.

In this paper we introduce essentially retractable acts. Also we give a classification of monoids using essentially retractable acts. First we give general properties of essential subacts.

Lemma 1.1. *For a monoid $S$ the following hold:*

i) *If $B_1 \subseteq' A_1$ and $B_2 \subseteq' A_2$, then $B_1 \cap B_2 \subseteq' A_1 \cap A_2$.*
ii) *The intersection of finitely many essential subacts of an $S$-act $A_S$ is essential.*

*Speaker

iii) *If $f : A_S \longrightarrow B_S$ is an $S$-morphism and $B' \subseteq' B$, then $f^{-1}(B') \subseteq' A_S$.*
iv) *If $B \subseteq' A$ and $B$ is indecomposable, then $A$ is indecomposable or $A = A' \cup \Theta$ such that $A'$ is indecomposable.*
v) *If $A = \coprod_{i\in I} A_i$, $|A_i| \geq 2$ and $B \subseteq' A$, then $B = \coprod_{h\in I} B_I$ with $B_i \subseteq' A_i$ for each $i \in I$.*

## 2. Main Results

As we mentioned before a right $S$-act $A$ is called retractable if for any subact $B$ of $A$, $\hom(A, B) \neq \emptyset$. Also, if for any subact $B$ of $A$, there exists an epimorphism (resp. a monomorphism) from $A$ into $B$, then A is called epi-retractable (resp. mono-retractable). Also a right $S$-act $A$ is called largely (or essentially) mono-retractable if $A$ embeds in each of its intersection large subacts. As we know, in the category of S-acts congruences and essential subacts play more important roles than subacts and intersection large subacts, respectively. So we introduce a general class of essentially retractable $S$-acts as follows.

Definition 2.1. A right $S$-act $A_S$ is called *essentially retractable* if for any essential subact $B_S$ of $A_S$, $\mathrm{Hom}(A_S, B_S) \neq \emptyset$.

Two following results are easily checked.

Lemma 2.2. *An $S$-act $A$ is essentially retractable if and only if* Im(f) *is an essentially retractable $S$-act for some $f \in$* End(A).

Lemma 2.3. *The following hold for a monoid $S$.*
i) *$S$ and $\Theta$ are essentially retractable.*
ii) *Every essential subact of an essentially retractable right $S$-act is essentially retractable.*
iii) *A retract of an essentially retractable $S$-act is essentially retractable.*
iv) *Let $\{A_i\}_{i\in I}$ be a family of essentially retractable $S$-acts and $|A_i| \geq 2$. Then $\coprod_{i\in I} A_i$ is essentially retractable.*
v) *If $A_S$ is essentially retractable, then $\coprod_I^B A_S$ is essentially retractable for any subact $B_S$ of $A_S$.*
vi) *If $S$ contains a left zero and $A$ is a right $S$-act, then $S \coprod A$ is essentially retractable.*

Obviously, every retractable right $S$-act is essentially retractable. But the converse is not valid. For example $S$ and $\Theta$ are essentially retractable, and so by Lemma 2.3, $S \coprod \Theta \coprod \Theta$ is essentially retractable. But for a monoid $S$ with no left zero $S \coprod \Theta \coprod \Theta$ is not retractable. The following proposition deduces that to prove an $S$-act is retractable, it suffices to show that all of its factor acts are essentially retractable.

Proposition 2.4. *Let $A$ be a right $S$-act. If any non-zero factor of $A$ is essentially retractable then $A$ is retractable.*

Proposition 2.5. *Let $S$ be a monoid with a left zero. If $A$ is essentially retractable right $S$-act and $B$ is an essential subact of $A$ with $Hom(A/B, B) = \{0\}$, then, $B$ is essentially retractable.*

Similar to rectactable $S$-act, essentially retractable $S$-acts are not preserved under product, coproduct and factor. By [1, Lemma 2], if an $S$-act $A$ has no fixed element, then $A\coprod\Theta$ is an essential extension of $A$. So if $S$ contains no left zero, then $S\subseteq' S\coprod\Theta$ with $\hom(S\coprod\Theta,S)=\emptyset$. Hence, we deduce the following result.

PROPOSITION 2.6. *The following are equivalent for a monoid $S$.*

i) *Every right $S$-act is essentially retractable.*

ii) *$S$ contains a left zero.*

iii) *Every coproduct of a family of essentially retractable right $S$-acts is essentially retractable.*

iv) *Every factor of an essentially retractable right $S$-act is essentially retractable.*

v) *Let $\{A_i\}_{i\in I}$ be a family of essentially retractable $S$-acts. If $\prod_{i\in I}A_i$ is essentially retractable, then each $A_i$ is also essentially retractable.*

As in [2, V.3.4], two monoids $S$ and $T$ are called Morita equivalent if the two categories **Act-S** and **Act-T** are equivalent. Also, a property (P) of a monoid $S$ is called a Morita invariant property, if each monoid $T$ which is Morita equivalent to $S$ has also property (P).

THEOREM 2.7. *Assume that $S$ is a monoid on which all right acts are essentially retractable. If $T$ is a monoid which is Morita equivalent to $S$, then, all right $T$-acts are essentially retractable, that is, essential retractablity is a Morita invariant property.*

PROPOSITION 2.8. *Assume that $S\subseteq T$ are monoids such that $T=\coprod_{i=1}^{n}S$ is a finitely generated free $S$-act, for some positive integer $n$. Let $A$ be an indecomposable $S$-act and $B$ an essential subact of $A$. Then, $Hom_S(A,B)\neq\emptyset$ if and only if $Hom_T(A\otimes T,B\otimes T)\neq\emptyset$.*

PROOF. First note that by using [2, Proposition II.5.13] we can show, for any subact $B$ of a right $S$-act $A$, $B\subseteq' A$ if and only if $B\otimes T\subseteq' A\otimes T$. Moreover by [2, Propositions II.5.19 and II.5.13],

$$Hom_T(A\otimes T,B\otimes T)\cong Hom_S(A,Hom_T(T,B\otimes T))\cong Hom_S(A,B\otimes T).$$

Also, by [2, Proposition II.5.14], $B\otimes T=\coprod_{i=1}^{n}(B\otimes S)$. Moreover, since $A$ is indecomposable, by [2, Propositions II.5.13 and II.1.22 ],

$$Hom_S(A,\coprod_{i=1}^{n}(B\otimes S))\cong\coprod_{i=1}^{n}Hom_S(A,B\otimes S)\cong\coprod_{i=1}^{n}Hom_S(A,B).$$

So $Hom_S(A,B)\neq\emptyset$, if and only if $Hom_S(A,\coprod_{i=1}^{n}(B\otimes S))\neq\emptyset$, if and only if $Hom_T(A\otimes T,B\otimes T)\neq\emptyset$. □

In the rest of this section we give some classifications of monoids and acts by essentially retractable $S$-acts.

PROPOSITION 2.9. *The following are equivalent for a monoid $S$.*

i) *Every right $S$-act is retractable.*

ii) *Every right $S$-act is essentially retractable.*

iii) *Every injective right $S$-act is essentially retractable.*

iv) *Every injective right $S$-act is retractable.*

Proof. The implications (i)⇒(ii)⇒(iii) are clear. We prove (iii)⇒(iv)⇒(i). To prove (iii)⇒(iv), let $B$ be a subact of an injective right $S$-act $A$ and $E(B)$ be the injective envelope of $B$. Since $B \subseteq' E(B)$, by (iii) there exists a homomorphism from $E(B)$ into $B$. Also there exists a homomorphism from $A$ into $E(B)$ by injectivity of $E(B)$. Thus $Hom(A, B) \neq \emptyset$, that is, $A$ is retractable. To prove (iv)⇒(i), let $A$ be a right $S$-act and $B$ be a subact of $A$. First note that since $E(B)$ is injective, the embeding $f : A \cap E(B) \longrightarrow E(B)$ can be extended to $\bar{f} : A \longrightarrow E(B)$. Also by (iv), there exists $g : E(B) \longrightarrow B$. Therefore $g\bar{f}$ is a homomorphism from $A$ to $B$, that is, $A$ is retractable. □

Proposition 2.10. *The following are equivalent for a monoid $S$.*

i) *Every essentially retractable right $S$-act is torsion free.*
ii) *Every essentially retractable right $S$-act with two generating elements is torsion free.*
iii) *Any right cancellable element of $S$ is right invertible.*
iv) *All right $S$-acts are torsion free.*

Proposition 2.11. *The following are equivalent for a monoid $S$.*

i) *Every essentially retractable right $S$-act is principally weakly flat.*
ii) *$S$ is a regular monoid.*
iii) *Every right $S$-act is principally weakly flat.*

Theorem 2.12. *The following are equivalent for a monoid $S$.*

i) *Every essentially retractable right $S$-act is weakly flat.*
ii) *Every right $S$-act is weakly flat.*
iii) *$S$ is a regular monoid which satisfies condition (R).*

Proposition 2.13. *Let $S$ be a monoid. Then, every essentially retractable right $S$-act is flat if and only if every right $S$-act is flat.*

Proposition 2.14. *The following are equivalent for a monoid $S$.*

i) *Every essentially retractable right $S$-act satisfies condition (P).*
ii) *$S$ is a group.*
iii) *Every right $S$-act satisfies condition (P).*

Theorem 2.15. *The following are equivalent for a monoid $S$.*

i) *Every essentially retractable right $S$-act is equalizer flat.*
ii) *Every essentially retractable right $S$-act satisfies condition (E).*
iii) *$S = \{1\}$ or $S = \{0, 1\}$.*
iv) *Every right $S$-act satisfies condition (E).*

Proposition 2.16. *The following are equivalent for a monoid $S$.*

i) *Every essentially retractable right $S$-act is free.*
ii) *Every essentially retractable right $S$-act is projective.*
iii) *Every right $S$-act is strongly flat.*
iv) *$S = \{1\}$.*
v) *Every right $S$-act is free.*

## References

1. H. Barzega, *Essentiality in the category of S-acts*, Eur. J. Pure Appl. Math. **9** (1) (2016) 19–26.
2. M. Kilp, U. Knauer and A. Mikhalev, *Monoids, Acts and Categories*, W. de Gruyter. Berlin, 2000.
3. R. Khosravi, *On retractable S-acts*, J. Math. Res. Appl. **32** (4) (2012) 392–398.
4. S. M. Khuri, *Endomorphism rings and lattice isomorphism*, J. Algebra **56** (2) (1979) 401–408.
5. M. A. Naghipoor and M. Ershad, *Retractable and coretractable acts over semigroups*, Southeast Asian Bull. Math. **40** (2016) 887–903.
6. M. R. Vedadi,*Essentially retractable modules*, J. Sci. I. R. Iran **18** (4) (2007) 355–360.

E-mail: khosravi@fasau.ac.ir
E-mail: ma_naghipoor@yahoo.com

# Injectivity in the Category $\mathbf{Set_F}$

Mahdieh Haddadi
Department of Mathematics, Statistics and Computer Sciences, Semnan University, Semnan, Iran
and Seyed Mojteaba Naser Sheykholislami*
Department of Mathematics, Statistics and Computer Sciences, Semnan University, Semnan, Iran

Abstract. In this research, we investigate the notion of injectivity in an arbitrary covariety and we show that the injectivity in the category of F-coalgebras, for every functor $F$, is well-behaved.
**Keywords:** F-Coalgebra, Injectivity.
**AMS Mathematical Subject Classification [2010]:**
18B20,46M10.

## 1. Introduction

Universal coalgebra is one of the most important branches of mathematics that has been widely used in various fields of theoretical computer science such as transition systems, automata, object oriented specification, and lazy functional programming languages, in a common and general explanation. The study of a certain subject in category theory, is called injectivity, is interested to many people, including the author who had worked with injectivity in the category of $F$-coalgebras. In this paper, we show that the notion of injective $F$-coalgebra in the category $\mathbf{Set_F}$ is well-behaved in the sense of the paper [2].

Now let us recall some necessary notions in this paper. The readers may consult [1, 5, 7] for the facts about category theory and universal $F$-coalgebra used in this paper. Here we also follow the notations and terminologies used there.

Given a functor $F : \mathbf{Set} \to \mathbf{Set}$, a *coalgebra of type* $F$, or simply, an *$F$-coalgebra* is a pair $(A, \alpha_A)$ consisting of a set $A$ and a map $\alpha_A : A \to F(A)$. The set $A$ is called the *underlying set* or *carrier* of the coalgebra, $\alpha$ is often called the *structure map* of $A$, and $F$ is called the *type* of it. An *$F$-homomorphism* between two $F$-coalgebras $(A, \alpha_A)$, $(B, \alpha_B)$ is a map $f : A \to B$ with $F(f) \circ \alpha_A = \alpha_B \circ f$. The class of $F$-coalgebras together with the $F$-homomorphisms form a category which is denoted by $\mathbf{Set_F}$.

For every $F$-coalgebra $(A, \alpha_A)$, an *$F$-subcoalgebra* of $(A, \alpha_A)$ is a subset $B$ of $A$ with a structure map $\alpha_B$ such that the inclusion map $\iota : B \to A$ is an $F$-homomorphism. We write $(B, \alpha_B) \leq (A, \alpha_A)$ whenever $(B, \alpha_B)$ is an $F$-subcoalgebra of $(A, \alpha_A)$. It is worth noting that with the natural structure maps $\alpha_{f(A')} = F(f)\circ\alpha_{A'}\circ f^{-1}$ and $\alpha_{f^{-1}(B')} = F(f)^{-1}\circ\alpha_B\circ f$, for every $F$-homomorphism

*Speaker

$f : A \to B$ between $F$-coalgebras, $(A', \alpha_{A'}) \leq (A, \alpha_A)$, and $(B', \alpha_{B'}) \leq (B, \alpha_B)$, the inclusion maps $f(A') \hookrightarrow B$ and $f^{-1}(B') \hookrightarrow A$ are $F$-homomorphism.

A *terminal $F$-coalgebra* is an $F$-coalgebra $(\Theta, \alpha_\Theta)$ for which there exists precisely one $F$-homomorphism $\theta_A : A \to \Theta$, so-called *terminal $F$-homomorphism*, for every $F$-coalgebra $(A, \alpha)$. Terminal $F$-coalgebras are uniquely determined up to isomorphism, so we can speak of "the" terminal $F$-coalgebra. The *initial $F$-coalgebra* is dually defined. In $\mathbf{Set_F}$, the initial object always exists, it is the empty $F$-coalgebra, see [5], while the terminal $F$-coalgebra need not always exist. But in [7, Theorem 10.4], it is shown that for every bounded functor $F$, the terminal $F$-coalgebra exists. A functor $F$ is called *bounded* if there is some cardinality $\kappa$ so that for every $F$-coalgebra $(A, \alpha_A)$ and every $a \in A$ one can find an $F$-subcoalgebra $(U_a, \alpha_{U_a})$ of $(A, \alpha_A)$ such that the cardinal number of $U_a$ is less than or equal to $\kappa$ and $a \in U_a$. Throughout this paper we only consider coalgebras of type $F$ for which $F$ is bounded and preserves weak pullbacks; i.e. transforms weak pullbacks into weak pullbacks.

For every $(A, \alpha_A) \in \mathbf{Set_F}$, a *terminal $F$-subcoalgebra* $(B, \alpha_B)$ of $(A, \alpha_A)$ is an $F$-subcoalgebra of $A$ such that the terminal $F$-homomorphism $\theta_B$ is an injection map.

The category $\mathbf{Set_F}$ is cocomplete, in particular, the coproduct of a family $\{(A_i, \alpha_{A_i})\}_{i\in I}$ is the disjoint union of $A_i$'s, $(\sum_{i\in I} A_i, \alpha_{\sum_{i\in I} A_i})$, and it is called *sum*.

Since we have assumed that $F$ preserves weak pullbacks, an arbitrary intersection of $F$-subcoalgebras is again an $F$-subcoalgebra, [7]. So for every $F$-coalgebra $A$ and every $a \in A$, we have $< a >= \bigcap\{B \leq A | a \in B\}$ with the structure map $\alpha_A|_{<a>}$ is an $F$-subcoalgebra of $A$.

It is worth noting that, in the category $\mathbf{Set_F}$ the $F$-epimorphisms are onto $F$-homomorphisms. Also, the embeddings are one-to-one $F$-homomorphisms and $F$-monomorphisms are left cancelable $F$-homomorphisms and they do not necessarily coincide. But here since $F$ preserves weak pullbacks, they coincide, see [7]. Whenever the structure map is clear from the context, we shall use the same notation for a coalgebra and for its carrier.

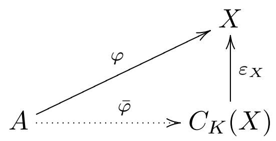
A *$\kappa$-source* is an $F$-coalgebra $P$ together with a family $\{\varphi_k : P \to A_k\}_{k\in\kappa}$ of $F$-homomorphisms. A *$\kappa$-simulation* $R$ between $F$-coalgebras $\{A_k\}_{k\in\kappa}$ is a subset of the cartesian product $\{A_k\}_{k\in\kappa}$, $\times_{k\in\kappa} A_k$, on which an $F$-coalgebra structure can be defined so that all projections $\pi_k : R \to A_k$ become $F$-homomorphisms.

An equivalence relation $\chi$ on an $F$-coalgebra $A$ is called a *congruence* on $A$ if $\chi$ is the kernel of an $F$-homomorphism $f : A \to B$. We denote the set of all congruences on $A$ by $\mathrm{Con}(A)$ which forms a bounded lattice in which the diagonal relation $\Delta_A = \{(a, a) \mid a \in A\}$ is the smallest element.

A major theme in universal coalgebra is the study of covariety. Here a *covariety* is a class of $F$-coalgebras closed under the operators $\mathcal{H}$ ($F$-homomorphic images), $\mathcal{S}$ ($F$-subcoalgebras), and $\Sigma$ (sums).

Let $X$ be a set. We refer to the elements of $X$ as colors and to every set map from an $F$-coalgebra $A$ to $X$ as a coloring. An $F$-coalgebra $C_K(X)$ together with a coloring $\varepsilon_X : C_K(X) \to X$ is called *cofree* over $X$, with respect to a class $K$ of $F$-coalgebras, if the following universal property is valid for them. For every $F$-coalgebra $A$ in $K$ and for any coloring $\varphi : A \to X$ there exists a unique $F$-homomorphism $\bar{\varphi} : A \to C_K(X)$ such that $\varphi = \varepsilon_X \circ \bar{\varphi}$.

(1)




We write $C(X)$ for $C_{\mathbf{Set_F}}(X)$.

Lemma 1.1. [7]

i) *Every covariety $\mathcal{CV}$ has a cofree $C_{\mathcal{CV}}(X)$ contained in $C(X)$, for every set $X$.*

ii) *Every sub-covariety $\mathcal{CV}'$ of covariety $\mathcal{CV}$ has a cofree $C_{\mathcal{CV}'}(X)$ contained in $C_{\mathcal{CV}}(X)$, for every set $X$.*

Now we use the terminology of Banaschewski [2, 3, 4] and we give the following definitions in the context of $F$-coalgebras.

Definition 1.2. An $F$-coalgebra $Q$ is *injective* if for every embedding $i : B \to A$ and every $F$-homomorphism $f : B \to Q$, there exists an $F$-homomorphism $\bar{f} : A \to Q$ such that $\bar{f} \circ i = f$.

Obviously, the definition of injectivity is up to isomorphism, i.e. every $F$-coalgebra in the definition of injective $F$-coalgebra may be replaced by an isomorphic $F$-coalgebra. Hence we can assume that $i$ is the inclusion map rather than embedding.

For a given subclass of $F$-monomorphisms $\mathcal{M}$, an $\mathcal{M}$-morphism $m$ is called to be $\mathcal{M}$-*essential* if for every $F$-homomorphism $f : B \to C$, $fm \in \mathcal{M}$ implies $f \in \mathcal{M}$.

One says that injectivity relative to a class $\mathcal{M}$ is well-behaved if the following propositions are established.

Proposition 1.3 (First well-behaviour Theorem [2]). *For an $F$-coalgebra $A$, the following conditions are equivalent:*

i) *$A$ is $\mathcal{M}$-injective.*

ii) *$A$ is $\mathcal{M}$-absolute retract.*

iii) *$A$ has no proper $\mathcal{M}$-essential extension.*

Proposition 1.4 (Second well-behaviour Theorem [2]). *Every $F$-coalgebra $A$ has an $\mathcal{M}$-injective hull.*

Proposition 1.5 (Third well-behaviour Theorem [2]). *The following conditions are equivalent, for an $\mathcal{M}$-morphism $m : A \to B$ in $\mathbf{Set_F}$.*

i) *$B$ is an $\mathcal{M}$-injective hull of $A$.*

ii) *$B$ is a maximal $\mathcal{M}$-essential extension of $A$.*

iii) *$B$ is a minimal $\mathcal{M}$-injective extension of $A$.*

In [2] Banaschewski has proved that the following notions and conditions are necessary for having well-behaved $\mathcal{M}$-injectivity in a category $\mathcal{C}$.

$B_1$ - The class $\mathcal{M}$ is composition closed.

$B_2$ - The class $\mathcal{M}$ is isomorphism closed and left regular; that is, for $f \in \mathcal{M}$ with $fg = f$ we have $g$ is an isomorphism.

$B_3$ - $\mathcal{C}$ satisfies Banaschewski's $\mathcal{M}$-condition, meaning that for every $\mathcal{M}$-homomorphism $f : A \to B$ in $\mathcal{C}$ there exists a homomorphism $g : B \to C$ such that $g \circ f$ is $\mathcal{M}$-essential.

$B_4$ - $\mathcal{C}$ satisfies $\mathcal{M}$-transferability conditions; that is, pushouts preserve $\mathcal{M}$-monomorphisms.

$B_5$ - $\mathcal{C}$ has $\mathcal{M}$-direct limits of well ordered direct systems.

$B_6$ - $\mathcal{C}$ is $\mathcal{M}^*$-cowell powered; that is, for every object $A \in \mathcal{C}$, the class

$$\{m : A \to B \mid B \in Obj(\mathcal{C}),\ m \text{ is an } \mathcal{M}\text{-essential monomorphism}\},$$

up to isomorphism, is a set.

## 2. Injectivity of $F$-Coalgebra

In this section, we discuss the notion of injectivity in $\mathbf{Set_F}$ and give some properties concerning injective $F$-coalgebras to identify this kind of $F$-coalgebras. We also show that the notion of injectivity in the category of $F$-coalgebras well-behaves.

It is easy to check that every injective $F$-subcoalgebra of an $F$-coalgebra $A$ is a retract of $A$ and cofree $F$-coalgebras and terminal $F$-coalgebras are injective. In [6] it is shown that how one can construct the cofree $F$-coalgebra over an arbitrary set $X$, when $F$ is bounded. So, for every $F$-coalgebra $A$, using $\varphi = id_A$ in Diagram (1), we get the embedding $\overline{\varphi} : A \to C(A)$. Therefore every F-coalgebra is embedded into an injective F-coalgebra. Also, for every terminal $F$-coalgebra $\Theta$, every $F$-homomorphism $f : \Theta \to A$ is embedding. Now we have the following theorem.

Theorem 2.1. *Every injective $F$-coalgebra contains a copy of terminal $F$-coalgebra.*

Immediately, using the above theorem we have the following corollary.

Corollary 2.2. *Every cofree $F$-coalgebra $C(X)$ over a non-empty set $X$, contains a copy of terminal $F$-coalgebra.*

Definition 2.3. An $F$-subcoalgebra $B$ of an $F$-coalgebra $A$ is called *large* in $A$, if $A$ is an essential extension of $B$. We denote this situation by $B \subseteq' A$.

Lemma 2.4. *A non-empty $F$-subcoalgebra $B$ of an $F$-coalgebra $A$ is large in $A$ if and only if for every congruence $\chi \neq \Delta_A$ on $A$, $\chi \cap B \times B \neq \Delta_B$ and it is a congruence on $B$.*

One can easily check that:

- Let $B \leq B' \leq A$. Then $B \subseteq' A$ if and only if $B \subseteq' B'$ and $B' \subseteq' A$.
- If $B \subseteq' A$ and $B$ is embedded in an injective $F$-coalgebra $Q$, then $A$ can be embedded in $Q$.

Now we give the following theorem.

Theorem 2.5. *Let $B$ be a proper retract of $A$; that is, $B \subsetneqq A$ and the inclusion map $\iota : B \to A$ has a left inverse $\pi : A \to B$. Then $B$ can not be large in $A$.*

Lemma 2.6. *For every $F$-coalgebra $A$ and every congruence $\chi \in \mathrm{Con}(A)$, there exists a maximal congruence $\kappa$ with $\kappa \cap \chi = \Delta_A$.*

Lemma 2.7. *Let $A$ be an $F$-coalgebra and $\Phi = \{(B_i, \alpha_{B_i})\}_{i\in I}$ be a family of disjoint $F$-subcoalgebra of $A$. Then there exists a structure map $\alpha_{A/\varrho_\Phi}$ on $A/\varrho_\Phi := (\sum_{i\in I}\theta_A(B_i)) + A \setminus (\cup_{i\in I}B_i)$ such that the map $\pi_{A/\varrho_\Phi} : A \to A/\varrho_\Phi$ defined by $\pi_{A/\varrho_\Phi}(a) = a$, for $a \in A \setminus (\cup_{i\in I}B_i)$, and $\pi_{A/\varrho_\Phi}(a) = \iota_i(\theta_A(a))$, for $a \in B_i$, in which $\iota_i : \theta_A(B_i) \to A/\varrho_\Phi$ is the inclusion map, is an $F$-epimorphism.*

Corollary 2.8. *Let $\mathcal{CV}$ be a covariety and $\mathcal{CV}'$ be a subcovariety of $\mathcal{CV}$. Then there exists a structure map $\alpha_{C^*_{\mathcal{CV}}(X)}$ on $C^*_{\mathcal{CV}}(X) := C_{\mathcal{CV}}(X)/\varrho_{\{C_{\mathcal{CV}'}(X)\}}$ such that $\pi_{C^*_{\mathcal{CV}}(X)} : C_{\mathcal{CV}}(X) \to C^*_{\mathcal{CV}}(X)$ defined by $\pi_{C^*_{\mathcal{CV}}(X)}(c) = c$, for $c \in C_{\mathcal{CV}}(X) \setminus C_{\mathcal{CV}'}(X)$, and $\pi_{C^*_{\mathcal{CV}}(X)}(c) = \theta_{C_{\mathcal{CV}}(X)}(c)$, for $c \in C_{\mathcal{CV}'}(X)$, is an $F$-epimorphism.*

Definition 2.9. For every $F$-coalgebra $A$ and family $\Phi = \{B_i\}_{i\in I}$ of $F$-subcoalgebras of $A$, the congruence $\varrho_\Phi = \ker(\pi_{A/\varrho_\Phi})$ is called the *Rees congruence generated* by $\Phi$ and $A/\varrho_\Phi$ is called the *Rees factor* of $A$ on $\varrho_\Phi$.

Remark 2.10. If $\mathcal{M}ono$ is the class of all monomorphisms in the category $\mathbf{Set_F}$, then $\mathcal{M}ono$ is isomorphism closed, by the left cancelability of monomorphisms in the category $\mathbf{Set_F}$. Also Gumm in [5, Lemma 3.7] shows that monomorphisms in the category $\mathbf{Set_F}$ is closed under composition and in [5, Lemma 4.6] shows that $\mathbf{Set_F}$ satisfies $\mathcal{M}ono$-transferability conditions. Also, by [5, Theorem 4.2], the category $\mathbf{Set_F}$ has $\mathcal{M}ono$-direct limits. Finally, since $\mathbf{Set_F}$ is a subcategory of $\mathbf{Set}$, $\mathbf{Set_F}$ is $\mathcal{M}ono^*$-cowell powered. So, to prove that injectivity is well-behaviored in $\mathbf{Set_F}$, it is enough to show that $\mathbf{Set_F}$ satisfies Banaschewski's condition for monomorphisms in the category of $F$-coalgebras.

Now we give Banaschewski's condition for monomorphisms in the category of $F$-coalgebras, but first, let us note the following Lemma.

Lemma 2.11. *Let $B$ be an $F$-subcoalgeebra of an $F$-coalgebra $A$, $\varrho_B$ be the Rees congruence generated by $B$ on $A$ and $\kappa_B$ be the maximal congruence on $A$ with $\kappa_B \cap \varrho_{\{B\}} = \Delta_A$. Then $\kappa_B \cap B \times B = \Delta_B$.*

Theorem 2.12. *For every $F$-homomorphism $f : B \to A$, there is an $F$-homomorphism $g : A \to C$ such that $g \circ f$ is an essential $F$-monomorphism.*

By Theorem 2.12 and Remark 2.10, the class $\mathcal{M}ono$ satisfies in conditions $B_1$ to $B_6$. So, the notion of injectivity in the category $\mathbf{Set_F}$ is well-behaved and every $F$-coalgebra $A$ has an injective hull.

## References

1. J. Adámek, H. Herrlich and G. E. Strecker, *Abstract and Concrete Categories*, John Wiley & Sons, New York, 1990.
2. B. Banaschewski, *Injectivity and essential extensions in equational classes of algebras*, 1970 Proc. Conf. on Universal Algebra (Queen's Univ., Kingston, Ont., 1969) pp. 131–147 Queen's Univ., Kingston, Ont.
3. M. M. Ebrahimi, M. Haddadi and M. Mahmoudi, *Injectivity in a category: an overview on well behavior theorems*, Algebras Groups Geom. **26** (4) (2009) 451–472.
4. M. M. Ebrahimi, M. Haddadi and M. Mahmoudi, *Injectivity in a category: An overview on smallness conditions*, Categ. General Algebraic Structure. App. **2** (1) (2014) 83–112.
5. H. P. Gumm, *Elements of the General Theory of Coalgebras*, LUATCS'99, Rand Afrikaans University, Johannesburg, South Africa, 1999.

6. H. P. Gumm and T. Schröder, *Coalgebras of bounded type*, Math. Struct. Comput. Sci. **12** (5) (2002) 565–578.
7. J. J. M. M. Rutten, *Universal coalgebra: A theory of systems*, Theoret. Comput. Sci. **249** (1) (2000) 3–80.

E-mail: m.haddadi@semnan.ac.ir
E-mail: s.m.naser@semnan.ac.ir

# Line Graphs with a Sequentially Cohen-Macaulay Clique Complex

Ashkan Nikseresht*
Department of Mathematics, Shiraz University, Shiraz, Iran

Abstract. Let $H$ be a simple undirected graph and $G = \mathrm{L}(H)$ be its line graph. Assume that $\Delta(G)$ denotes the clique complex of $G$. We show that $\Delta(G)$ is sequentially Cohen-Macaulay if and only if it is shellable if and only if it is vertex decomposable. Furthermore, we state a complete characterization of those $H$ for which $\Delta(G)$ is sequentially Cohen-Macaulay.

**Keywords:** Line graph, Stanley-Reisner ideal, Sequentially Cohen-Macaulayness, Edge ideal, Squarefree monomial ideal.
**AMS Mathematical Subject Classification [2010]:** 13F55, 05E40, 05E45.

## 1. Introduction

In this paper, $K$ denotes a field and $S = K[x_1, \dots, x_n]$. Let $G$ be a simple graph on vertex set $\mathrm{V}(G) = \{v_1, \dots, v_n\}$ and edge set $\mathrm{E}(G)$. Then the *edge ideal* $I(G)$ of $G$ is the ideal of $S$ generated by $\{x_i x_j | v_i v_j \in \mathrm{E}(G)\}$. A graph $G$ is called Cohen-Macaulay (CM, for short) when $S/I(G)$ is CM for every field $K$. Many researchers have tried to combinatorially characterize CM graphs in specific classes of graphs, see for example, [2, 3, 4, 5, 9]).

The family of cliques of a graph $G$ forms a simplicial complex which is called the *clique complex of* $G$ and is denoted by $\Delta(G)$. Algebraic properties of simplicial complexes in general also has got a wide attention recently, see for example [3, 7] and the references therein. If we denote the Stanley-Reisner ideal of $\Delta$ by $I_\Delta$, then we have $I_{\Delta(G)} = I(\overline{G})$, where $\overline{G}$ denotes the complement of the graph $G$. Thus studying clique complexes of graphs algebraically, is another way to study algebraic properties of graphs.

Here we say a simplicial complex $\Delta$ is CM over $K$, when $S/I_\Delta$ is CM . If $\Delta$ is CM over every field $K$, then we simply say that $\Delta$ is CM. Recall that $\Delta^{[i]} = \langle F | F \in \Delta, \dim F = i \rangle$ is called the *pure i-skeleton* of $\Delta$ and if each $\Delta^{[i]}$ is CM for $i \le \dim \Delta$, then $\Delta$ is called *sequentially CM.*

Suppose that $H$ is a simple undirected graph and $G = \mathrm{L}(H)$ is the *line graph* of $H$, that is, edges of $H$ are vertices of $G$ and two vertices of $G$ are adjacent if they share a common endpoint in $H$. Line graphs are well-known in graph theory and have many applications (See for example [10, Section 7.1]). In particular, [10, Theorems 7.1.16 to 7.1.18], state some characterizations of line graphs and methods that, given a line graph $G$, can find a graph $H$ for which $G = \mathrm{L}(H)$.

In [8], the author investigated when $\Delta(G)$ is CM, where $G = \mathrm{L}(H)$. A characterization of all $H$ such that $\Delta(G)$ ic CM was given. The family of such graphs

*Speaker

was proved to be a very limited family of graphs. Here we study when $\Delta(G)$ is sequentially CM and will show that the family of graphs $H$ for which $\Delta(G)$ is sequentially CM is a much larger class of graphs.

For definitions and basic properties of simplicial complexes and graphs one can see [3] and [10], respectively. In particular, all notations used in the sequel without stating the definitions are as in these two references.

## 2. Main Results

In this section, we always assume that $\Delta = \Delta(G)$, where $G = \mathrm{L}(H)$. Note that every 0-dimensional complex is CM and a pure 1-dimensional complex is CM if and only if it is connected (See for example [1, Exercise 5.1.26]). The following result considers $\Delta^{[i]}$ for $i \geq 3$.

Proposition 2.1. *Suppose that $H$ is connected. Then all nonempty $\Delta^{[i]}$ for $i \geq 3$ are CM if and only if $H$ has at most one vertex $v$ with degree $\geq 4$.*

Suppose that $v$ is a vertex of $H$ with degree 2 adjacent to vertices $a$ and $b$. By *splitting* $v$, we get the graph $H'$ with vertex set $(\mathrm{V}(H) \setminus \{v\}) \cup \{v_1, v_2\}$, where $v_1$ and $v_2$ are new vertices, and the same edge set as $H$, where we identify the edges $av$ and $bv$ of $H$ with $av_1$ and $bv_2$ in $H'$. Note that $v_1$ and $v_2$ are both leaves (vertices of degree 1) in $H'$. Also recall that if $\Delta$ is shellable then it is sequentially CM and if $\Delta$ is vertex decomposable, then it is shellable (for definitions of shellability and vertex decomposability see [3, Section 8.2] and [7], respectively).

Proposition 2.2. *Suppose that $H$ is connected. Then the following are equivalent.*

1) *$\Delta(G)$ is sequentially CM.*
2) *If $H'$ is obtained by splitting all vertices of degree 2 of $H$ which are not in a triangle, then every connected component of $H'$ is an edge except at most one component whose line graph has a sequentially CM clique complex.*
3) *$H$ can be obtained by consecutively applying the following two operations on a graph $H_0$ in which every vertex of degree two is in a triangle and whose line graph has a sequentially CM clique complex:*
   a) *attaching a new leaf to an old leaf of the graph;*
   b) *unifying two leaves whose distance is at least 4.*

*Moreover, if any the above statements holds, $H_0$ is as in Proposition 2.2 and $\Delta(\mathrm{L}(H_0))$ is vertex decomposable (resp. shellable), then $\Delta(G)$ is vertex decomposable (resp. shellable).*

In the sequel, unless stated otherwise explicitly, we assume that $H_0$ is a connected graph with exactly one vertex $v$ with degree $r > 3$ and also suppose that every vertex of degree 2 in $H_0$ is in a triangle. We also let $G_0 = \mathrm{L}(H_0)$ and $\Delta_0 = \Delta(G_0)$. According to Proposition 2.2 and its corollary, by characterizing those $H_0$ for which $\Delta_0$ is sequentially CM, we can derive a characterization of all graphs whose line graphs have a sequentially CM clique complex. Noting that for $i > 2$, $\Delta_0^{[i]}$ is either empty or the pure $i$-skeleton of a simplex and for $i < 2$, $\Delta_0^{[i]}$ is CM since $\Delta_0$ is connected, we just need to see when $\Delta_0^{[2]}$ is CM. If $\Delta$ is pure and for any two facets $F$ and $G$ of $\Delta$, there is a sequence $F = F_1, \ldots, F_t = G$ of facets of $\Delta$, such that $|F_i \cap F_{i+1}| = |F_i| - 1$ for all $i$, we say that $\Delta$ is *strongly connected* (or connected in codimension 1). By [3, Lemma 9.1.12], every CM complex is strongly connected so first we study when $\Delta_0^{[2]}$ is strongly connected.

Suppose that $l_0 = \{v\}$ and define $L_i = \mathrm{N}_{H_0}(L_{i-1}) \setminus (\cup_{j=0}^{i-1} L_j)$ to be the set of vertices of *level* $i$ in $H_0$. Here $\mathrm{N}_{H_0}(A)$ is the set of all vertices adjacent to a vertex in $A$ inside the graph $H_0$. Thus indeed, the level of a vertex is its distance to $v$. Note that a vertex with level $i$ can be adjacent only to vertices with levels $i-1, i, i+1$. Suppose that $H_0[L_i]$ is the induced subgraph of $H_0$ on the vertex set $L_i$. Then if $H' = H_0[L_1]$, every $u \in L_1$ has degree at most 2 in $H'$, since it is also adjacent to $v$ in $H_0$. Therefore each connected component of $H'$ is either an isolated vertex or a cycle or a path of length $\geq 1$. We call these isolated vertices, cycles and paths with positive lengths of $H_0[L_1]$, the *level 1 isolated vertices*, *level 1 cycles* and *level 1 paths*, respectively.

Proposition 2.3. *The complex $\Delta_0^{[2]}$ is strongly connected, if and only if $H_0$ satisfies both of the following conditions (See an example in Figure 1).*

1) *Every level 3 vertex of $H_0$ is a leaf.*
2) *A level 2 vertex $x$ of $H_0$ satisfies one of the following:*
   a) *$x$ is a leaf adjacent to an endpoint of a level 1 path;*
   b) *$\deg(x) = 2$ and $x$ is adjacent to both endpoints of a level 1 path with length 1;*
   c) *$\deg(x) = 3$ and $x$ is adjacent to both endpoints of a level 1 path with length 1 and the other neighbor of $x$ is either a level 3 vertex or a level 2 vertex with degree 3 or the endpoint of a level 1 path.*

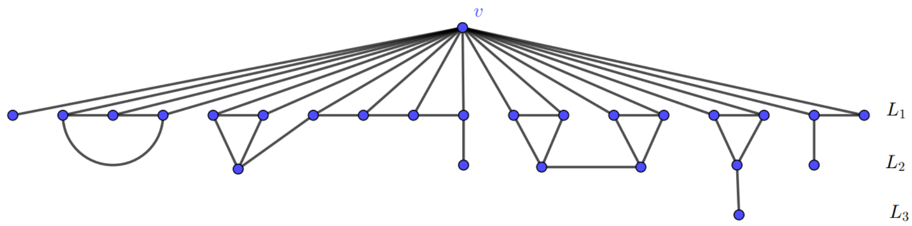



Figure 1. An example of $H_0$ satisfying conditions of Proposition 2.3.

Definition 2.4. Suppose that $C$ is a graph, $v$ is a vertex of $C$ and $r$ is a positive integer. We say that $C$ is an $r$-graph rooted at $v$ or simply an $r$-graph, if $C$ is connected, $\deg(v) = r$, all other vertices of $C$ have degree at most $\min\{r, 3\}$, all vertices of $C$ with degree 2 are in some triangles and also $C$ satisfies the conditions of Proposition 2.3, where the level of a vertex of $C$ is defined by $L_0 = \{v\}$ and $L_i = \mathrm{N}(L_{i-1}) \setminus (\cup_{j=0}^{i-1} L_j)$.

Theorem 2.5. *Suppose that $H$ is a connected graph with at least 1 edge. Let $\Delta = \Delta(\mathrm{L}(H))$. Then the following are equivalent.*

1) *$\Delta$ is vertex decomposable.*
2) *$\Delta$ is shellable.*
3) *$\Delta$ is sequentially CM (over some field).*

4) *For some positive integer $r$, there is an $r$-graph $H_0$ in which every level 2 vertex with degree 3 has a leaf neighbor and $H$ can be constructed from $H_0$ by consecutively applying the operations (3a) and (3b) of Proposition 2.2(3).*
5) *If $H'$ is the graph obtained by splitting all vertices of $H$ with degree 2 which are not in any triangle, then every connected component of $H'$ is an edge except at most one. The only non-edge connected component of $H'$, if exists, is an $r$-graph for a positive integer $r$, in which every level 2 vertex with degree 3 has a leaf neighbor.*

Remark 2.6 (*A "visual description" of graphs whose line graphs have sequentially CM clique complexes*). Suppose that $G = \mathrm{L}(H)$. Then according to the previous theorem, $\Delta(G)$ is sequentially CM if and only if $H$ can be drawn in the following way (See Figure 2).

First we draw some (maybe zero) paths and cycles and call them the level 1 paths and cycles (these are exactly the level 1 paths and cycles of $H_0$ in the previous theorem). Then we add a new vertex $v$ and join this vertex to all vertices of these path and cycles. For each path with length 1 we may also add a new vertex and join this vertex to both endpoints of the path (the level 2 vertices of $H_0$ with degree $\geq 2$). We call these vertices, level 2 vertices. Now we attach some paths with lengths at least one to the following vertices (these paths denote applying (3a) of Proposition 2.2(3) several times to the leaves of $H_0$): at most one path to each endpoint of a level 1 path, except those adjacent to a level 2 vertex; at most one path to each level 2 vertex; some (maybe zero) paths to $v$. Finally, we may "tie" some pairs of these new paths together, by unifying their degree 1 ends, but as we must not make any new triangles, the distance of the degree 1 ends should be at least 4 (this is applying (3b) of Proposition 2.2(3)).

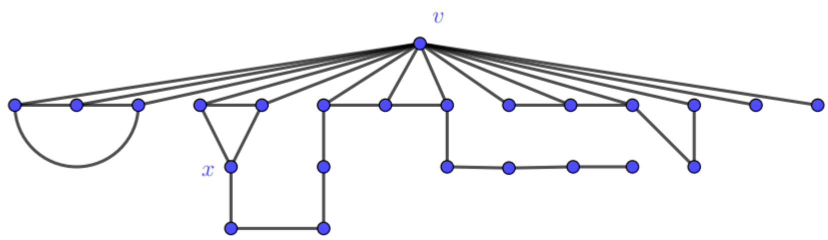



Figure 2. A graph whose line graph has a sequentially CM clique complex.

**An Algorithm.** At the end of this paper, we show that using Theorem 2.5(5), we can present a linear time algorithm which takes as input a graph $G$ and checks whether $G$ is a line graph or not and if yes, says whether $\Delta(G)$ is sequentially CM. Checking if $G$ is a line graph and even returning an $H$ such that $G = \mathrm{L}(H)$ has been previously done by Lehot in [6] in a linear time. Thus we can assume that $H$ is given and we must find out if $\Delta(\mathrm{L}(H))$ is sequentially CM. Here we state an algorithm, the correctness of which is ensured by Theorem 2.5 and its worst case time complexity is $\Theta(n)$. In this algorithm, we use breadth-first search (BFS) which can be found in for example [10].

Step 1: Run through the vertices of $H$ and compute the degree of each vertex. If for a second time a vertex with degree more than three is visited, return false. Also for each vertex $x$ with degree 2 and with neighbors $a$ and $b$, check if $a$ is a neighbor of $b$. If not, split the vertex $x$ by removing the edge $xb$ and adding a new vertex adjacent only to $b$.

Step 2: Compute the connected components of the obtained graph (say, by BFS). If more than one connected component is not an edge return false. If all connected components are edges, return true. Else let $H_0$ be the only connected component which is not an edge.

Step 3: Find a vertex $v$ with maximum degree in $H_0$. Run a BFS starting at $v$ and mark each visited vertex with its level which is the distance of the vertex from $v$. When visiting a level 2 vertex $y$ consider the following cases.

**$\deg(y) = 1$:** Let $a$ be the neighbor of $y$ (which has level 1). If $a$ has no level 1 neighbor (so that $a$ is not the endpoint of a level 1 path), return false.

**$\deg(y) = 2$:** The neighbors of $y$ should have level 1 and be adjacent. If not, return false.

**$\deg(y) = 3$:** Then its neighbors should be two level 1 adjacent vertices and a vertex not yet visited. If not, return false.

Also when visiting a level 3 vertex $x$, if $x$ has not degree 1, return false.

Step 4: Return true.

## References

1. W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, Cambridge, 1993.
2. M. Crupi, G. Rinaldo and N. Terai, *Cohen-Macaulay edge ideals whose height is half of the number of vertices,* Nagoya Math. J. **201** (2011) 116–130.
3. J. Herzog and T. Hibi, *Monomial Ideals*, Springer-Verlag, London, 2011.
4. T. Hibi, A. Higashitani, K. Kimura and A. B. O'Keefe, *Algebraic study on Cameron-Walker graphs,* J. Algebra **422** (2015) 257–269.
5. D. T. Hoang, N. C. Minh and T. N. Trung, *Cohen-Macaulay graphs with large girth*, J. Algebra Appl. **14** (7) (2015) 1550112.
6. P. G. H. Lehot, *An optimal algorithm to detect a line graph and output its root graph,* J. ACM **21** (1974) 569–575.
7. A. Nikseresht, *Chordality of clutters with vertex decomposable dual and ascent of clutters,* J. Combin. Theory Ser. A **168** (2019) 318–337.
8. A. Nikseresht, *On algebraic properties of line graphs and their complements*, The 10th Graph Theory and Algebraic Combinatorics Conference, Yazd University, Yazd, Iran, (2018) pp. 179–182.
9. K. N. Vander Meulen, A. Van Tuyl and C. Watt, *Cohen-Macaulay circulant graphs,* Comm. Algebra **42** (2014) 1896–1910.
10. D. B. West, *Introduction to Graph Theory*, 2nd ed., Prentice Hall,. Inc., Upper Saddle River, NJ, 2001.

E-mail: ashkan_nikseresht@yahoo.com

# Action of Automorphism Group on a Certain Subgroup

Shirin Fouladi
Department of Mathematics, Faculty of Mathematical Sciences and Computer, Kharazmi University, Tehran, Iran
and Reza Orfi*
Department of Mathematics, Faculty of Mathematical Sciences and Computer, Kharazmi University, Tehran, Iran

ABSTRACT. Let $G$ be a group and $L(G)$ be the set of all elements of $G$ fixed by all automorphisms of $G$. In this talk we find $L(G)$ for all $p$-groups of maximal class of order less than $p^6$ and $p$-groups of maximal class for $p = 2, 3$.
**Keywords:** Automorphism group, $p$-Group of maximal class.
**AMS Mathematical Subject Classification [2010]:** 20D45, 20D15.

## 1. Introduction

Let $G$ be a group, $x \in G$ and $\alpha \in Aut(G)$ is an automorphism of $G$. The autocommutator of $x$ and $\alpha$ is defined as $[x, \alpha] = x^{-1}x^{\alpha}$. In 1994, Hegarty [5] considered the following definition for $Z(G)$, the center of group $G$,

$$Z(G) = \{g \in G \mid g^{\alpha} = g \text{ for all } \alpha \in Inn(G)\}.$$

Hegarty also introduced $L(G)$, the absolute center of a group $G$ as follows:

$$L(G) = \{g \in G \mid g^{\alpha} = g \text{ for all } \alpha \in \mathrm{Aut}(G)\}.$$

It is clear that the subgroup $L(G)$ is characteristic and $L(G) \leq Z(G)$. Schur's theorem states that the derived subgroup of a group is finite whenever the central factor of the group is finite. Hegarty proved an analogue to Schur's theorem for the absolute center and the autocommutator subgroup of a group, that is, if $G$ is a group such that $G/L(G)$ is finite, then $\langle g^{-1}g^{\alpha} \mid g \in G, \alpha \in \mathrm{Aut}(G)\rangle$ is also finite. Moreover Chaboksavar et al. [2] classified all finite groups $G$ whose absolute central factors are isomorphic to a cyclic group, $Z_p \times Z_p, D_8, Q_8$, or a non-abelian group of order $pq$ for some distinct primes $p$ and $q$. Meng and Guo [9] explored the relationship between $L(G)$ and the Frattini subgroup $\Phi(G)$ for a finite group $G$. They also determined the structure of the absolute center of all finite minimal non-abelian $p$-groups.

In this talk we study $L(G)$ for $p$-groups of maximal class, where $p \in \{2, 3\}$ and all $p$-groups of maximal class of order less than $p^6$. As the definition of $L(G)$ shows, studying $L(G)$ directly depends on the structure of $\mathrm{Aut}(G)$.

Throughout this paper the following notation is used. The terms of the lower and the upper central series of $G$ are denoted by $\gamma_i(G)$ and $Z_i(G)$, respectively. The center of $G$ is denoted by $Z = Z(G)$. If $\alpha$ is an automorphism of $G$ and $x$ is an element of $G$, then we write $x^{\alpha}$ for the image of $x$ under $\alpha$. For a normal subgroup $N$ of $G$, we let $\mathrm{Aut}^N(G)$ denote the group of all automorphisms of $G$

*Speaker

centralizing $G/N$. Let $H \leq G$ and $A \leq \mathrm{Aut}(G)$, we note that $\mathcal{C}_A(H) = \{\alpha \in A \mid h^\alpha = h, \forall h \in H\}$ and $\mathcal{C}_H(A) = \{h \in H \mid h^\alpha = h, \forall \alpha \in A\}$.

## 2. Main Results

Let $G$ be a $p$-group of maximal class of order $p^n$ $(n \geq 3)$, where $p$ is a prime. If $n = 3$, then $L(G) = 1$ for $p > 2$ and $L(G) = Z(G)$ for $p = 2$. Therefore, in the rest of the paper we assume that $n \geq 4$. Following [7], we define the 2-step centralizer $K_i$ in $G$ to be the centralizer in $G$ of $\gamma_i(G)/\gamma_{i+2}(G)$ for $2 \leq i \leq n-2$ and define $P_i = P_i(G)$ by $P_0 = G$, $P_1 = K_2$, $P_i = \gamma_i(G)$ for $2 \leq i \leq n$. The degree of commutativity $l = l(G)$ of $G$ is defined to be the maximum integer such that $[P_i, P_j] \leq P_{i+j+l}$ for all $i, j \geq 1$ if $P_1$ is not abelian and $l = n-3$ if $P_1$ is abelian. Take $s \in G - \bigcup_{i=2}^{n-2} K_i$, $s_1 \in P_1 - P_2$ and $s_i = [s_{i-1}, s]$ for $2 \leq i \leq n-1$. It is easily seen that $\{s, s_1\}$ is a generating set for $G$ and $P_i(G) = \langle s_i, \ldots, s_{n-1}\rangle$ for $1 \leq i \leq n-1$ and so $Z(G) = P_{n-1}(G) = \langle s_{n-1}\rangle$. For the rest of the paper we fix the above notation. By [7, Corollary 3.2.7] and [1, Corollary p.59], we have the following result.

Lemma 2.1. *Let $G$ be a $p$-group of maximal class of order $p^n$.*

i) *The degree of commutativity of $G$ is positive if and only if the 2-step centralizers of $G$ are all equal.*

ii) *If $G$ is metabelian then $G$ has positive degree of commutativity.*

Lemma 2.2. *If $G$ is a $p$-group of maximal class of order $p^n$, then $\mathrm{Aut}_p(G)$ fix $Z(G)$ elementwise.*

Proof. Consider the action of $\mathrm{Aut}_p(G)$ on $Z(G)$. It is obvious that $\mathcal{C}_{Z(G)}(\mathrm{Aut}_p(G)) \neq 1$ since $\mathrm{Aut}_p(G)$ and $Z(G)$ are $p$-groups. As $|Z(G)| = p$, we have $\mathcal{C}_{Z(G)}(\mathrm{Aut}_p(G)) = Z(G)$, which completes the proof. □

Corollary 2.3. *If $G$ is a $p$-group of maximal class of order $p^n$ and $\mathrm{Aut}(G)$ is also a $p$-group, then $L(G) = Z(G)$.*

Corollary 2.4. *Let $G$ be a 2-group of maximal class of order $2^n$, then $L(G) = Z(G)$.*

Proof. By [4, Theorem 5.9], we can see that $\mathrm{Aut}(G)$ is also a 2-group which completes the proof by using Corollary 2.3. □

In what follows first we find the absolute center for all finite 3-groups of maximal class and finally we obtain the absolute center for all $p$-groups of maximal class of order $p^n$, where $4 \leq n \leq 5$.

Lemma 2.5. *Let $G$ be a $p$-group of maximal class of order $p^n$. If $\delta \in \mathrm{Aut}(G)$ with $s^\delta = s^a x$ and $s_1^\delta = s_1^c y$, $x, y \in \Phi(G)$ and $0 < a, c < p$, then $s_{n-1}^\delta = s_{n-1}^{a^{n-2}c}$.*

Lemma 2.6. *Let $G$ be a 3-group of maximal class of order $3^n$ $(n \geq 4)$, then $L(G) = 1$.*

Proof. First we see that for $n = 4$, $G$ is metabelian; and for $n \geq 5$, $G$ has degree of commutativity $n-4$ by [1, Theorem 3.13] and so it is metabelian. Moreover, by [4, Theorem 5.8], we have $H \neq 1$. Now if $P_1$ is abelian, then by Lemma 2.9, $L(G) = 1$. Furthermore if $P_1$ is not abelian, then by observing the proof of [4, Theorem 5.6 (i)], we have $H = \langle \beta_2 \rangle$ when $n$ is odd and $H = \langle \beta_3 \rangle$ when $n$ is even, where $s^{\beta_2} = s^{-1}$, $s_1^{\beta_2} = s_1$, $s^{\beta_3} = s^{-1}$ and $s_1^{\beta_3} = s_1^{-1}$. Note that $s^{-1} = s^2 s^{-3}$ and $s^{-3} \in \Phi(G)$. Therefore, Lemma 2.5 completes the proof. □

LEMMA 2.7. *Let $G$ be a $p$-group of maximal class of order $p^4$ $(p > 2)$. Then $L(G) = 1$.*

PROOF. By [8, Lemma 9] we see that $Aut(G)$ is not $p$-group. Since $P_1 = \mathcal{C}_G(\gamma_2(G))$, $\gamma_2(G) \leq Z(P_1) \leq P_1$ which implies that $P_1/Z(P_1)$ is cyclic and so $P_1$ is abelian, as desired. □

Now for $p > 3$, Curran [3, Corollary 5] proved that there is only one group of order $p^5$ whose automorphism group is also a $p$-group in which $(p-1, 3) = 1$. The presentation of this group is as follows:

$$G_0 = \langle a_1, a \mid a^p = [a_1, a]^p = [a_1, a, a]^p = [a_1, a, a, a]^p = [a_1, a, a, a, a] = 1$$
$$a_1^p = [a_1, a, a, a] = [a_1, a, a_1]^{-1} \rangle.$$

We note that $G_0$ is of maximal class. By this observation we state the following theorem.

THEOREM 2.8. *Let $G$ be a $p$-group of maximal class of order $p^5$ with $p > 3$. If $G = G_0$ then $L(G) = Z(G)$, for otherwise $L(G) = 1$.*

PROOF. First we claim that $G$ is metabelian. To prove this we have $[\gamma_2(G), Z_2(G)] = 1$ and so $\gamma_3(G) = Z_2(G) \leq Z(\gamma_2(G)) \leq \gamma_2(G)$, which implies that $\gamma_2(G)$ is abelian. If $G = G_0$ then Corollary 2.3 completes the proof. Therefore for the rest of the proof we may assume that $H \neq 1$. Since $p \geq 5$, by using [7, Proposition 3.3.2] we have $\exp(G/Z(G)) = \exp(G') = p$ which yields that $\mho_1(G) \leq Z(G) \cong \mathbb{Z}_p$. Moreover by [7, Lemma 1.2.11] $G$ is regular. Now if $\mho_1(G) = Z(G)$, then $|\Omega_1(G)| = p^4$. Hence $\Omega_1(G)$ is a maximal subgroup of $G$ and $\Omega_1(G) = \{x \in G | x^p = 1\}$ since $G$ is regular. On setting $s \in G - (P_1 \cup \Omega_1(G))$, we have $|s| = p^2$ and so $L(G) = 1$. If $\mho_1(G) = 1$, then $\exp(G) = p$. Now from Jame's list [6], there are only two families $\Phi_9$ and $\Phi_{10}$ of groups of maximal class of order $p^5$. By observing the presentation of these groups, we see that only $\Phi_9(1^5)$ and $\Phi_{10}(1^5)$ are of exponent $p$. Now if $G = \Phi_9(1^5)$ with the following presentation:

$$\langle s, s_1, \dots, s_4 \mid [s_i, s] = s_{i+1}, s^p = s_i^p = 1 \ (1 \leq i \leq 4) \rangle,$$

then obviously $P_1$ is abelian and so $L(G) = 1$. Furthermore if $G = \Phi_{10}(1^5)$ with the presentation below:

$$\langle s, s_1, \dots, s_4 \mid [s_i, s] = s_{i+1}, [s_1, s_2] = s_4, s^p = s_i^p = 1 \ (1 \leq i \leq 4) \rangle,$$

then the map $\alpha$ defined by $s^\alpha = s^{-1}, s_1^\alpha = s_1$ is an automorphism of order 2 and it is easily seen that $s_4^\alpha = s_4^{-1}$, which completes the proof. □

Now for the rest of paper we may assume that $G$ is a metabelian $p$-group of maximal class of order $p^n (p > 2)$ and $\mathrm{Aut}(G)$ is not $p$-group. It is straightforward to see that when $p$ is odd, $\mathrm{Aut}(G)$ is supersolvable and is a split extension of $\mathrm{Aut}_p(G)$ by a subgroup of the direct product of two cyclic groups of order $p-1$. On the other hand, if $H$ be a $p'$-subgroup of $\mathrm{Aut}(G)$, then we have $\mathrm{Aut}(G) = \mathrm{Aut}_p(G) \rtimes H$ and $H$ is embedded in $\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}$. Since $P_1(G)$ and $\Phi(G)$ are characteristic subgroups of $G$, $G/\Phi(G)$ and $P_1/\Phi(G)$ are invariant under $H$. So by Maschke's Theorem there exists $s \in G - P_1$ such that $G/\Phi(G) = P_1/\Phi(G) \oplus$

$\langle\Phi(G), s\rangle/\Phi(G)$ and $\langle\Phi(G), s\rangle/\Phi(G)$ is invariant under $H$. In the rest of the paper $s$ will be as above. Therefore, if $\delta \in H$ then $s^\delta = s^a x$ and $s_1^\delta = s_1^c y$, where $x, y \in \Phi(G)$ and $0 < a, c < p$. We recall that if $G$ is metabelian $p$-group of maximal class, then $G$ has positive degree of commutativity and $|s|$ divides $p^2$. In the next theorem we find the absolute center for finite metabelian $p$-group of maximal class when $H \neq 1$.

THEOREM 2.9. *Let $G$ be a metabelian $p$-group of maximal class of order $p^n (p > 2)$ and $H \neq 1$. If $P_1$ is abelian, then $L(G) = 1$.*

PROOF. First we may assume that $|s| = p$. Now we see that any element of $G$ is uniquely determined by $s^t u$, where $0 \leq t < p$ and $u \in P_1$. Assume that $1 < b < p$ and define $\beta : G \to G$ by $(s^t u)^\beta = s^t u^b$. We will show that $\beta$ is an automorphism. Let $g_1 = s^t u$ and $g_2 = s^{t'} u'$, where $0 \leq t, t' < p$ and $u, u' \in P_1$. We may write $g_1 g_2 = s^{t+t'}[s^{t'}, u^{-1}]uu'$. If $t + t' \equiv r(\bmod p)$, then $s^{t+t'} = s^r$ since $|s| = p$ and so $(g_1 g_2)^\beta = s^r([s^{t'}, u^{-1}]uu')^b$. Moreover $g_1^\beta g_2^\beta = s^{t+t'}[s^{t'}, u^{-b}]u^b u'^b$. We have $[s^{t'}, u^{-b}] = [s^{t'}, u^{-1}]^b$ since $P_1$ is abelian and so $\beta$ is a homomorphism. Also $\beta$ is onto since $G = \langle s, s_1^b\rangle$. Thus $\beta$ is an automorphism. Furthermore $s_{n-1}^\beta = s_{n-1}^b \neq s_{n-1}$, which completes the proof since $L(G) \leq Z(G) = \langle s_{n-1}\rangle$. □

THEOREM 2.10. *Let $G$ be a metabelian $p$-group of maximal class of order $p^n (p > 2)$. If $H$ is not cyclic, then $L(G) = 1$.*

PROOF. Since $H$ is a non-cyclic abelian group, there exists a Sylow $q$-subgroup $Q$ of $H$ such that $Q$ is not cyclic and so $\Omega_1(Q)$ is not cyclic. First if $q$ is odd, then $\mathcal{C}_Q(Z(G))$ is cyclic and so $Q \nleq \mathcal{C}_Q(Z(G))$. Let $\alpha \in Q - \mathcal{C}_Q(Z(G))$. Then $\alpha$ moves some elements of $Z(G)$ which yields that $L(G) = 1$. Now if $q = 2$ and $\delta$ is an automorphism of $G$ of order 2, then $s^\delta = s^a x$ and $s_1^\delta = s_1^c y$, where $x, y \in \Phi(G)$ and $0 < a, c < p$. By the same argument there exists an automorphism $\alpha$ such that $s^\alpha = s^a, s_1^\alpha = s_1^c$ and $\delta \equiv \alpha \pmod{\mathrm{Aut}^\Phi(G)}$. Hence $\alpha^2 \in \mathrm{Aut}^\Phi(G)$ and so $a^2 \equiv 1 \pmod p$ and $c^2 \equiv 1 \pmod p$ or equivalently $a, c \in \{1, p-1\}$. Therefore any automorphism $\delta$ of order 2 has the form $s^\delta = s^a x$ and $s_1^\delta = s_1^c y$, where $x, y \in \Phi(G)$ and $a, c \in \{1, p-1\}$. Thus we have three types of automorphisms of order 2, since $\mathrm{Aut}^\Phi(G)$ is a $p$-group. We consider type I by setting $(a, c) = (1, p-1)$, type II with $(a, c) = (p-1, 1)$ and type III with $(a, c) = (p-1, p-1)$. Moreover $\Omega_1(Q) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ which means that there exist exactly three automorphisms of order 2. If $\delta$ is of order 2 and of type I, then Lemma 2.9 completes the proof. Now suppose that there is no automorphism of order 2 of type I, this yields that there exist at least two automorphisms $\delta_1$ and $\delta_2$ of order 2 of type II or III. If $\delta_1$ and $\delta_2$ are both of type II or if $\delta_1$ and $\delta_2$ are both of type III , then $\delta_1\delta_2 \in \mathrm{Aut}^\Phi(G) \cap H = 1$ or equivalently $\delta_1 = \delta_2$, a contradiction. □

## References


1. N. Blackburn, *On a special class of p-groups*, Acta Math. **100** (1958) 45–92.
2. M. Chaboksavar, M. Farrokhi Derakhshandeh Ghouchan and F. Saeedi, *Finite groups with a given absolute central factor group*, Arch. Math. **102** (2014) 401–409.
3. M. J. Curran, *Automorphisms of certain p-groups (p odd)*, Bull. Aust. Math. Soc. **38** (2) (1988) 299–305.
4. S. Fouladi and R. Orfi, *Automorphisms of metabelian prime power order groups of maximal class*, Bull. Aust. Math. Soc. **77** (2008) 261–276.

5. P. Hegarty, *The absolute center of a group*, J. Algebra **169** (1994) 929–935.
6. R. James, *The groups of order* $p^6$ ($p$ *an odd prime*), Math. Comp. **34** (150) (1980) 613–637.
7. C. R. Leedham-Green and S. McKay, *The Structure of Groups of Prime Power Order*, Oxford University Press, 2002.
8. D. MacHale, *Some finite groups which are rarely automorphism groups-II*, Math. Proc. R. Ir. Acad. A **83** (2) (1983) 189–196.
9. H. Meng and X. Guo, *The absolute center of finite groups*, J. Group Theory **18** (2015) 887–904.

E-mail: s_fouladi@khu.ac.ir
E-mail: orfi@khu.ac.ir

# Geometric Reflections and Cayley Graph-Reflections (Type $A_1$)

Saeed Azam
Department of Mathematics and IPM, University of Isfahan, Isfahan, Iran
and Fatemeh Parishani*
Department of Mathematics, University of Isfahan, Isfahan, Iran

Abstract. In this work, we consider geometric reflections based on elements of a reflectable base of an extended affine root system $R$, and prove that in type $A_1$, any geometric reflection of a reflectable base is a Cayley graph-reflection if and only if the nullity of $R$ is less than or equal one. Also we show that any extended affine root system $R$, is a union of extended affine root systems of type $A_1$ with the same nullities as the nullity of $R$.
**Keywords:** Cayley graph, Extended affine root systems, Geometric reflection, Normalized dart.
**AMS Mathematical Subject Classification [2010]:** 17B22, 20F55, 94C15.

## 1. Introduction

In the past three decades there has been an intensive investigation on the theory of extended affine Lie algebras and related objects such as root systems and Weyl groups, see for example [1, 2, 3]. Root systems and Weyl groups occupy a big portion of the theory of extended affine Lie algebras; in addition to their importance in the study of the structure of Lie algebras and their classification, they are of much interest because of their combinatorial nature and independent applications in other branches of mathematics and theoretical physics.

Weyl groups are a subclass of groups generated by (geometric) reflections. In this work we present a new characterization of geometric reflections by merging the theory of extended affine Weyl groups, the covering theory of Cayley graphs in the sense of [6, 7] and the theory of Coxeter systems, see [5].

In [6], the authors give a new characterization of Coxeter groups by using a refined notion of a Cayley graph, introduced in 2000 by Malnic, Nedela and Skoviera [7]. An application of this new notion of graph appears in the theory of Cayley graphs. In 2007, Gramlich, Hofmann and Neeb used the new notion of graph to show that any Cayley graph is a regular 1-cover of a monopole and vice versa [6].

To achieve are main result, we need to introduce some notions. We use [1, 4, 6] for these notions. In this work we assume that all vector spaces are finite dimensional real vector spaces. We denote by $\mathcal{V}^*$, the dual space of the vector space $\mathcal{V}$. Let $\mathcal{V}$ be a vector space equipped with a positive semi-definite symmetric

*Speaker

bilinear form $(\cdot,\cdot)$, and $\mathcal{V}^0$ denote the radical of the form. Also assume that $\dim(\mathcal{V}^0)=\nu$. Let $R\subseteq\mathcal{V}$. Set $R^0=R\cap\mathcal{V}^0$ and $R^\times=R\setminus R^0$.

Definition 1.1. [1, Definition II.2.1] $R$ is called an *irreducible reduced extended affine root system* if $0\in R$, $R=-R$, $R$ spans $\mathcal{V}$, if $\alpha\in R^\times$, then $2\alpha\notin R$, $R$ satisfies in the *root string property*, $R^\times$ can not be decomposed as $R_1\uplus R_2$, where $R_1$ and $R_2$ are non-empty subsets of $R^\times$ satisfying $(R_1,R_2)=\{0\}$ (here $R$ is called *connected*), and finally if $\sigma\in R^0$, then there exists $\alpha\in R^\times$ such that $\alpha+\sigma\in R$.

One can check that $R^0=\{\alpha\in R\mid(\alpha,\alpha)=0\}$ and $R^\times=\{\alpha\in R\mid(\alpha,\alpha)\neq 0\}$. The integer $\nu$ is called the *nullity* of $R$. It is clear from axioms that irreducible reduced finite root systems are extended affine root systems of nullity zero. From [1, Chapter II], one can always find a finite root system $\dot R$ contained in $R$. The type and the rank of $\dot R$ is called the *type* and the *rank* of $R$ respectively.

From now on, we want to focus on type $A_1$. Let $\{0,\pm\epsilon\}$ be a finite root system of type $A_1$. By [1, Chapter II], if $R$ is an extended affine root system of type $A_1$ and nullity $\nu\geq 0$, then $R$ has the following structure

$$(S+S)\bigcup(\pm\epsilon+S),$$

where $S$ is a semilattice(lattice) in $\mathcal{V}^0$ (See [1, Definition II.1.2]). From [1, Proposition II.1.11], if $S$ is a semilattice in $\mathcal{V}^0$, then the lattice $\Lambda:=\langle S\rangle$ have a basis consists of elements of $S$. We show this basis with $B=\{\sigma_1,\ldots,\sigma_\nu\}$ and fix it in this work. By [1], we have $S=\cup_{i=0}^m(\tau_i+2\Lambda)$, where $m\geq\nu$, $\tau_0=0$ and for $1\leq i\leq\nu$, $\tau_i=\sigma_i$ and for $i>\nu$, $\tau_i=\sum_{r=1}^{\nu}n_{i,r}\sigma_r$ with $n_{i,r}\in\{0,1\}$ and at least two $n_{i,r}\neq 0$. Furthermore $\tau_1,\ldots,\tau_m$ generate $\Lambda$. Set

$$\Pi=\ \{\alpha_0:=\epsilon,\alpha_i=\tau_i-\epsilon\mid 1\leq i\leq m\}. \tag{1}$$

We want to use (1) in the sequel.

To define the notion of an *extended affine Weyl group*, we set $\dot{\mathcal{V}}=\mathrm{span}_{\mathbb{R}}\dot R$; then $\mathcal{V}=\dot{\mathcal{V}}\oplus\mathcal{V}^0$. Now set $\tilde{\mathcal{V}}=\dot{\mathcal{V}}\oplus\mathcal{V}^0\oplus(\mathcal{V}^0)^*$, and extend the form on $\mathcal{V}$ to a nondegenerate form on $\tilde{\mathcal{V}}$. Now for $\alpha\in\tilde{\mathcal{V}}$ with $(\alpha,\alpha)\neq 0$, we define $w_\alpha\in\mathrm{End}(\tilde{\mathcal{V}})$ such that $w_\alpha(\beta)=\beta-(\beta,\alpha^\vee)\alpha$ where $\alpha^\vee=2\alpha/(\alpha,\alpha)$.

Definition 1.2. The extended affine Weyl group $\mathcal{W}$ of $R$ is defined to be the subgroup of $\mathrm{GL}(\tilde{\mathcal{V}})$ generated by elements $w_\alpha$, $\alpha\in R^\times$. Furthermore any elements $w_\alpha$, $\alpha\in R^\times$, is called a geometric reflection. We denote the center of $\mathcal{W}$ with $Z(\mathcal{W})$.

It is known that, if $R$ is an extended affine root system of type $A_1$ of nullity $\nu$, then any elements of $\mathcal{W}$ has the unique expression as follow:

$$w=w_\epsilon^n\prod_{r=1}^{\nu}t_r^{m_r}z, \tag{2}$$

where $n\in\{0,1\}$, $m_r\in\mathbb{Z}$ and $t_r:=w_{\epsilon+\sigma_r}w_\epsilon$.

Definition 1.3. [4, Definition 1.9] Assume that $R$ is an extended affine root system and $\mathcal{W}$ is the corresponding Weyl group. A subset $\Pi$ of $R^\times$ is called a reflectable base if $\mathcal{W}_\Pi\Pi=R^\times$ and no proper subset of $\Pi$ has this property. We mean $\mathcal{W}_\Pi=\langle w_\alpha\mid\alpha\in\Pi\rangle$.

From [2, Proposition 4.26], [4, Theorem 3.1], if $\Pi$ is as (1), then $\Pi$ is a reflectable base of $R$.

We need to introduced the notion of a graph in the sense of [7]. A graph $\Gamma$ is a 4- tuple $(V, D, \iota, \lambda)$ where $V$ is a non-empty set of *vertices*, $D$ is a set, which might be empty, called the set of *darts*. Also $\iota : D \to V$ is a map and $\lambda : D \to D$ is a permutation of order 2. For every dart $d$, $\iota(d)$ is called the *initial vertex* of $d$ and $\lambda(d)$ , denoted by $d^{-1}$, is called the *reverse* of $d$. The vertex $\iota(d^{-1})$ is called *terminal vertex* of $d$.

Definition 1.4. For an automorphism $\sigma$ of a connected graph $\Gamma = (V, D, \iota, \lambda)$ set $\mathrm{Fix}_\sigma(V) := \{v \in V \mid \sigma(v) = v\}$ and $\mathrm{Norm}_\sigma(D) := \{d \in D \mid d \neq \sigma(d) = d^{-1}\}$. The sets $\mathrm{Fix}_\sigma(V)$ and $\mathrm{Norm}_\sigma(D)$ are called the *set of fixed vertices* and the set of *normalized darts* of $\Gamma$ with respect to the automorphism $\sigma$, respectively.

Definition 1.5. An automorphism $\sigma$ of a connected graph $\Gamma = (V, D, \iota, \lambda)$ is called a *graph-reflection* on $\Gamma$, if $\sigma^2 = 1$, $\mathrm{Fix}_\sigma(V) = \emptyset$ and the graph $\Gamma_\sigma = (V, D_\sigma, \iota_\sigma, \lambda_\sigma)$ with $D_\sigma = D \setminus \mathrm{Norm}_\sigma(D)$ and $\iota_\sigma = \iota\mid_{D_\sigma}$, $\lambda_\sigma = \lambda\mid_{D_\sigma}$, is disconnected.

Definition 1.6. Let $G$ be a group and $X \subset G \setminus \{1_G\}$ be a symmetric generating set of $G$, that is, $X = X^{-1}$ and $G = \langle X\rangle$. The Cayley graph $\mathrm{Cay}(G, X)$ is the 4-tuple $(G, G \times X, \iota, -1)$ where $\iota(g, x) := g$ and $(g, x)^{-1} = (gx, x^{-1})$.

The following theorem gives a new characterization of a Coxeter group in terms of its Cayley graph (See [5] for definition of a Coxeter group).

Theorem 1.7. [6, Theorem 7.6] *The following statements are equivalent:*

1) $(G, X)$ *is a Coxeter system.*
2) *The elements of* $X$ *act as graph-reflections on* $\mathrm{Cay}(G, X)$.

## 2. Main Results

Let $\Gamma := \mathrm{Cay}(G, X)$ be the Cayley graph of $(G, X)$. The group $G$ acts on $\Gamma$ by left multiplication and this action is regular, so we can consider $G$ as a subgroup of $\mathrm{Aut}(\Gamma)$. Suppose $1 \neq \sigma \in \mathrm{Aut}(\Gamma)$ is such that $\sigma^2 = 1$. From Definition 1.4, we have

$$\mathrm{Norm}_\sigma(G \times X) = \{(g, x) \in G \times X \mid (g, x) \neq \sigma(g, x) = (gx, x)\}.$$

We note that $d \in \mathrm{Norm}_\sigma(G \times X)$ if and only if $d^{-1} \in \mathrm{Norm}_\sigma(G \times X)$.

Lemma 2.1. *Let* $x' \in X$. *Then* $(g, x) \in \mathrm{Norm}_{x'}(G\times X)$ *if and only if* $x'g = gx$.

Lemma 2.2. *Suppose* $g$ *is an arbitrary vertex of the Cayley graph* $\Gamma = \mathrm{Cay}(G, X)$. *Then with respect to an involution* $x \in X$, *there is at most one normalized dart in* $\Gamma$ *with initial vertex* $g$.

Let $R$ be an extended affine root system of type $A_1$ and nullity $\nu > 0$ with extended affine Weyl group $\mathcal{W}$. Consider (1) and (2). Assume that $\Gamma$ is the Cayley graph of $\mathcal{W}$ with respect to the generating set $S_\Pi := \{w_\alpha \mid \alpha \in \Pi\}$.

Theorem 2.3. *Suppose* $\mathcal{W}$ *is an extended affine Weyl group of type* $A_1$ *with nullity* $\nu$, *and* $\Gamma$ *is the Cayley graph of* $\mathcal{W}$ *with respect to the generating set* $S_\Pi$, *then for* $0 \le i \le m$ *we have,*

$$Norm_{w_{\alpha_i}}(\mathcal{W} \times S_\Pi) = \{(w, w_{\alpha_i}) \mid w \in w_{\alpha_i}^n z,\ z \in Z(\mathcal{W}),\ n \in \{0, 1\}\}.$$

Theorem 2.4. *Let $R$ be an extended affine root system of type $A_1$ of nullity $\nu$, $\Pi$ be the reflectable base of $R$ introduced in (1) and $\alpha \in \Pi$. Then the geometric reflection $w_\alpha$ is a graph-reflection of the Cayley graph of $(\mathcal{W}, S_\Pi)$ if and only if $\nu \leq 1$.*

Now as a consequence of Theorems 1.7 and 2.4 we have the following theorem.

Theorem 2.5. *Let $R$ be an extended affine root system of type $A_1$ of nullity $\nu$, and $\mathcal{W}$ be its corresponding Weyl group. Assume $\Pi$ is a reflectable base of $R$. Then $(\mathcal{W}, S_\Pi)$ is a Coxeter system if and only if $\nu \leq 1$.*

Remark 2.6. Note that in this paper, we consider an especial reflectable base of an extended affine root system $R$ of type $A_1$, but we can prove that any reflectable base of $R$ is of the form $\Pi = \{r_i\tau_i + s_i\epsilon \mid 0 \leq i \leq m\}$, where $r_i, s_i \in \{\pm 1\}$ and $\{\tau_0, \ldots, \tau_m\}$ is a set of coset representatives for $S$, namely $S = \uplus_{i=0}^{m}(\tau_i + 2\Lambda)$. Thus we can extend Theorems 2.3 and 2.4 for general case.

We focus on type $A_1$ because, by using the following theorem we have any extended affine root system is a union of extended affine root systems of type $A_1$.

Theorem 2.7. *Let $R$ be an extended affine root system of type $X$ of nullity $\nu$ and $\alpha \in R^\times$. Set $S_\alpha := \{\sigma \in \mathcal{V}^0 \mid \alpha + \sigma \in R\}$ and $R_\alpha := (S_\alpha + S_\alpha) \cup (\pm\alpha + S_\alpha)$. Then $R_\alpha$ is an extended affine root system of type $A_1$ of nullity $\nu$ and $R = \cup_{\alpha \in R^\times} R_\alpha$.*

Corollary 2.8. *Let $R$ be an extended affine root system of type $X \neq BC_1$ and nullity $\nu > 1$, with extended affine Weyl group $\mathcal{W}$ and assume $\Pi \subseteq R^\times$ such that $S_\Pi$ is a generating set of $\mathcal{W}$. Then there exist geometric reflections in $S_\Pi$, which are not Cayley graph-reflections on* $\mathrm{Cay}(\mathcal{W}, S_\Pi)$.

## 3. Examples

This section is devoted to some examples elaborating on the results in the previous sections.

Example 3.1. The following graphs in Figure 1, show the Cayley graphs of extended affine Weyl groups of nullities $\nu = 0, 1, 2$, respectively. The normalized darts of some geometric reflections show in dashed lines.

Example 3.2. This example extends Example 3.1 to simply laced extended affine Weyl groups of rank and nullity $> 1$, namely it shows that any geometric reflection corresponding to the considered underlying reflectable base, is not a Cayley graph-reflection. To show this, let $R$ be an extended affine root system of simply laced type $X$, rank $\ell > 1$ and nullity $\nu > 1$. We know that $R = \dot{R} + \Lambda$ where $\dot{R}$ is an irreducible finite root system of type $X$ and $\Lambda$ is a lattice of rank $\nu$. We fix a basis $\dot{\Pi} = \{\alpha_1, \ldots, \alpha_\ell\}$ of $\dot{R}$ and a $\mathbb{Z}$-basis $\{\sigma_1, \ldots, \sigma_\nu\}$ of $\Lambda$. Set $\alpha := \alpha_i$ for some $1 \leq i \leq \ell$ and fix it. From [2, Lemma 4.24] (also see [4, Lemma 1.21(i)]), we know that

$$\Pi(X) := \{\alpha_1, \ldots, \alpha_\ell, \sigma_1 - \alpha, \ldots, \sigma_\nu - \alpha\},$$

is a reflectable base for $R$. We set $\sigma_0 = 0$, and

$$S := \cup_{i=0}^{\nu}(\sigma_i + 2\Lambda) \text{ and } R_b = (S + S) \cup (\pm\alpha + S).$$

Then $S$ is a semilattice in $\Lambda$, and $R_b$ is an extended affine root system of type $A_1$ and nullity $\nu$. By Remark 2.6, $\Pi_b := \{\alpha, \sigma_1 - \alpha, \ldots, \sigma_\nu - \alpha\}$ is a reflectable base for $R_b$. We denote the Weyl group of $R_b$ by $\mathcal{W}_b$. Since $\mathcal{W}_b \subseteq \mathcal{W}$ and $\Pi_b \subseteq \Pi(X)$, the Cayley graph $\Gamma_b := \mathrm{Cay}(\mathcal{W}_b, S_{\Pi_b})$ is a subgraph of the Cayley graph $\Gamma := \mathrm{Cay}(\mathcal{W}, S_{\Pi(X)})$. Since $\nu > 1$, we see from Theorem 2.4 that for $\beta \in \Pi_b$ the geometric reflection $w_\beta$ is not a Cayley graph-reflection of $\Gamma_b$. It is easy to see that $w_\beta$ is not a Cayley graph reflection of $\Gamma$, too.

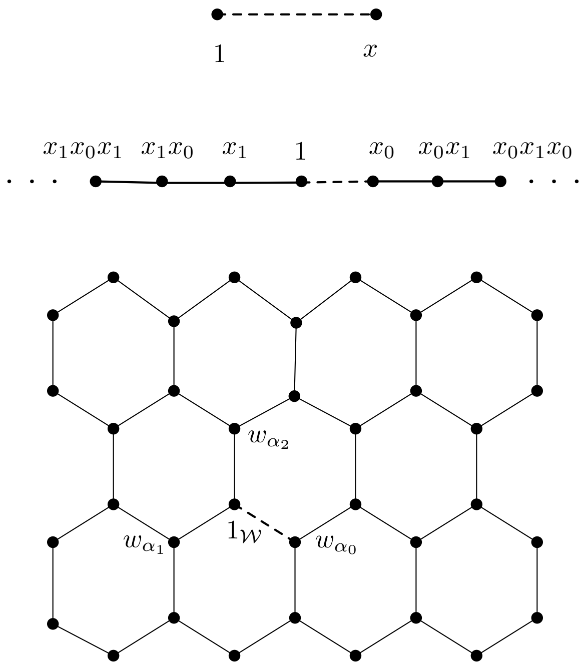



FIGURE 1. The Cayley graphs of extended affine Weyl groups, type $A_1$.

## References

1. B. Allison, S. Azam, S. Berman, Y. Gao and A. Pianzola, *Extended affine Lie algebras and their root systems*, Mem. Amer. Math . Soc. **126** (603) (1997) 1–122.
2. S. Azam, *Extended affine Weyl groups*, J. Algebra **214** (2) (1999) 571–624.
3. S. Azam, *Extended affine root systems*, J. Lie Theory **12** (2) (2002) 515–527.
4. S. Azam, H. Yamane and M. Yousofzadeh, *Reflectable bases for affine reflection systems*, J. Algebra **371** (2012) 63–93.
5. N. Bourbaki, *Lie Groups and Lie Algebras: Chapters* $4-6$ *(Elements of Mathematics)*, Springer, Berlin, 2002.
6. R. Gramlich, G. W. Hofmann and K. H. Neeb, *Semi-edges, reflections and Coxeter groups*, Tran. Amer. Math. Soc. **359** (8) (2007) 3647–3668.
7. A. Malnic, R. Nedela and M. Skoviera, *Lifting graph automorphisms by voltage assignments*, Eur. J. Combin. **21** (7) (2000) 927–947.

E-mail: azam@ipm.ir
E-mail: f.parishani93@sci.ui.ac.ir

# On the $t-$Nacci Sequences of Some Finite Groups of Nilpotency Class Two

Mansour Hashemi
Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran
and Mina Pirzadeh*
Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran

Abstract. We consider finite groups $H_m$ and $G_{mn}$ as follows:

$$H_m = \left\langle x, y | x^{m^2} = y^m = 1, y^{-1}xy = x^{1+m} \right\rangle,\ m \geq 2,$$
$$G_{mn} = \langle x, y | x^m = y^n = 1, [x, y]^x = [x, y], [x, y]^y = [x, y] \rangle\ \ m, n \geq 2.$$

In this paper, we first study the groups $H_m$ and $G_{mn}$. Then by using the properties of $H_m$, $G_{mm}$ and $t-$nacci sequences in finite groups, we show that the period of $t-$nacci sequences in these groups are a multiple of Wall number $K(t, m)$.

**Keywords:** Finite group, Nilpotent groups, $t-$Nacci sequences, Wall number.
**AMS Mathematical Subject Classification [2010]:** 05C25, 20F05, 20D60.

## 1. Introduction

Fibonacci numbers $F_n$ are defined by the recurrence relation $F_0 = 0, F_1 = 1, F_n = F_{n-2} + F_{n-1}, n \geq 2$. For any given integer $t \geq 2$, the $t-$step Fibonacci sequence $F_n(t)$ is defined [5] by the following recurrence formula:

$$F_n(t) = F_{n-t}(t) + F_{n-t+1}(t) + \cdots + F_{n-1}(t),$$

with initial conditions $F_0(t) = 0, F_1(t) = 0, \ldots, F_{t-2}(t) = 0$ and $F_{t-1}(t) = 1$. Reducing the $t-$step Fibonacci sequence $F_n(t)$ by a modulus $m$, we can get a periodic sequence defined by $F_n(t, m) = F_n(t) \pmod{m}$. Following Wall [1], one may also prove that $F_n(t, m)$ is periodic sequence. We use $K(t, m)$ to denote the minimal length of the period of the sequence $F_n(t, m)$ and call it Wall number of $m$ with respect to $t-$step Fibonacci sequence. For example, for

$$\{F_n(4)\}_{n=0}^{n=\infty} = \{0, 0, 0, 1, 1, 2, 4, 8, 15, 29, \ldots\},$$

by considering

$$\{F_n(4) \bmod 2\}_{n=0}^{n=\infty} = \{0, 0, 0, 1, 1, \underline{0}, 0, 0, 1, 1, \ldots\},$$

we get $K(4, 2) = 5$.

We now introduce a generalization of Fibonacci sequence in finite groups which first presented in [5] by Knox.

*Speaker

Definition 1.1. Let $j \leq t$. A $t-$nacci sequence in a finite group is a sequence of group elements $x_0, x_1, \ldots, x_n, \ldots$ for which, given an initial set $\{x_0, x_1, \ldots, x_{j-1}\}$, each element is defined by

$$x_n = \begin{cases} x_0 x_1 \ldots\ x_{n-1}, & j \leq n < t, \\ x_{n-t} x_{n-t+1} \ldots\ x_{n-1}, & n \geq t. \end{cases}$$

Note that the initial set $\{x_0, x_1, \ldots, x_{j-1}\}$, generate the group. The $t-$nacci sequence of $G$ with seed in $X = \{x_0, x_1, \ldots, x_{j-1}\}$ is denoted by $F_t(G; X)$ and its period is denoted by $LEN_t(G; X)$, see [3].

Now, we consider

$$\begin{aligned} H_m =& \left\langle x, y | x^{m^2} = y^m = 1, y^{-1}xy = x^{1+m} \right\rangle,\ m \geq 2, \\ G_m =& G_{mm} = \langle x, y | x^m = y^m = 1, [x, y]^x = [x, y], [x, y]^y = [x, y] \rangle,\ m \geq 2. \end{aligned}$$

For every $t \geq 3$, to study the $t-$nacci sequences of $H_m$ and $G_m$, we define the sequences $\{T_n(t)\}_0^\infty$ and $\{g_n(t)\}_0^\infty$ of numbers, respectively, as follows:

$$\begin{aligned} T_0(t) =& T_0(t-1), \ldots, T_t(t) = T_t(t-1), T_{t+1}(t) = F_{n+t-4}(t-1) + T_{t+1}(t-1), \\ T_n(t) =& T_{n-t}(t) + T_{n-t+1}(t) + \cdots + T_{n-1}(t) \\ & + F^2_{n+t-4}(t) + F^2_{n+t-5}(t) \\ & \vdots \\ & + F^2_{n-2}(t) - F_{n-3}(t) \left( F_{n+t-2}(t) - F_{n+t-3}(t) \right),\ n > t+1. \end{aligned}$$

$$\begin{aligned} g_0(t) =& g_1(t) = g_2(t) = 0, g_3(t) = g_3(t-1), \ldots, g_{t+1}(t) = g_{t+1}(t-1), \\ g_n(t) =& g_{n-t}(t) + g_{n-t+1}(t) + g_{n-t+2}(t) + \cdots + g_{n-1}(t) \\ & + F_{n-3}(t)(F_{n-1}(t) - F_{n-2}(t)) \\ & + (F_{n-3}(t) + F_{n-2}(t))(F_n(t) - F_{n-1}(t)) \\ & + (F_{n-3}(t) + F_{n-2}(t) + F_{n-1}(t))(F_{n+1}(t) - F_n(t)) \\ & + (F_{n-3}(t) + F_{n-2}(t) + F_{n-1}(t) + F_n(t))(F_{n+2}(t) - F_{n+1}(t)) \\ & \vdots \\ & + (F_{n-3}(t) + \cdots + F_{n+t-5}(t))(F_{n+t-3}(t) - F_{n+t-4}(t)),\ n > t+1. \end{aligned}$$

The 2−nacci length and 3−nacci length of $H_m$ and $G_m$ were investigated in [2, 3]. In this paper, we study the $t-$nacci sequences of $H_m$ and $G_m$. In Section 2, we state some preliminary results which are needed for proving our main results. In Section 3, we generalize 3−nacci sequences idea to $t-$nacci sequences ($t \geq 4$).

## 2. Some Preliminaries

We have collected the technical results that lead to the main results of this Section. For given integers $m \geq 2$ and $t \geq 4$, let $F_i = F_i(t, m), K(m) = K(t, m)$. Then we have the following results:

Lemma 2.1. *For integers $n, i$ and $m$ with $m \geq 2$, we have*

(i) $F_{K(m)+i} \equiv F_i \pmod{m}$,

(ii) $F_{nK(m)+i} \equiv F_i \pmod{m}$.

Corollary 2.2. *For integers $n$ and $m \geq 2$, if*

$$\begin{cases} F_n & \equiv 0 \pmod{m}, \\ \vdots & \vdots \quad \vdots \\ F_{n+t-2} & \equiv 0 \pmod{m}, \\ F_{n+t-1} & \equiv 1 \pmod{m}. \end{cases}$$

*Then $K(m)\,|n$.*

We need some results concerning the groups $H_m$ and $G_{mn}$.

Lemma 2.3. *If $G$ is a group and $G' \subseteq Z(G)$, then the following hold for every integer $k$ and $u, v, w \in G$ where $[u, v] = u^{-1}v^{-1}uv$ denotes the commutator of $u$ and $v$:*

(i) $[uv, w] = [u, w][v, w]$ *and* $[u, vw] = [u, v][u, w]$.
(ii) $[u^k, v] = [u, v^k] = [u, v]^k$.
(iii) $(uv)^k = u^k v^k [v, u]^{k(k-1)/2}$.

Corollary 2.4. *Let $G = H_m$. Then*

(i) *every element of $H_m$ can be uniquely presented by $y^r x^s$, where $0 \leq r \leq m-1$ and $0 \leq s \leq m^2 - 1$.*
(ii) $|G| = m^3$.
(iii) $x^s y^r = y^r x^{s+mrs}$.

The following propositions are of interest to consider and one may see the proof in [2].

Proposition 2.5. *Let $G = H_m$. Then $Z(G) = G' \simeq \langle z | z^m = 1 \rangle$.*

Proposition 2.6. *Let $G = G_{mn}$ Then*

(i) $G' = \langle [a, b] \rangle$.
(ii) *Every element of $G$ is in the form $x^i y^j g$, where $0 \leq i \leq m-1$, $0 \leq j \leq n-1$ and $g \in G'$.*
(iii) $Z(G) = \langle x, y, z | x^{m/d} = y^{n/d} = z^d = [x, y] = [x, z] = [y, z] = 1 \rangle$.

For the particular case, consider $m = n$ then for $m \geq 2$ we get

$$G_m = G_{mm} = \langle x, y | x^m = y^m = 1, [x, y]^x = [x, y], [x, y]^y = [x, y] \rangle .$$

Corollary 2.7. *With the above facts, we have*

(i) $|G_m| = m^3, Z(G_m) = G'_m, |Z(G_m)| = m$.
(ii) *Every element of $G_m$ can be written uniquely in the form $x^r y^s [y, x]^t$, where $0 \leq r, s, t \leq m-1$.*

## 3. The $t$−Nacci Sequences of $H_m$ and $G_m$

In this section, we examine the $t$−nacci sequences of $H_m$ and $G_m$ with respect to the ordered generating set $X = \{x, y\}$. First, we show that every element of $F_t(G; X)$ has a standard form. The following Lemma is of interest to consider and one may see the proof in [4].

Lemma 3.1. *For every $t \geq 4$ and $n \geq 3$, each element $x_n(t)$ of the $t-$nacci sequences of groups $H_m$ can be written in the form*

$$x_n(t) = y^{F_{n+t-3}(t)} x^{F_{n+t-2}(t) - F_{n+t-3}(t) + mT_n(t)}.$$

We denote the period of $F_t(H_m; x, y)$ by $P$, *i.e.* $P_t(H_m; x, y) = P$ and we have the following corollary:

Corollary 3.2. *For every $m \geq 2$, $K(t, m) | P$.*

In what follow, we study the $t-$nacci sequence of $G_m$.

Theorem 3.3. *For every $t \geq 4$ and $n \geq 3$, each element $x_n$ of the $t-$nacci sequences of groups $G_m$ can be written in the form*

$$x_n(t) = x^{F_{n+t-2}(t) - F_{n+t-3}(t)} y^{F_{n+t-3}(t)} [y, x]^{g_n(t)}.$$

Theorem 3.4. *If $LEN_t(G_m; X) = P$ then the equations*

$$\begin{cases} F_P & \equiv 0 \pmod{m}, \\ \vdots & \vdots \quad \vdots \\ F_{P+t-2} & \equiv 0 \pmod{m}, \\ F_{P+t-1} & \equiv 1 \pmod{m}. \end{cases}$$

*hold. Moreover, $K(t, m)$ divides $P$.*

Here, by using a program written in the computer algebra system GAP [6], we checked that for every $t = 3, 4$ and $2 \leq m \leq 10$

$$LEN_t(H_m) = K(t, m^2).$$

Also, for every prime number $p$

$$LEN_t(G_p) = \begin{cases} 2K(t, p), & p = 2, \\ K(t, p), & p \neq 2. \end{cases}$$

Note that this formula, may be generalized for $n = p_1^{\alpha_1} \dots p_s^{\alpha_s}$; i.e. in this case we have $LEN_t(G_n) = l.c.m\{LEN_t(G_{p_1^{\alpha_1}}), \dots, LEN_t(G_{p_1^{\alpha_s}})\}$. Some of these results are shown below:

Table 1: The period of $t-$nacci sequences of $H_m$.

| $m$ | $LEN_3(H_m)$ | $K(3, m^2)$ | $LEN_4(H_m)$ | $K(4, m^2)$ |
|---|---|---|---|---|
| 2 | 8 | $K(3, 2^2)$ | 10 | $K(4, 2^2)$ |
| 3 | 39 | $K(3, 3^2)$ | 78 | $K(4, 3^2)$ |
| 4 | 32 | $K(3, 4^2)$ | 40 | $K(4, 4^2)$ |
| 5 | 155 | $K(3, 5^2)$ | 1560 | $K(4, 5^2)$ |
| 6 | 312 | $K(3, 6^2)$ | 390 | $K(4, 6^2)$ |
| 7 | 336 | $K(3, 7^2)$ | 2394 | $K(4, 7^2)$ |
| 8 | 128 | $K(3, 8^2)$ | 160 | $K(4, 8^2)$ |
| 9 | 351 | $K(3, 9^2)$ | 702 | $K(4, 9^2)$ |
| 10 | 1240 | $K(3, 10^2)$ | 1560 | $K(4, 10^2)$ |

Table 2: The period of $t-$nacci sequences of $G_m$ code.

| $m$ | $LEN_3(G_m)$ | $K(3,m)$ | $LEN_4(G_m)$ | $K(4,m)$ |
|---|---|---|---|---|
| 2 | 8 | 4 | 10 | 5 |
| 3 | 13 | 13 | 26 | 26 |
| 4 | 16 | 8 | 20 | 10 |
| 8 | 32 | 16 | 40 | 20 |

**Acknowledgement**

The authors would like to thank reviewers for the reading and their useful comments in this paper.

## References

1. D. D. Wall, *Fibonacci series modulo m*, Amer. Math. Monthly. **67** (6) (1960) 525–532.
2. H. Doostie and M. Hashemi, *Fibonacci lenghts involving the wall number $k(n)$*, J. Appl. Math. Computing. **20** (2006) 171–180.
3. M. Hashemi and E. Mehraban, *On the $3-$nacci sequences of finitely generated groups*, Menemui Matematik. **37** (2015) 20–27.
4. M. Hashemi and M. Pirzadeh, *$t-$nacci sequences in some special groups of order $m^3$*, Math. Reports. **20(70)** (4) (2018) 389–399.
5. S. W. Knox, *Fibonacci sequences in finite groups*, Fibonacci Quarterly. **30** (1992) 116–120.
6. The GAP Group, GAP-Groups, Algorithms, and Programming, Version 4.4.12. http://www.gap-system.org.

E-mail: m_hashemi@guilan.ac.ir
E-mail: m.pirzadeh.math@gmail.com

# When Gelfand Rings are Clean

Esmaeil Rostami*
Faculty of Mathematics and Computer & Mahani Mathematical Research Center,
Shahid Bahonar University of Kerman, Kerman, Iran

Abstract. In this paper, we consider a special class of ideals of a commutative ring called "lifting ideals"and comaximal factorizations of ideals of a ring into this class of ideals. Then by using Pierce stalks we characterize the Gelfand rings whose ideals can be written as a product of comaximal lifting ideals. Finaly, we characterize completely regular topological spaces $X$ such that $C(X)$ is a clean ring.
**Keywords:** Lifting idempotents, Gelfand rings, Clean rings.
**AMS Mathematical Subject Classification [2010]:** 13A15.

## 1. Introduction

Throughout this paper, all rings are assumed to be commutative with identity. Over the years man authors were interested and have investigated some class of ring such as clean and Gelfand rings. Also, "lifting idempotents" is an example of tichniques that are important in rings. Nicholson in [9] studied lifting idempotents in a noncommutative ring. He showed that idempotents of a (*exchange*) clean ring $R$ can be lifted by each ideal (teft ideal for nonmummutative cases) of $R$. Also he showed that the converse of this result holds when its idempotents are central. Note that a ring $R$ is called *clean* if every element of $R$ is the sum of a unit element and an idempotent element. We recall that a ring $R$ is called a *Gelfand ring* if whenever $x+y=1_R$ there are $a,b\in R$ with $(1+ax)(1+by)=0$. Moreover, a ring is $R$ called a *pm-ring* if each prime ideal is contained in a unique maximal ideal. It was proved that a commutative ring is Gelfand if and only if it is pricesely a *pm*-ring, see [7].

Representing ideals, modules, and subring of a ring as a direct sum, sum, an intersection, or direct product of a special class of algebraic structure is an important problem.

Representation of ideals, modules, and subring as a product and intersection are more interesting. Noether's is a beginer in this topic and we can traced back some important works in her papers, for example she proved that in Noetherian rings each proper ideal has a (completely, up to order) unique comaximal factorization.

McAdam-Swan, Hedayat-Rostami, in [6, Section 5] and [4, 5], respectively considered and studed these type of factorizations and decompositions

In this paper, for a generalization of McAdam-Swan, Hedayat-Rostami, in [6, Section 5] and [4, 5], respectively, we will define and consider some special class of ideals of ideals and rings rings called "lifting ideals"and "lifting rings". Also,

*Speaker

we consider some factorizations of proper ideals into this a product and other reoesentations into of some ideals. And by useing Pierce stalks we characterize the Gelfand rings whose ideals are of some important shape ideals.

## 2. Main Results

Lifting idempotents modulo ideals and module over rings (not necessarily commutative) is a technique employed in the proofs of most of the results concerning clean rings, locally compact (commutative or not) rings, and top strongly clean rings, see [10].

Hence we consider, study and generalized a special type of ideals in a (noncommutative or commutative) ring called “lifting ideals”.

Definition 2.1. We recall an ideal $I$ is called *a lifting ideal* if each idempotent of $R/I$ can be lift to an idempotent of $R$.

Let $R$ be a ring and $I$ be an ideal of $R$. The ideal $I$ is said to have a *comaximal factorization* if there are proper ideals $I_1, \dots, I_n$ of $R$ such that $I = I_1 \dots I_n$ and $I_i + I_j = I$, when $i \neq j$. McAdam-Swan, Hedayat-Rostami, in [6, Section 5] and [4, 5], respectively considered and studed these type of factorizations and decompositions.

Definition 2.2. Let $R$ be a ring and $I$ be an ideal of $R$. We say that $I$ is *ideal* (LFCCII) if it has a lifting comaximal factorization. A ring $R$ is called a *LFCCR* whenever each proper ideal is a LFCCII.

In the following we consider maximal and prime (maxspectrume and spectrume) of a rinf $R$ with the zariski topology.

McGovern in [7], give a list of the following four equivalent algebric and geometric conditions for a ring.

Set $T' := \langle \{e \in T \mid e^2 = e\} \rangle$ for an ideal $T$. Now let $ID(R) := \{T' \mid T \trianglelefteq R\}$ Clearly $ID(R)$ is non-empty and $ID(R)$ contains maximal elements. The maximal elements of $ID(R)$ are precisely $\mathfrak{m}'$, for $\mathfrak{m}$ is a prime or (maximal) ideal of $R$ by [8, Proposition 3.2]. The ring $R/\mathfrak{m}'$ is called a *Pierce stalk* of $R$ for each maximal (or prime) ideal $\mathfrak{m}$ of $R$. See [8] for more information.

Now we have the following proposition.

Proposition 2.3. *Let $R$ be a Gelfand LFCCRR. Then its Pierce stalks are semilocal.*

Proof. Since Pierce stalks of any ring are indecomposable, we have the Pierce stalks of an LFCCRR are rings whose proper ideals have complete comaximal factorizations. Now since $R$ is Gelfand, the Pierce stalks of $R$ are semilocal by [5, Proposition 4.6]. □

Proposition 2.4. *Let $X$ be any topological space and $Y$ be any Hausdorff subspace of $X$ such that for every connected component $C$ of $X$ the set $C \cap Y$ is a finite set. Then for every connected component $A$ of $Y$, we must have $|A| = 1$. In particular, $Y$ is totally disconnected.*

Proof. Let $A$ be any connected component of the topological space $Y$. Then $A$ is connected in $X$. So there is a connected component $C$ of $X$ such that $A \subseteq C$.

By assumption, since $A \subseteq C \cap Y$, $A$ must be finite and since $Y$ is Hausdorff, $A$ has exactly one element. So $|A| = 1$ and $Y$ is totally disconnected. □

By [8, Proposition 3.2], every connected component of $Spec(R)$ as a (unnecessarily Hausdorff) topological space is homeomorphic to $Spec(R/\mathfrak{m}')$. Now we have the following theorem.

Recall that a complete comaximal factorization for an ideal of a ring is a comaximal factorization whose factors are indecomposable quotient.

Theorem 2.5. *Let $R$ be a Gelfand or pm-ring. Then $R$ is a LFCCRR if and only if it is a clean ring.*

Proof. (⇒). By [2, Proposition 1.2] since $R$ is a Gelfand ring, Max($R$) is Hausdorff as a subspace of Spec($R$). Now by Proposition 2.3, the Pierce stalks of $R$ are semilocal. Thus by Proposition 2.4, each connected component of the commutative ring Spec($R$) has a unique maximal ideal. Therefore by [1, Proposition 1.2], $R$ is a clean ring.

(⇐). If $R$ is a clean ring, then every ideal of $R$ is a lifting ideal and so $R$ is a LFCCRR. □

Now for a completely regular topological space $X$ we bring the following theorem for when $C(X)$ is LFCCRR, that is, in the last theorem of this paper, we consider a completely regular topological space $X$ such that $C(X)$ is an LFCCRR.

Theorem 2.6. *Let $X$ be completely regular topological space. Then $C(X)$ is clean if and only if it is an LFCCRR.*

Proof. By the reference [3, Theorem 2.11], the mentioned ring is a Gelfand (pm-) ring. Thus, the result follows from Theorem 2.5. □

## References


1. W. D. Burgess and W. Stephenson, *Rings all of whose Pierce stalks are local*, Canad. Math. Bull. **22** (1979) 159–164.
2. M. Contessa, *On pm-rings*, Comm. Algebra **10** (1982) 93–108.
3. L. Gillman and M. Jerison, *Rings of Continuous Functions*, Springer-Verlag, New York, 1976.
4. S. Hedayat and E. Rostami, *A characterization of commutative rings whose maximal ideal spectrum is Noetheria*, J. Algebra Appl. **17** (1) (2018) 1850003.
5. S. Hedayat and E. Rostami, *Decomposition of ideals into pseudo irreducible ideals*, Comm. Algebra **45** (2017) 1711–1718.
6. S. McAdam and R. G. Swan, *Unique comaximal factorization*, J. Algebra **276** (2004) 180–192.
7. W. Wm. McGovern, *Neat rings*, J. Pure Appl. Algebra **205** (2006) 243–265.
8. A. R. Magid, *The Separable Galois Theory of Commutative Rings*, Pure and Applied Mathematics, No. 27. Marcel Dekker, Inc., NewYork, 1974.
9. W. K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc. **229** (1977) 269–278.
10. W. K. Nicholson and Y. Zhou, *Strong lifting*, J. Algebra **285** (2005) 795–818.


E-mail: e_rostami@uk.ac.ir

# Some Properties of Generalized Groups

Mohammad Reza Ahmadi Zand
Faculty of Mathematical Sciences, University of Yazd, Yazd, Iran
and Salimeh Rostami*
Department of Mathematics, University of Yazd, Yazd, Iran

Abstract. In this paper, we study some properties of generalized groups and generalized normal subgroups. Moreover, we recall the notion of relativization in resolvability and irresolvability of topological space and obtain an important results about them.
**Keywords:** Generalized groups, Normal generalized groups, Generalized normal subgroups, Resolvable relative to $X$.
**AMS Mathematical Subject Classification [2010]:** 22F05, 54-XX.

## 1. Introduction

Generalized groups are an interesting extension of groups. This notion was first introduced by Molaei in [5]. A generalized group is a nonempty set $G$ admitting an operation called multiplication, which satisfies the following conditions:

1) $(xy)z = x(yz)$ for all $x, y, z \in G$,
2) for each $x \in G$ there exists a unique element $z \in G$ such that $zx = xz = x$ (we denote $z$ by $e(x)$),
3) for each $x \in G$ there exists an element $y \in G$ called inverse of $x$ such that $xy = yx = e(x)$.

It is well known that each $x$ in $G$ has a unique inverse in $G$, the inverse of $x$ is denoted by $x^{-1}$ [5]. Moreover, for a given $x \in G$, $e(e(x)) = e(x)$, $(x^{-1})^{-1} = x$ and $e(x^{-1}) = e(x)$.

Definition 1.1. [3] If $G$ and $H$ are two generalized groups, then a map $f : G \to H$ is called a homomorphism if $f(ab) = f(a)f(b)$ for all $a, b \in G$.

Theorem 1.2. [3] *Let $f : G \to H$ be a generalized group homomorphisms. Then*

1) *$f(e(a)) = e(f(a))$, is an identity element in $H$ for all $a \in G$;*
2) *$f(a^{-1}) = (f(a))^{-1}$, for all $a \in G$;*
3) *if $K$ is a generalized subgroup of $G$, then $f(K)$ is a generalized subgroup of $H$;*
4) *if $D$ is a generalized subgroup of $H$ and $f^{-1}(D) \neq \emptyset$, then $f^{-1}(D)$ is a generalized subgroup of $G$.*

Definition 1.3. [3] A generalized group $G$ is called a normal generalized group if $e(ab) = e(a)e(b)$ for all $a, b \in G$.

*Speaker

REMARK 1.4. For every $a$, $b$ belong to a generalized group $G$ we have $e(e(a)e(b)) = e(ab)$ [1].

DEFINITION 1.5. [3] A nonempty subset $H$ of a generalized group $G$ is called a generalized subgroup if, it is a generalized group under the operations of $G$.

THEOREM 1.6. [3] *If $G$ is a generalized group and $a \in G$, Then $G_a = \{x \in G : e(x) = e(a)\}$ is a generalized subgroup of $G$. In fact, $G_a$ is a group.*

DEFINITION 1.7. [3] A generalized subgroup $N$ of a generalized group $G$ is called a generalized normal subgroup if there exist a generalized group $E$ and a homomorphism $f : G \to E$ such that for all $a \in G$ we have $N_a = \emptyset$ *or* $N_a = ker(f_a)$, where $N_a := N \cap G_a$ , $f_a := f|_{G_a}$ and $ker(f_a) = \{x \in G_a \ : \ f(x) = f(e(a))\}$.

## 2. Main Results

PROPOSITION 2.1. *If $f : G \to H$ is a generalized groups homomorphisms and $G$ is a normal generalized group, then $f(G)$ is a normal generalized subgroup of $H$.*

PROOF. We know that for all $f(x), f(y) \in f(G)$ it follows that $e(f(x)f(y)) = e(f(xy)) = f(e(xy)) = f(e(x)e(y)) = f(e(x))f(e(y)) = e(f(x))e(f(y))$. So, $f(G)$ is a normal generalized subgroup of $H$. □

PROPOSITION 2.2. *Let $G$ be a generalized group in which $e(a)b = be(a)$ for any $a, b \in G$. Then $G$ is a normal generalized group and even more, $(ab)^{-1} = b^{-1}a^{-1}$.*

PROOF. We know that $ab = abe(b)$ and by assumption, $e(b)ab = ab$. So $e(ab) = e(b)$. Similarly, we obtain $e(ab) = e(a)$. Then, $G$ is a group and proof is complete. In fact, we show more than it was claimed. □

PROPOSITION 2.3. *$G$ is a normal generalized group if and only if $e(x)e(y)e(x) = e(x)$ for every $x, y \in G$.*

PROOF. It's clear that $e(x)e(y)e(x) \in G_{e(x)}$ for every $x, y \in G$. Since $G$ is normal generalized group, we have

$$\begin{aligned}(e(x)e(y)e(x))(e(x)e(y)e(x)) &= e(x)e(y)e(x)e(y)e(x)\\ &= e(xy)e(xy)e(x)\\ &= e(xy)e(x)\\ &= e(x)e(y)e(x).\end{aligned}$$

Then, $e(x)e(y)e(x)$ is an idempotent element of the group $G_{e(x)}$ and so, $e(x)e(y)e(x) = e(x)$. Conversely, let $e(x)e(y)e(x) = e(x)$ for every $x, y \in G$. Then we have

$$(e(x)e(y))(e(x)e(y)) = (e(x)e(y)e(x))e(y) = e(x)e(y).$$

Since $e(e(x)e(y)) = e(xy)$, So $e(x)e(y)$ is an idempotent element of the group $G_{e(xy)}$. Now, it is obvious that $e(x)e(y) = e(xy)$ and $G$ is a normal generalized group. □

Proposition 2.4. *If $A$ and $B$ are generalized normal subgroups of $G$, then $A \cap B$ is also a generalized normal subgroup of $G$.*

Proof. Since $A$ and $B$ are generalized normal subgroups of $G$, there exist generalized groups homomorphisms $f : G \to E$ and $g : G \to F$, respectively, such that for every $a \in G$

$$A_a = \emptyset \ or \ A_a = ker(f_a),$$

and

$$B_a = \emptyset \ or \ B_a = ker(g_a).$$

Now consider mapping $h : G \to E \times F$ defined by $x \mapsto (f(x), g(x))$. $h$ is direct product of two maps $g$ and $h$ and so, it is a generalized groups homomorphism. It is clear to see that , if $(A \cap B)_a \neq \emptyset$, then $(A \cap B)_a = A_a \cap B_a = ker(f_a) \cap ker(g_a) = ker(h_a)$. Therefore, $A \cap B$ is a generalized normal subgroup of $G$. □

Proposition 2.5. *Let $f : G \to H$ be a onto homomorphism between generalized groups and $N$ is a generalized normal subgroups of $H$. Then $f^{-1}(N)$ is a generalized normal subgroup of $G$.*

Proof. Since $N$ is a generalized normal subgroup of $H$, there exists a generalized groups homomorphism $g : H \to E$ such that for every $b \in H$, $N_b = \emptyset$ or $N_b = ker(g_b)$. Suppose the mapping $gof : G \to E$. $gof$ is a homomorphism. Let $(f^{-1}(N))_a \neq \emptyset$, then $(f^{-1}(N))_a = \{x \in G_a \mid f(x) \in N\}$. Since $x \in G_a$, so $e(f(x)) = f(e(x)) = f(e(a)) = e(f(a))$. In the following we have

$$\begin{aligned}(f^{-1}(N))_a &= \{x \in G_a \mid f(x) \in N_{f(a)} = ker(g_{f(a)})\\ &= \{x \in G_a \mid g(f(x)) = g(e(f(a)))\}\\ &= \{x \in G_a \mid (gof)(x) = (gof)(e(a))\}\\ &= ker(gof)_a.\end{aligned}$$

Therefore, $f^{-1}(N)$ is a generalized normal subgroup of $G$. □

Proposition 2.6. *Normality is preserved on taking direct product, i.e. if $A$ is a generalized normal subgroup of $G$ and $B$ is a generalized normal subgroup of $H$, then $A \times B$ is a generalized normal subgroup of $G \times H$.*

Proof. Since $A$ is a generalized normal subgroup of $G$, there exists a generalized groups homomorphism $f : G \to E_1$ such that, $A_a = \emptyset$ or $A_a = ker(f_a)$. Since $B$ is a generalized normal subgroup of $H$, there exists a generalized groups homomorphism $g : H \to E_2$ such that, $B_b = \emptyset$ or $B_b = ker(g_b)$. Now suppose the mapping $l : G \times H \to E_1 \times E_2$ defined by $(x, y) \mapsto (f(x), g(y))$. It is clear that $l$ is a generalized groups homomorphism. if for $(a, b) \in G \times H$, $(A \times B)_{(a,b)} \neq \emptyset$, then we have

$$\begin{aligned}(A \times B)_{(a,b)} = (A \times B) \cap (G_a \times H_b) = (A \cap G_a) \times (B \cap H_b) &= A_a \times B_b\\ &= ker(f_a) \times ker(g_b)\\ &= ker(l_{(a,b)}).\end{aligned}$$

So $A \times B$ is a generalized normal subgroup of $G \times H$. □

## 3. Resolvability of Topological Generalized Groups

E. Hewitt in 1943 [2] introduced the notion of resolvability. He defined a topological space $X$ is resolvable if it can be represented as the union of two disjoint dense sets, otherwise it is irresolvable. In the same paper [2], it is defined that a space is hereditarily irresolvable if every nonempty subspace of it is irresolvable. We also know that a homogeneous space with a resolvable subspace is itself resolvable [6].

Theorem 3.1. [2] *Every topological space $X$ has the unique representation $X = F \cup E$, where $F$ is closed and resolvable, $E$ is open and hereditarily irresolvable and $F \cap E = \emptyset$. This representation is called the "Hewitt representation" of $X$.*

In the main reference, for the Hewitt representation of a topological space $X$, open and hereditarily irresolvable space is denoted by $G$. But in this paper, we show that by $E$, because we took $G$ for generalized groups.

Sh. Modak in his paper "Relativization in resolvability and irresolvability" [4] in 2011, influenced by the famous mathematician A. Arkhagel'skii, relativized the property of resolvability and irresolvability. In this paper, he states that a nonempty subset $A$ of a topological space $(X, \tau)$ is called *resolvable relative to $X$* or *resolvable in $X$* if there are two dense subsets $D_1$ and $D_2$ of $(X, \tau)$ with $D_1 \cap A \neq \emptyset$, $D_2 \cap A \neq \emptyset$ such that $D_1 \cap D_2 \cap A = \emptyset$; otherwise, it is called *irresolvable relative to $X$* or *irresolvable in $X$*.
In the section 2 of [4], it is mentioned that for $Y \subset X$, resolvability of $Y$ with respect to its relative topology does not necessarily imply resolvability of $Y$ in $X$, and it is also given an example that unfortunately doesn't work for it. In the next proposition, we fail this statement.

Proposition 3.2. *Let $X$ be a topological space. Then every resolvable subset $A$ of $X$ is resolvable relative to $X$(or resolvable in $X$).*

Proof. Suppose that $A \subseteq X$ be resolvable. So, there exist two dense subsets $D_1$ and $D_2$ of $A$ which satisfy $\overline{D_1} = A = \overline{D_2}$ and $D_1 \cap D_2 = \emptyset$. Now, we get $D_{\acute{1}} = D_1 \cup (X - A)$ and $D_{\acute{2}} = D_2 \cup (X - A)$ that satisfy the following conditions:

i) $\overline{D_{\acute{1}}} = X = \overline{D_{\acute{2}}}$.
ii) $D_{\acute{1}} \cap A \neq \emptyset$, $D_{\acute{2}} \cap A \neq \emptyset$.
iii) $D_{\acute{1}} \cap D_{\acute{2}} \cap A = \emptyset$.

Therefore, we can say that $A$ is resolvable relative to $X$. □

This proposition is justified by the following example.

Example 3.3. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\}$. It is clear that

$$C(\tau)(closed\ subsets) = \{\emptyset, X, \{b, c, d\}, \{a, d\}, \{d\}\}.$$

Let $Y = \{b, c, d\} \subset X$. Then $\tau_Y(relative\ topology) = \{\emptyset, Y, \{b, c\}\}$ and $C(\tau_Y) = \{\emptyset, Y, \{d\}\}$. Now, we can see that $\{b\}$ , $\{c, d\}$ with relative topology are dense in $Y$ and $\{b\} \cap \{c, d\} = \emptyset$. Therefore $(Y, \tau_Y)$ is a resolvable space. On the other hand, we have

$$D(X, \tau) = \{X, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}.$$

It is obvious that $D_1 = \{a, b\}$, $D_2 = \{a, c\}$ both are dense in $(X, \tau)$. Also $D_1 \cap Y \neq \emptyset$, $D_2 \cap Y \neq \emptyset$ and $D_1 \cap D_2 \cap Y = \emptyset$. Hence $Y$ is resolvable in $X$.

Note that one can easily verify that for every open subset $Y$ of a topological space $X$, resolvability of $Y$ in $X$ and resolvability of $Y$ with respect to the relative topology are equivalent.

The next result is closely related to [4, Theorem 2.12] and previous proposition.

Proposition 3.4. *Let $X$ be a irresolvable topological space with the Hewitt representation $X = F \cup E$. Then a non-empty homogeneous subset $A$ of $X$ with $intA \neq \emptyset$ is irresolvable if and only if $int(A \cap E) \neq \emptyset$.*

Proof. Suppose that $int(A \cap E) = \emptyset$. Since that $intA \neq \emptyset$, it follows that $intA \subset X - E = F$. The resolvability of $F$ implies that $intA$ is resolvable. Hence $A$ is also resolvable, a contradiction. Thus $int(A \cap E) \neq \emptyset$.
Conversely, suppose that for $A \subset X$ with $intA \neq \emptyset$, $int(A \cap E) \neq \emptyset$. Then by in [4, Theorem 2.12], we have that $A$ is irresolvable in $X$. Now by contraposition of Proposition 3.2, it is obtained that $A$ is irresolvable. □

**Acknowledgement**

The authors thank referees for their careful reading and valuable comments.

## References

1. F. Fatehi and M. R. Molaei, *On completely simple semigroups*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **28** (2012) 95–102.
2. E. Hewitt, *A problem of set-theoretic topology*, Duke Math. J. **10** (2) (1943) 309–333.
3. M. R. Mehrabi, M. R. Molaei and A. Oloomi, *Generalized subgroups and homomorphisms*, Arab J. Math. Sci. **6** (2) (2000) 1–7.
4. S. Modak, *Relativization in resolvability and irresolvability*, Int. Math. Forum **6** (22) (2011) 1059–1064.
5. M. R. Molaei, *Generalized groups*, Buletinul Institului Polithnic Din Iaşi **65** (1999) 21–24.
6. Y. Zelenyuk, *Almost maximal spaces*, Topol. Appl. **154** (2007) 339–357.

E-mail: mahmadi@yazd.ac.ir
E-mail: salimehrostami66@yahoo.com

# Associated Primes of Formal Local Cohomology Modules

Behruz Sadeqi*
Departement of Sciences, Marand Branch, Islamic Azad university, Marand, Iran
and Ebrahim Ejabi
Departement of Sciences, Yazd Branch, Islamic Azad university, Yazd, Iran

Abstract. Let $\mathfrak{a}$ be an ideal of a commutative Noetherian ring $R$ and $M$, a finitely generated $R$-module. In this paper, we proved that if $\operatorname{Supp}\mathfrak{F}^i_{\mathfrak{a}}(M)$ is finite for all $i < t$, then $\operatorname{Ass}(\mathfrak{F}^t_{\mathfrak{a}}(M))$ is finite.
**Keywords:** Formal local cohomology, Associated prime ideals, Cofinitness, Weakly laskerian modules.
**AMS Mathematical Subject Classification [2010]:** 13D45, 13E99.

## 1. Introduction

In this paper, $(R, \mathfrak{m})$ is commutative Noterian local ring with nonzero identity and all modules are finitely generated. Recall that the $i$-th formal local cohomology module of $M$ with respect to $\mathfrak{a}$ is denoted by $\mathfrak{F}^i_{\mathfrak{a}}(M)$ for all $i \in \mathbb{N}_0$ (See [2]). The notions of weakly Laskerian modules were introduced by Divaani-Aazar and Mafi in [1]. An $R$ module $M$ is said to be weakly Laskerian if the set of associated primes of any quotient module of $M$ is finite. Moreover it is closed under taking submodules, quotients and extensions, i.e., it is a Serre subcategory of the category of $R$-modules.

In this paper, we proved that if $\operatorname{Supp}\mathfrak{F}^i_{\mathfrak{a}}(M)$ is finite for all $i < t$, then $\operatorname{Ass}(\mathfrak{F}^t_{\mathfrak{a}}(M))$ is finite. Our terminology and notation of formal local cohomology modules come from [4].

Lemma 1.1. *Let $M$ be an $\mathfrak{a}$-torsion $R$-module: $M = \bigcup_{n=1}^{\infty}(0 :_M \mathfrak{a}^n)$. Then*

$$\operatorname{Ass} M = \operatorname{Ass}(0 :_M \mathfrak{a}).$$

Proof. By $(0 :_M \mathfrak{a}) \subseteq M$ then $\operatorname{Ass}(0 :_M \mathfrak{a}) \subseteq \operatorname{Ass} M$. But $\mathfrak{p} \in \operatorname{Ass} M$ then there is $(0 \neq x) \in M$ such that $\mathfrak{p} = (0 :_M x)$, but$M$ is an $\mathfrak{a}$-torsion $R$-module,then $(0 :_M \mathfrak{a}) \subseteq \bigcup_{n=1}^{\infty}(0 :_M \mathfrak{a}^n)$, and then $\operatorname{Ass} M \subseteq \operatorname{Ass}(0 :_M \mathfrak{a})$. □

Lemma 1.2. *Let $R$ be a ring, and $M$, an $R$-module. If $N$ is submodule of $M$ then $\operatorname{Ass}(M/N) \subseteq \operatorname{Ass} M \cup \operatorname{Supp} N$. In particular, if the set $\operatorname{Supp}(N)$ is finite, then $\operatorname{Ass}(M/N)$ is finite if and only if $\operatorname{Ass} M$ is finite.*

Proof. Let $\mathfrak{p} \in \operatorname{Ass}(M/N) \setminus \operatorname{Supp} N$. So $(0 \neq x) \in M$ such that $\mathfrak{p} = (N :_R x)$; we have $\mathfrak{p}x \subseteq N$. Set $Rad(\operatorname{Ann}(\mathfrak{p}x)) = \bigcap_{i=1}^{n} \mathfrak{q}_i$. Then there exists a posotive integer $t$ such that $(\mathfrak{q}_1 \dots \mathfrak{q}_n)^t \mathfrak{p}x = 0$. Set $\mathfrak{q} := (\mathfrak{q}_1 \dots \mathfrak{q}_n)^t$, then $\mathfrak{q}\mathfrak{p}x = 0$, therefore $\mathfrak{p} \subseteq \operatorname{Ann}(\mathfrak{q}x) \subseteq (N :_R \mathfrak{q}x)$. Let $a \in (N :_R \mathfrak{q}x)$,then $a\mathfrak{q}x \subseteq N$ and $a\mathfrak{q} \subseteq \mathfrak{p}$.Then $a \in \mathfrak{p}$ and $\mathfrak{p} = \operatorname{Ann}(\mathfrak{q}x)$; therefore $\mathfrak{p} \in \operatorname{Ass}(\mathfrak{q}x)$, and hence $\mathfrak{p} \in \operatorname{Ass}(M)$. □

*Speaker

## 2. Main Results

THEOREM 2.1. *Let $(R, \mathfrak{m})$ be a local ring. $M$ is a finitely generated $R$-module. Suppose that there is an integer $t \in \mathbb{N}_0$ such that for all $i < t$ the set* $\mathrm{Supp}(\mathfrak{F}^i_{\mathfrak{a}}(M))$ *is finite. Then* $\mathrm{Ass}(\mathfrak{F}^t_{\mathfrak{a}}(M))$ *is finite.*

PROOF. We proceed by induction on $t$. If $t = 0$, then $\mathfrak{F}^0_{\mathfrak{a}}(M)$, by [3, Lemma 2.1], is artinian, and hence $\mathrm{Ass}(\mathfrak{F}^0_{\mathfrak{a}}(M))$ is finite. So, suppose that $t > 0$. Let $\mathrm{Supp}(\mathfrak{F}^i_{\mathfrak{a}}(M))$ is finite for all $i < t$. We prove(by induction) that $\mathrm{Ass}(\mathfrak{F}^t_{\mathfrak{a}}(M))$ is finite. By [4, Theorem 3.11] there is the exact sequence:

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(M) \longrightarrow M \longrightarrow M/\Gamma\mathfrak{a}(M) \longrightarrow 0,$$

then

$$\cdots \longrightarrow \mathfrak{F}^i_{\mathfrak{a}}(M) \longrightarrow \mathfrak{F}^i_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) \longrightarrow \mathfrak{F}^{i+1}_{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(M)) \longrightarrow \cdots,$$

is exact sequence. But

$$\mathrm{Ass}(\mathfrak{F}^i_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))) \subseteq \mathrm{Ass}(\mathfrak{F}^i_{\mathfrak{a}}(M)) \cup \mathrm{Ass}(\mathfrak{F}^{i+1}_{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(M))).$$

Let $i = t-1$, and hence $\mathrm{Ass}(\mathfrak{F}^{t-1}_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)))$ is finite if and only if $\mathrm{Ass}(\mathfrak{F}^t_{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(M)))$ is finite and $\mathrm{Ass}(\mathfrak{F}^{t-1}_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))) = \mathrm{Ass}(\mathfrak{F}^{t-1}_{\mathfrak{a}}(M))$, Thus there is an $M$-regular element $x \in \mathfrak{a}$. The exact sequence

$$0 \longrightarrow M \xrightarrow{.x} M \longrightarrow M/xM \longrightarrow 0,$$

induces [4, Theorem 3.11] the long exact sequence

$$\cdots \longrightarrow \mathfrak{F}^{t-1}_{\mathfrak{a}}(M) \xrightarrow{.x} \mathfrak{F}^{t-1}_{\mathfrak{a}}(M) \xrightarrow{g} \mathfrak{F}^{t-1}_{\mathfrak{a}}(M/xM) \xrightarrow{f} \mathfrak{F}^t_{\mathfrak{a}}(M) \longrightarrow \cdots.$$

It can be seen that $\mathrm{Supp}(\mathfrak{F}^i_{\mathfrak{a}}(M/xM))$ is finite set for all $i < t$. From induction hypothesis, we deduce $\mathrm{Ass}(\mathfrak{F}^{t-1}_{\mathfrak{a}}(M/xM))$ is finite. By applying Lemma 1.2 to the exact sequence

$$0 \longrightarrow Im\, g \longrightarrow \mathfrak{F}^{t-1}_{\mathfrak{a}}(M/xM) \longrightarrow Im\, f \longrightarrow 0,$$

we deduce that $\mathrm{Ass}(Im\, f)$ is finite. By noting that $Im\, f = (0 :_{\mathfrak{F}^t_{\mathfrak{a}}(M)} x)$ and using Lemma 1.2 the result now follows. □

COROLLARY 2.2. *Let* $\mathrm{Supp}(\mathfrak{F}^i_{\mathfrak{a}}(M))$ *be finite for all $i < t$, and $N$, a submodule of* $\mathfrak{F}^i_{\mathfrak{a}}(M)$ *such that* $\mathrm{Ass}(\mathrm{Tor}^R_1(R/\mathfrak{a}, N))$ *is finite, then* $\mathrm{Ass}(\mathfrak{F}^t_{\mathfrak{a}}(M)/N)$ *is finite.*

PROOF. The exact sequence

$$0 \longrightarrow N \longrightarrow \mathfrak{F}^t_{\mathfrak{a}}(M) \longrightarrow \mathfrak{F}^t_{\mathfrak{a}}(M)/N \longrightarrow 0,$$

induces the long exact sequence

$$\cdots \longrightarrow \mathrm{Tor}^R_i(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M)/N) \longrightarrow \mathrm{Tor}^R_{i+1}(R/\mathfrak{a}, N) \longrightarrow \mathrm{Tor}^R_{i+1}(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M)) \cdots.$$

The $\mathrm{Ass}(R/\mathfrak{a} \otimes \mathfrak{F}^t_{\mathfrak{a}}(M))$ is finite by Lemma 1.1 and $\mathrm{Ass}(\mathrm{Tor}^R_1(R/\mathfrak{a}, N))$ is finite by hypothesis, hence $\mathrm{Ass}(\mathfrak{F}^n_{\mathfrak{a}}(M)/N)$ is finite. □

COROLLARY 2.3. *Let $(R, \mathfrak{m})$ be a local ring. $M$ is a finitely generated $R$-module. Suppose that there is $n \in \mathbb{N}$ such that for all $i < n$, $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is Artinian. Then* $\mathrm{Ass}(\mathfrak{F}^n_{\mathfrak{a}}(M))$ *is finite.*

PROOF. Since Artinian modules have finite support and by Theorem 2.1, corollary is an immediate consequence. □

**Acknowledgement**

The authors are deeply grateful to the referee for careful reading of the original manuscript and valuable suggestions.

**References**

1. K. Divaani-Aazar and A. Mafi, *Associated primes of local cohomology modules of weakly Laskerian modules*, Comm. Algebra **34** (2) (2006) 681–690.
2. M. Asgharzadeh and K. Divaani-Aazar, *Finiteness properties of formal local cohomology modules and Cohen-Macaulayness*, Commun. Alg. **39** (3) (2011) 1082–1103.
3. M. H. Bijan-Zadeh and S. Rezaei, *Artinianness and attached primes of formal local cohomology modules*, Algebra Colloq. **21** (2) (2014) 307–316.
4. P. Schenzel, *On formal local cohomology and connectedness*, J. Algebra **315** (2) (2007) 894–923.

E-mail: b-sadeqi@marandiau.ac.ir; behruz.sadeqi@gmail.com
E-mail: EIgabi55@gmail.com

# Methods for Constructing Shellable Simplicial Complexes

Mohammad Farrokhi
Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS),
The Center for Research in Basic Sciences and Contemporary Technologies, IASBS,
Zanjan 45137-66731, Iran
Alireza Shamsian*
Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS),
Zanjan 45137-66731, Iran
and Ali Akbar Yazdan Pour
Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS),
Zanjan 45137-66731, Iran

Abstract. A clutter $\mathcal{C}$ with vertex set $X$ is an antichain of $2^X$ such that $X = \cup\mathcal{C}$. For any clutter $\mathcal{C}$, we consider the independence complex of $\mathcal{C}$ whose faces are independent sets in $\mathcal{C}$. In this paper, we introduce some methods to obtain clutters $\mathcal{C}'$ containing a given clutter $\mathcal{C}$ as an induced subclutter such that the independence complex of $\mathcal{C}'$ is shellable. Consequently, for a given squarefree monomial ideal $I \subset S = \mathbb{K}[x_1, \ldots, x_n]$, we obtain a squarefree monomial ideal $J \supseteq I$ in an extension ring $S'$ of $S$ such that the ring $S'/J$ is Cohen-Macaulay.
**Keywords:** Hybrid clutter, Simplicial complex, Shellable clutter, Cohen-Macaulay complex.
**AMS Mathematical Subject Classification [2010]:** 05E40, 05E45.

## 1. Introduction

Shellable simplicial complexes play an important role in both combinatorics and commutative algebra. In combinatorial setting, the notion of shellebility gives rise to an inductive proof for the Euler-Poincaré formula in any dimension. If $f_i$ denotes the number of $i$-faces of a $d$-dimensional polytope (with $f_{-1} = f_d = 1$), the Euler-Poincaré formula states that

$$\sum_{i=-1}^{d} (-1)^i f_i = 1.$$

Earlier inductive proofs" of the above formula were proposed, notably a proof by Schläfli in 1852, but it was later observed that all these proofs assume that the boundary of every polytope can be built up inductively in a nice way, what is called shellability. A striking application of shellability of polytopes was made by McMullen in 1970, who gave the first proof of the so-called "upper bound theorem" for polytopes [3].

*Speaker

In algebraic setting, as it is quoted in Stanley's outstanding book, "shellability is a simple but powerful tool for proving the Cohen-Macaulay property, and almost all Cohen-Macaulay complexes arising 'in nature' turn out to be shellable. Moreover, a number of invariants associated with Cohen-Macaulay complexes can be described more explicitly or computed more easily in the shellable case" (See [4]). Indeed, the Stanley-Reisner ring of a pure shellable simplicial complex turns out to be Cohen-Macaulay. Moreover, the Stanley-Reisner ideal of Alexander dual of a shellable simplicial complex has linear quotients and hence linear resolution.

From geometric point of view, shellable complexes are bouquets of spheres [1]. Indeed, if $\Delta$ is shellable, then $\Delta$ is homotopic equivalent to wedge some of some spheres, namely

$$\Delta \cong \bigwedge_{F_j} \mathbb{S}^{\dim F_j}.$$

In this paper, we introduce some combinatorial methods to transform an arbitrary clutter $\mathcal{C}$ to a clutter $\mathcal{C}' \supseteq \mathcal{C}$ (by adding some points and circuits) such that the independence complex of the new clutter $\mathcal{C}'$ is shellable, generalizing the case introduced by Villarreal [6]. Our results also generalize the result of Cook and Nagel in [2] who show that the graph obtained by adding a vertex to each clique partition of $G$ is Cohen-Macaulay.

First, we recall some combinatorial tools and their relations to commutative algebra.

**1.1. Simplicial Complexes.** A *simplicial complex* $\Delta$ on a set $V = \{v_1, \ldots, v_n\}$ of vertices is a collection of subsets of $V$ such that $\{v_i\} \in \Delta$ for all $i$ and, $F \in \Delta$ implies that all subsets of $F$ are also in $\Delta$. The elements of $\Delta$ are called *faces* and the maximal faces under inclusion are called *facets* of $\Delta$. We denote by $\mathcal{F}(\Delta)$ the set of facets of $\Delta$. The *dimension* of a face $F$ is $\dim F = |F| - 1$, where $|F|$ denotes the cardinality of $F$. A simplicial complex is called *pure* if all its facets have the same dimension. The *dimension* of $\Delta$, is defined as

$$\dim(\Delta) = \max\{\dim F\colon\ F \in \Delta\}.$$

Given a simplicial complex $\Delta$ on the vertex set $\{v_1, \ldots, v_n\}$. For $F \subseteq \{v_1, \ldots, v_n\}$, let $\mathbf{x}_F = \prod_{v_i \in F} x_i$. The *non-face ideal* or the *Stanley-Reisner ideal* of $\Delta$, denoted by $I_\Delta$, is an ideal of $S$ generated by square-free monomials $\mathbf{x}_F$, where $F \notin \Delta$.

Definition 1.1 (Shellable simplicial complexes). A simplicial complex $\Delta$ is called *shellable* if there is a total order of the facets of $\Delta$, say $F_1, \ldots, F_t$, such that $\langle F_1, \ldots, F_{i-1}\rangle \cap \langle F_i\rangle$ is generated by a set of maximal proper faces of $F_i$ for $2 \le i \le t$.

**1.2. Clutters and their Associated Ideals.** In this section, we recall some definitions about clutters and their associated ideals in a polynomial ring.

Definition 1.2 (Clutter). A *clutter* $\mathcal{C}$ with vertex set $X$ is an antichain of $2^X$ such that $X = \cup\mathcal{C}$. The elements of $\mathcal{C}$ are called *circuits* of $\mathcal{C}$. A *d-circuit* is a circuit consisting of exactly $d$ vertices, and a clutter is called *d-uniform*, if every circuit has $d$ vertices.

For a non-empty clutter $\mathcal{C}$ on the vertex set $[n]$, we define the ideal $I(\mathcal{C})$ to be

$$I(\mathcal{C}) = (\mathbf{x}_T\colon\ T \in \mathcal{C}),$$

and we set $I(\varnothing) = 0$. The ideal $I(\mathcal{C})$ is called the *circuit ideal* of $\mathcal{C}$.

Let $n$, $d$ be positive integers and let $V$ be a set consisting $n$ elements. For $n \geq d$, let

$$\mathcal{C}_{n,d} = \{F \subset V\colon \quad |F| = d\}.$$

This clutter is called the *complete d-uniform clutter* on $V$ with $n$ vertices.

The *complement* $\bar{\mathcal{C}}$ of a $d$-uniform clutter $\mathcal{C}$ with vertex set $[n]$ is defined as

$$\bar{\mathcal{C}} = \mathcal{C}_{n,d} \setminus \mathcal{C} = \{F \subseteq [n]\colon \quad |F| = d,\ F \notin \mathcal{C}\}.$$

Let $\mathcal{C}$ be a clutter on the vertex set $[n]$ and let $\Delta_{\mathcal{C}}$ be the simplicial complex on $[n]$ with $I_{\Delta_{\mathcal{C}}} = I(\mathcal{C})$. The simplicial complex $\Delta_{\mathcal{C}}$ is called the *independence complex* of $\mathcal{C}$ and a face $F \in \Delta_{\mathcal{C}}$ is called an *independent set* in $\mathcal{C}$. The clutter $\mathcal{C}$ is said to be shellable (resp. Cohen-Macaulay) if $\Delta_{\mathcal{C}}$ is shellable (resp. $\mathbb{K}[x_1, \ldots, x_n]/I(\mathcal{C})$ is Cohen-Macaulay). If $\mathcal{C}$ is a $d$-uniform clutter, then the simplicial complex $\Delta(\mathcal{C})$ whose Stanley-Reisner ideal is $I(\bar{\mathcal{C}})$ is called the *clique complex* of $\mathcal{C}$ and a face $F \in \Delta(\mathcal{C})$ is called a *clique* in $\mathcal{C}$. It is easily seen that $F \subseteq [n]$ is a clique in $\mathcal{C}$ if and only if either $|F| < d$ or else all $d$-subsets of $F$ belongs to $\mathcal{C}$.

**1.3. Hybrid Clutters.** Let $\mathcal{C}$ be a $d$-uniform clutter, and $A_1, \ldots, A_\theta$ be a clique partition of $V(\mathcal{C})$. Let the non-null hypergraphs $\mathcal{B}_1, \ldots, \mathcal{B}_\theta$ be such that $\mathcal{C}, \mathcal{B}_1, \ldots, \mathcal{B}_\theta$ are pairwise disjoint. Define the $d$-uniform clutter $\mathcal{C}^{\mathcal{B}_1,\ldots,\mathcal{B}_\theta}_{A_1,\ldots,A_\theta}$ as follows:

$$\mathcal{C}^{\mathcal{B}_1,\ldots,\mathcal{B}_\theta}_{A_1,\ldots,A_\theta} = \mathcal{C} \cup \bigcup_{i=1}^{\theta} \big\{F \subseteq A_i \cup V(\mathcal{B}_i)\colon \quad |F| = d \text{ and } F \cap V(\mathcal{B}_i) \in \mathcal{B}_i\big\}.$$

The clutter $\mathcal{C}^{\mathcal{B}_1,\ldots,\mathcal{B}_\theta}_{A_1,\ldots,A_\theta}$ is called a *hybrid* clutter of $\mathcal{C}$ with respect to $\mathcal{B}_1, \ldots, \mathcal{B}_\theta$ and the clique partition $A_1, \ldots, A_\theta$.

## 2. Main Results

The main aim of our work is to generate several shellable simplicial complexes from a given clutter. Indeed, for a given clutter $\mathcal{C}$, we apply some operation on $\mathcal{C}$ to obtain a clutter $\mathcal{C}'$ such that the simplicial complex $\Delta_{\mathcal{C}'}$ is shellable.

Definition 2.1. For a hypergraph $H$ of rank $r$, let $H^i$ be the $i$-uniform spanning subhypergraph of $H$ including all edges of size $i$, for $i = 1, \ldots, r$. The hypergraph $H$ is said to have property $\mathcal{P}$ if it satisfies the following conditions:

a) Any $G$ in $\mathcal{F}(\Delta_{H^i})$ is contained properly in some $G'$ in $\mathcal{F}(\Delta_{H^{i+1}})$, for $i = 1, \ldots, r$, and
b) Any $G'$ in $\mathcal{F}(\Delta_{H^{i+1}})$ contains properly some $G$ in $\mathcal{F}(\Delta_{H^i})$, for $i = 1, \ldots, r$.

Theorem 2.2. *Let $\mathcal{C}$ be a $d$-uniform clutter, $A_1, \ldots, A_\theta$ be a clique partition of $\mathcal{C}$, and let $\mathcal{B}_1, \ldots, \mathcal{B}_\theta$ be hypergraphs of ranks at least $d-1$ satisfying property $\mathcal{P}$. If $\mathcal{C}' = \mathcal{C}^{\mathcal{B}_1,\ldots,\mathcal{B}_\theta}_{A_1,\ldots,A_\theta}$, then*

i) $\dim(\Delta_{\mathcal{C}'}) = \sum_{i=1}^{\theta} \dim(\Delta_{\mathcal{B}_i^d}) - \theta - 1$,

ii) $\Delta_{\mathcal{C}'}$ *is pure if and only if* $\Delta_{\mathcal{B}_i^s}$ *is pure, for all* $d - |A_i| \leq s \leq d$, *and*

$$\dim \Delta_{B_i^t} - \dim \Delta_{B_i^s} = t - s,$$

*for all* $s \leq t \leq d$, *and*

iii) $\mathcal{C}'$ *is shellable if and only if* $\mathcal{B}_i^t$ *is shellable for all* $i = 1, \dots, \theta$ *and* $d - |A_i| \leq t \leq d$.

Example 2.3. Let $\mathcal{B} = \langle [n] \rangle^{(r-1)} \setminus \mathcal{D}$ $(1 \leq r \leq n)$, where $\mathcal{D}$ is an $r$-uniform clutter. It is evident that $\mathcal{B}$ satisfies the property $\mathcal{P}$. If $\Delta(\mathcal{D}) = \Delta_{\mathcal{B}^r}$ is shellable, then $\Delta_{\mathcal{B}^i}$ is shellable for all $1 \leq i \leq r$.

Corollary 2.4. *Let* $\mathcal{C}$ *be a* $d$*-uniform clutter,* $A_1, \dots, A_\theta$ *be a clique partition of* $\mathcal{C}$*, and let* $\mathcal{B}_1, \dots, \mathcal{B}_\theta$ *be disjoint simplexes of dimensions at least* $d - 2$*. If* $\mathcal{C}' = \mathcal{C}^{\mathcal{B}_1, \dots, \mathcal{B}_\theta}_{A_1, \dots, A_\theta}$*, then*

i) $\Delta_{\mathcal{C}'}$ *is pure shellable of dimension* $(d-1)\theta - 1$*, and*

ii) *the ring* $\mathbb{K}[V(\mathcal{C}')]/I(\mathcal{C}')$ *is Cohen-Macaulay of dimension* $(d-1)\theta$.

Corollary 2.5. *Let* $\mathbb{K}$ *be a field.*

i) *Let* $\Delta$ *a simplicial complex. Associated to ideal* $I_\Delta \subseteq \mathbb{K}[x_1, \dots, x_n]$*, we define the ideal*

$$I'(\Delta) = \langle I_\Delta, x_1y_1, x_2y_2, \dots, x_ny_n \rangle \subset \mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_n],$$

*and* $S(\Delta)$ *it's Stanley-Reisner complex. Then the simplicial complex* $S(\Delta)$ *is shellable* [5, Proposition 5.4.10].

ii) *Let* $\pi$ *be a clique vertex-partition of* $G$*. Then* $\Delta_{G^\pi}$ *is Cohen-Macaulay* [2, Corollary 3.5].

Definition 2.6. Let $\mathcal{C}$ be a clutter and $U$ be an induced subclutter of $\mathcal{C}$. Then $U$ is *independently embedded* in $\mathcal{C}$ if $X \cup F \in \mathcal{F}(\Delta_\mathcal{C})$ for $F \subseteq V(U)$ and $X \subseteq V(\mathcal{C}) \setminus V(U)$ implies that $F \in \mathcal{F}(\Delta_{\mathcal{C}|_U})$.

Theorem 2.7. *Let* $\mathcal{C}$ *be a clutter with vertex set* $U \dot\cup V$ *and* $\{\mathcal{C}_u\}_{u\in U}$ *be a family of pairwise disjoint non-empty clutters. Let* $(\mathcal{C}, \{\mathcal{C}_u\}_{u\in U})$ *be the clutter obtained from* $\mathcal{C}$ *as follows:*

$$(\mathcal{C}, \{\mathcal{C}_u\}_{u\in U}) = \mathcal{C} \cup \bigcup_{u\in U} \{e \cup \{u\} : \quad e \in \mathcal{C}_u\}.$$

*If* $\mathcal{C}' := (\mathcal{C}, \{\mathcal{C}_u\}_{u\in U})$ *and* $\Delta' := \Delta_{\mathcal{C}'}$*, then*

i) $\dim(\Delta') = \sum_{u\in U} |V(\mathcal{C}_u)| + \dim(\Delta_{\mathcal{C}_V})$,

ii) $\Delta'$ *is pure if and only if* $|\mathcal{C}_u| = 1$ *for all* $u \in U$*,* $\Delta_{\mathcal{C}|_V}$ *is pure, and* $\mathcal{C}|_V$ *is independently embedded in* $\mathcal{C}$*,*

iii) $\mathcal{C}'$ *is shellable if and only if* $\mathcal{C}|_V$ *and* $\mathcal{C}_u$ *are shellable for all* $u \in U$.

Corollary 2.8. *Let* $\mathcal{C}$ *be a clutter on* $[n]$ *and* $C_1, \dots, C_n$ *be non-empty sets such that* $V(\mathcal{C}), C_1, \dots, C_n$ *are pairwise disjoint. Let*

$$\mathcal{C}' := \mathcal{C} \cup \{C_i \cup \{i\} : \quad i \in [n]\}.$$

*Then* $\Delta_{\mathcal{C}'}$ *is pure shellable simplicial complex, hence Cohen-Macaulay.*

THEOREM 2.9. *Let $\mathcal{C}$ be a clutter with vertex set $U\dot{\cup}V$ and $\{\mathcal{C}_u\}_{u\in U}$ be a family of pairwise disjoint non-empty clutters. Let $(\mathcal{C},\{\mathcal{C}_u\}_{u\in U})^*$ be the clutter obtained from $\mathcal{C}$ as follows:*

$$(\mathcal{C},\{\mathcal{C}_u\}_{u\in U})^* = \mathcal{C} \cup \bigcup_{u\in U} \mathcal{C}_u \cup \bigcup_{u\in U} \{\{u\}\} * \mathcal{C}_u^*,$$

*where $\mathcal{C}_u^* := \min\{e\setminus x\colon\ \ x\in e,\ e\in\mathcal{C}_u\}$, for all $u\in U$. If $\mathcal{C}' := (\mathcal{C},\{\mathcal{C}_u\}_{u\in U})^*$ and $\Delta' := \Delta_{\mathcal{C}'}$, then*

i) $\dim(\Delta') = \sum_{u\in U} |V(\mathcal{C}_u)| + \dim(\Delta_{\mathcal{C}_V})$,
ii) *$\Delta'$ is pure if and only if $|\mathcal{C}_u| = 1$ for all $u\in U$, $\Delta_{\mathcal{C}|_V}$ is pure, and $\mathcal{C}|_V$ is independently embedded in $\mathcal{C}$,*
iii) *$\mathcal{C}'$ is shellable if and only if $\mathcal{C}|_V$, $\mathcal{C}_u$, and $\mathcal{C}_u^*$ are shellable for all $u\in U$.*

## References


1. A. Björner and M. L. Wachs, *Shellable nonpure complexes and posets*, Trans. Amer. Math. Soc. **348** (4) (1996) 1299–1327.
2. D. Cook and U. Nagel, *Cohen-Macaulay graphs and face vectors of flag complexes*, SIAM J. Discrete Math. **26** (1) (2012) 89–101.
3. J. Gallier, *Notes on convex sets, polytopes, polyhedra, combinatorial topology, Voronoi diagrams and Delaunay triangulations*, (2008). arXiv:0805.0292
4. R. P. Stanley, *Combinatorics and Commutative Algebra*, 2nd ed., Progress in Mathematics, Vol. 41, Birkhaüser Boston, Inc., Boston, MA, 1996.
5. R. H. Villareal, *Monomial Algebras*, Marcel Dekker Inc., New York, 2001.
6. R. H. Villareal, *Cohen-Macaulay graphs*, Manuscripta Math. **66** (1) (1990) 277–293.



E-mail: farrokhi@iasbs.ac.ir
E-mail: ashamsian@iasbs.ac.ir
E-mail: yazdan@iasbs.ac.ir

# The Cyclic and Normal Graphs of the Group $D_{2n} \times C_p$, where $p$ is an Odd Prime

Hussan Jeddoa Mehdi
Faculty of Computer Sciences and Mathematics, University of Kufa, Najaf, Iraq
and Hayder Baqer Ameen Shelash*
Faculty of Computer Sciences and Mathematics, University of Kufa, Najaf, Iraq

Abstract. Suppose $G$ is a group. The cyclic graph $\Gamma_C G$ is a simple graph with vertex set $G$ and the edge set $E(\Gamma_C(G)) = \{\{x, y\} \mid \langle x, y\rangle \leq_C G\}$, where $\langle x, y\rangle \leq_C G$ means that $\langle x, y\rangle$ is a cyclic subgroup of $G$. The normal graph $\Gamma_N G$ is anther graph with the same set of vertices and the edge set $E(\Gamma_N(G)) = \{\{x, y\} \mid \langle x, y\rangle \trianglelefteq G\}$. In this paper, we establish some properties of the cyclic and normal graphs defined on the group $D_{2n} \times C_p$, where $p$ is an odd prime.

**Keywords:** Cyclic graph, Normal graph, Split graph.
**AMS Mathematical Subject Classification [2010]:** 50B10, 05C07, 05C50.

## 1. Introduction

Throughout this paper, the word simple graph used for an undirected graph with no loops or multiple edges. Let $\Gamma$ be such a graph. We will denote by $V(\Gamma)$ and $E(\Gamma)$, the set of vertices and edges of $\Gamma$, respectively. The degree of a vertex $v \in V(\Gamma)$ is denoted by $deg(v)$, and it well-known that $deg(v) = |N(v)|$. The degree sequence of a graph with vertices $v_1, \ldots, v_n$ is the sequence $d = (deg(v_1), \ldots, deg(v_n))$. Every graph with degree sequence $d$ is called a realization of $d$. A degree sequence is uni-graphic if all of its realizations are isomorphic. It is usual to write the degree sequence of a graph $\Gamma$ as

$$d(\Gamma) = \begin{pmatrix} n_1 & n_2 & \cdots & n_s \\ \mu(n_1) & \mu(n_2) & \cdots & \mu(n_s) \end{pmatrix},$$

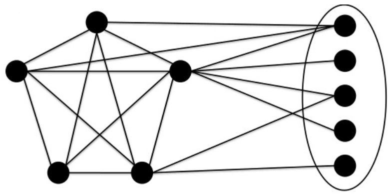
where $n_i$'s are denoted different degrees and $\mu(n_i)$'s are multiplicities of these vertices. The order of the largest clique in $\Gamma$ is its clique number denoted by $CN(\Gamma)$.

A locally cyclic group is a group in which every finitely generated subgroup is cyclic. It is easy to see that a group is locally cyclic if and only if every pair of elements in the group generates a cyclic subgroup. Also, every finite locally cyclic group is cyclic. Let $G$ be a group. The cyclicizer of an element $x$ of $G$, denoted $\mathbf{Cyc}_G(x)$, is defined as $\mathbf{Cyc}_G(x) = \{y \mid \langle x, y\rangle \leq_c G, y \in G\}$. We refer the interested readers to consult [**5, 6**] and references therein for more information on this topic. The cyclicizer of $G$ is defined by $\mathbf{Cyc}(G) = \bigcap_{x\in G} \mathbf{Cyc}_G(x)$ which is a normal subgroup of group $G$ [**2, 4, 5, 6**].

*Speaker

Suppose $G$ is a group. The cyclic graph $\Gamma_C G$ is a simple graph with vertex set $G$ and the edge set $E(\Gamma_C(G)) = \{\{x, y\} \mid \langle x, y\rangle \leq_C G\}$, where $\langle x, y\rangle \leq_C G$ means that $\langle x, y\rangle$ is a cyclic subgroup of $G$. The normal graph $\Gamma_N G$ is anther graph with the same set of vertices and the edge set $E(\Gamma_N(G)) = \{\{x, y\} \mid \langle x, y\rangle \trianglelefteq G\}$.

A graph $G$ is said to be split graph if its vertices can be partitioned into a clique and an independent set.



The present authors [1], computed the number of cyclic and normal subgroups of the group $D_{2n} \times C_p$, where $p$ is prime and $p \nmid n$, and presented the structure of the subgroups. If $p \nmid n$, then $\langle a^i\rangle, \langle a^i b\rangle, \langle a^i b, c\rangle$ and $\langle a^i, c\rangle$, $1 \leq i \leq n$, are all cyclic subgroups of the group and the number of these subgroups is $2(\tau(n) + n)$. If $p \mid n$, then $\langle a^i\rangle, \langle a^i b\rangle, \langle a^i b, c\rangle$, $1 \leq i \leq n$, and $\langle a^{\frac{n}{i}}, c\rangle$, when $i \mid \frac{n}{p^\alpha}$ and $\langle a^i c^j\rangle$, when $i \mid \frac{n}{p}, 1 \leq j \leq p-1$ are all cyclic subgroups of the group $D_{2n} \times C_p$. The normal subgroups are given by $\langle a^i\rangle, \langle a^i, c\rangle$, when $i \mid n$, $\langle a^i, a^j b\rangle, \langle a^i, a^j b, c\rangle$, when $1 \leq j \leq i$.

## 2. Main Results

For $n \geq 3$, the dihedral group $D_{2n}$ is an important example of finite groups. As is well known, the direct product of two finite groups $D_{2n}$ and $C_p$ is defined by

$$D_{2n} \times C_p = \langle a, b, c | a^n = b^2 = c^p = e, bab = a^{-1}, [a, c] = [b, c] = e\rangle.$$

Proposition 2.1. *Let $n = 2^r \prod_{i=}^{s} p_i^{\alpha_i}$ be an integer and $p \nmid n$. Then the cyclicizer $\mathbf{Cyc}(x)$ of $x$ in the group $D_{2n} \times C_p$ is given by the following:*

1) $\mathbf{Cyc}(a^i) = \{\{a^j c^k\} | 1 \leq j \leq n,\ 1 \leq k \leq p\}$.
2) $\mathbf{Cyc}(a^i b) = \begin{cases} \{c^k\}, & 1 \leq k \leq p, \\ \{a^i b c^k\}, & 1 \leq k \leq p. \end{cases}$
3) $\mathbf{Cyc}(c^k) = \{g | \ \forall g \in D_{2n} \times C_p, 1 \leq k \leq p\}$.
4) $\mathbf{Cyc}(a^i c^k) = \{a^j c^d | 1 \leq j \leq n,\ 1 \leq d \leq p\}$.
5) $\mathbf{Cyc}(a^i b c^k) = \begin{cases} c^d, & 1 \leq d \leq p, \\ a^i b c^d, & 1 \leq d \leq p-1. \end{cases}$

It follows from [3, Proposition 5] that for any group $G$, $deg_{\Gamma_G}(x) = |\mathbf{Cyc}_G(x) - 1|$, where $x \in G$.

Theorem 2.2. *The following are hold:*

1) $deg(a^i) = pn - 1$ *for all* $1 \leq i \leq n$.
2) $deg(a^i b) = 2p - 1$ *for all* $1 \leq i \leq n$.
3) $deg(c^k) = 2np - 1$ *for all* $1 \leq k \leq p$.
4) $deg(a^i c^k) = np - 1$ *for all* $1 \leq k \leq p$.
5) $deg(a^i b c^k) = 2p - 1$ *for all* $1 \leq k \leq p$.

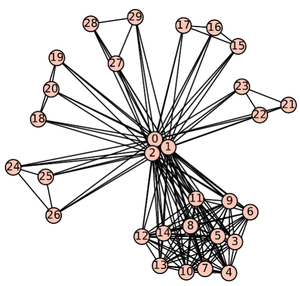
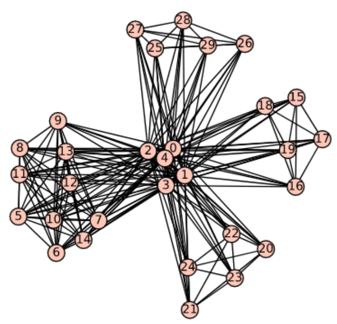
Proposition 2.3. *Let $n = 2^r \prod_{i=1}^{s} p_i^{\alpha_i}$ be a positive integer. The vertex degree sequences of the cyclic graph of the groups $D_{2n}$ and $D_{2n} \times C_p$ is given by the following:*

| $d(\Gamma_C D_{2n})$ | $d(\Gamma_C D_{2n} \times C_p)$ |
|---|---|
| $\begin{pmatrix} 1 & n-1 & 2n-1 \\ n & n-1 & 1 \end{pmatrix}$ | $\begin{pmatrix} 2np-1 & pn-1 & 2p-1 \\ p & p(n-1) & pn \end{pmatrix}$ |

Corollary 2.4. *Let $n \geq 3$. Then*

$$|E(\Gamma_C G)| = \begin{cases} \frac{n(n+1)}{2}, & if\ G \cong D_{2n}, \\ \\ \frac{pn(3p(n+1)-2)}{2}, & if\ G \cong D_{2n} \times C_p. \end{cases}$$

For example

| $G$ | $D_{10} \times C_3$ | $D_6 \times C_5$ |
|---|---|---|
| $\Gamma_C(G)$ |  |  |
| $d(\Gamma_C(G))$ | $\begin{pmatrix} 29 & 14 & 5 \\ 3 & 12 & 15 \end{pmatrix}$ | $\begin{pmatrix} 29 & 14 & 9 \\ 5 & 10 & 15 \end{pmatrix}$ |
| $\lvert E(\Gamma_C(G))\rvert$ | 390 | 435 |

Corollary 2.5. *Let $\Gamma_C$ be the cyclic graph of the group $D_{2n} \times C_p$. The following are holds:*

1) $\Gamma_C$ *is not bipartite.*
2) $\Gamma_C$ *is not Eulerian.*
3) $\Gamma_C$ *is not Hamiltonian.*
4) $\Gamma_C$ *is a split graph.*

We are now ready to present the normal graph. Let $G$ be a group. The normalizer of an element $x$ of $G$, $\mathbf{Nor}_G(x)$, defined as $\mathbf{Nor}_G(x) = \{y | \langle x, y\rangle \trianglelefteq G\}$.

Proposition 2.6. *Let $n = 2^r \prod_{t=}^{s} p_t^{\alpha_t}$ be an integer and $p \nmid n$. The set of neighborhood of vertices of normal graph of the group $D_{2n} \times C_p$ are given by the following:*

1) $\mathbf{Nor}_{D_{2n}}(a^i) = \begin{cases} \{y | \forall y \in D_{2n}\}, & p_t \nmid i \quad (\varphi(n) + \varphi(\frac{n}{2})), \\ \{\{a^j\}\}, & p_t \mid i \quad (n - (\varphi(n) + \varphi(\frac{n}{2}))). \end{cases}$

2) $\mathbf{Nor}_{D_{2n}}(a^i b) = \begin{cases} \{a^j\}, & 4p_s \nmid i, \forall s \\ \{a^{j+1}b\}, & \end{cases}$ .

3) $\mathbf{Nor}_{D_{2n}\times C_p}(a^ic^k) = \begin{cases} \{y|\forall y \in D_{2n}\times C_p\}, & p_t \nmid i \quad p(\varphi(n)+\varphi(\frac{n}{2})), \\ \{\{a^jc^k\}|1\le j\le n, 1\le k\le p\}, & p_t \mid i \quad p(n-(\varphi(n)+\varphi(\frac{n}{2}))). \end{cases}$

4) $\mathbf{Nor}_{D_{2n}\times C_p}(a^ibc^k) = \begin{cases} \{a^jc^d\}, & 1\le d\le p, 4p_s \nmid i, \forall s, \\ \{a^{j+1}bc^d\}, & 1\le d\le p. \end{cases}$

Theorem 2.7. *The following are hold:*

1) $deg(a^ic^k) = \begin{cases} 2np-1, & p_s \nmid i, \\ np-1, & p_s \mid i. \end{cases}$

2) $deg(a^ibc^k) = 2p(n - \sum_{i=1}^{s} O(a^{p_i})) - 1$ *for all* $1\le i\le n$.

3) $deg(c^k) = np-1$ *for all* $1\le k\le p$.

Proposition 2.8. *Let* $n = 2^r\prod_{1=1}^{s} p_i^{\alpha_i}$ *be positive integer. The degree sequence of the normal graph of the group* $D_{2n}$ *and* $D_{2n}\times C_p$ *are given by the following:*

1) If $2\nmid n$, then

| $d(\Gamma_N(D_{2n}))$ | $d(\Gamma_N(D_{2n}\times C_p))$ |
|---|---|
| $\begin{pmatrix} 2\varphi(n) & 2n-1 & n-1 \\ n & \varphi(n) & n-\varphi(n) \end{pmatrix}$ | $\begin{pmatrix} 2p\varphi(n) & 2np-1 & np-1 \\ pn & p\varphi(n) & p(n-\varphi(n)) \end{pmatrix}$ |

2) If $2\mid n$, then

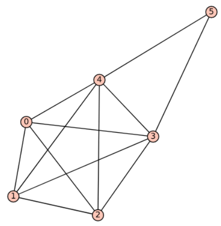
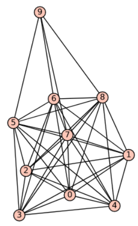
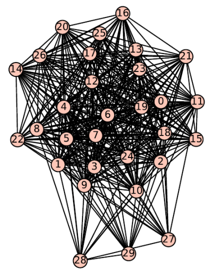
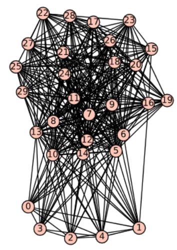
| $d(\Gamma_N(D_{2n}))$ |
|---|
| $\begin{pmatrix} 2(\varphi(n)+\varphi(\frac{n}{2})) & 2n-1 & n-1 \\ n & \varphi(n)+\varphi(\frac{n}{2}) & n-(\varphi(n)+\varphi(\frac{n}{2})) \end{pmatrix}$ |

| $d(\Gamma_N(D_{2n}\times C_p))$ |
|---|
| $\begin{pmatrix} 2p(\varphi(n)+\varphi(\frac{n}{2})) & 2np-1 & np-1 \\ pn & p(\varphi(n)+\varphi(\frac{n}{2})) & p(n-(\varphi(n)+\varphi(\frac{n}{2}))) \end{pmatrix}$ |

Corollary 2.9. *The following are holds:*

$$|E(\Gamma_N(G))| = \begin{cases} \frac{n(3\varphi(n)-1)+n^2}{2}, & G\cong D_{2n}, 2\nmid n, \\ \frac{n(n+3(\varphi(n)+\varphi(\frac{n}{2}))-1)}{2}, & G\cong D_{2n}, 2\mid n, \\ \frac{np(np+3p(\varphi(n)+\varphi(\frac{n}{2}))-p)}{2}, & G\cong D_{2n}\times C_p, 2\mid n, \\ \frac{pn(3p\varphi(n)-1)+n^2p^2}{2}, & G\cong D_{2n}\times C_p, 2\nmid n. \end{cases}$$

For example:

| $G$ | $D_6$ | $D_{10}$ | $D_{10} \times C_3$ | $D_6 \times C_5$ |
|---|---|---|---|---|
| $\Gamma_N$ |  |  |  |  |
| $d(\Gamma)$ | $\begin{pmatrix} 4 & 5 & 2 \\ 3 & 2 & 1 \end{pmatrix}$ | $\begin{pmatrix} 8 & 9 & 4 \\ 5 & 4 & 1 \end{pmatrix}$ | $\begin{pmatrix} 20 & 29 & 14 \\ 15 & 10 & 5 \end{pmatrix}$ | $\begin{pmatrix} 24 & 29 & 14 \\ 15 & 12 & 3 \end{pmatrix}$ |
| $\lvert E(\Gamma_N) \rvert$ | 12 | 40 | 330 | 255 |

Corollary 2.10. *Let $\Gamma_N$ be the normal graph of the graph $D_{2n} \times C_p$. The following are hold:*

1) *$\Gamma_N$ is not bipartite.*
2) *$\Gamma_N$ is not Eulerian.*
3) *$\Gamma_N$ is not Hamiltonian.*
4) *$\Gamma_N$ is a split graph.*

## References


1. H. J. Mehdi and H. B. Shelash, *Computing the certain of subgroups of the group $D_{2n} \times C_p$, p is an odd prime number*, IOP J. Phys. accepted.
2. H. B. Shelash, *Counting The Number of Subgroups, Normal Subgroups and Characteristic Subgroups in Certain Finite Groups*, Ph.D. Thesis, Universty of Kashan, Iran, 2018.
3. X. L. Ma, H. Q. Wei and G. Zhong, *The cyclic graph of a finite group,* Algebra **2013** (2013) 107265.
4. H. Kurzweil and B. Stellmacher, *The Theory of Finite Groups*, An Introduction, Universitext, Springer, New York, NY, USA, 2004.
5. S. J. Baishya, *A note on finite $C-$tidy groups*, Int. J. Group Theory **2** (3) (2013) 9–17.
6. D. Patrick and E. Wepsic, *Cyclicizers, centralizers and normalizers*, Tech. Rep. MS-TR 91-05, Rose-Hulman Institute of Technology, Terre Haute, Ind, USA, 1991.
7. The GAP Team, *GAP* – Groups, Algorithms, and Programming, Version 4.7.5, 2014.


E-mail: msc500604@gmail.com
E-mail: hayder.ameen@uokufa.edu.iq

# Simple Associative Algebras and their Corresponding Finitary Special Linear Lie Algebras

Hasan Mohammad Ali Saeed Shlaka*
Department of Mathematics, Faculty of Computer Science and Mathematics, University Kufa, Al-Najaf, Iraq

Abstract. Let $\mathbb{F}$ be a field of any characteristic. Given any associative algebra $A$ over $\mathbb{F}$, one can render it into a Lie algebra by defining a new product, the Lie product, for any two elements $a$ and $b$ in $A$ by means of $[a, b] = ab - ba$, where $ab$ is the associative product in $A$. It is natural to except that the Lie algebra so obtained has a structure which is closely connected with the associative structure of $A$. In this paper, we study the relation between simple associative algebras and their related finitary Special Linear Lie algebras.
**Keywords:** Associated Algebra, Lie Algebra.
**AMS Mathematical Subject Classification [2010]:** 17B69, 17B99.

## 1. Introduction

Throughout this paper, unless otherwise stated, we denote by $\mathbb{F}$, $V$, $V^*$, $\Pi$, $A$ and $L$ to be a field (algebraically closed) of characteristic $p \geq 0$, a vector space over $\mathbb{F}$, a dual space of $V$ over $\mathbb{F}$, a total subspace (See Definition 4.1 for more details) of $V$ over $\mathbb{F}$, an associative algebra over $\mathbb{F}$ and a Lie algebra over $\mathbb{F}$. The study of the structure of Lie algebras of simple rings were initiated in 1954 by the American mathematician I. N. Herstein in his papers [8, 9]. Recall that an associative algebra $A$ over a field $\mathbb{F}$ gives raise to become a Lie algebra $A^{(-)}$ under the Lie commutator

$$[x, y] := xy - yx \quad \text{for all} \quad x, y \in A, \tag{1}$$

where $xy$ is the usual multiplication in $A$. Put $A^{(0)} = A^{(-)}$ and $A^{(i)} = [A^{(i-1)}, A^{(i-1)}]$ $(i \geq 1)$. Then $L = A^{(i)}$, for some $i \geq 0$, is a Lie algebra. In several papers (See for example [10, 11, 12, 13] and [14]) from 1955 to 1975 Herstein studied Jordan and Lie structure of simple rings. A revision for Herstein's Lie theory was done by Martindale [17] in 1986. They examine the Lie ideals and the Lie subalgebras of simple associative rings. Despite the fact that simple Lie algebras have no ideals, the American mathematician Georgia Benkart [6] showed that all these Lie algebras have non-trivial inner ideals. In 1976, Benkart defined the inner ideal as a subspace $B$ of $L$ satisfies the property $[B, [B, L]] \in B$. In 1977, Benkart highlighted the relation between inner ideals and ad-nilpotent elements of Lie algebras [7]. Thus, a fundamental role in classifying Lie algebras are inner ideals because certain restrictions on the ad-nilpotent elements imply a criterion for distinguishing the non-classical from the classical simple Lie algebras in positive

*Speaker

characteristic. In 2008, Fernndez Lpez, Garcia, and Gmez Lozano [16] proved that inner ideals have role similar to that of one-sided ideals in associative algebras and can be used to improve Artinian structure theory for Lie algebras. In this paper, the structure of Lie algebras that obtained from the associative ones are studied. We start with some preliminaries on the second section. Section 3 is devoted to study the Lie algebras that come from the finite dimensional simple associative algebras. Section 4 consists of the infinite dimensional case where the finitary general and special linear Lie algebra are considered together with their inner ideals.

## 2. Preliminaries

Recall that all linear transformations of $V$ form the general linear Lie algebra $\mathfrak{gl}(V)$ under the commutator defined by $[x, y] = xy - yx$ for all $x, y \in \mathfrak{gl}(V)$. As an example of the Lie subalgebra and Lie ideal of $\mathfrak{gl}(V)$ is the special linear Lie algebra $\mathfrak{sl}(V)$, which defined as follows: $\mathfrak{sl}(V) = [\mathfrak{gl}(V), \mathfrak{gl}(V)]$. Recall that the linear transformation $x \in \mathfrak{gl}(V)$ is said to be finitary if $\dim(xV) < \infty$ [1]. The finitary general linear algebra is the Lie ideal $\mathfrak{fgl}(V)$ of $\mathfrak{gl}(V)$ consisting of all the finitary transformations of $V$, that is,

$$\mathfrak{fgl}(V) := \{x \in \mathfrak{gl}(V) \mid \dim(xV) < \infty\}.$$

Definition 2.1. A Lie algebra $L$ is called finitary if it is isomorphic to a subalgebra of $\mathfrak{fgl}(V)$.

We denote by $\mathfrak{fsl}(V)$ to be the finitary special linear Lie algebra, which is defined to be the set of all zero trace finitary linear transformations of $V$. Note that $\mathfrak{fsl}(V) = [\mathfrak{fgl}(V), \mathfrak{fgl}(V)]$. Baranov and Strada [4] classify the irreducible finitary Lie algebras. They proved the following results.

Theorem 2.2. [4] *Let $L$ be an infinite dimensional finitary simple Lie algebra over $\mathbb{F}$. Suppose that $p = 2, 3$. Then $L$ is isomorphic to either $\mathfrak{fsl}(V, \Pi)$, or $\mathfrak{fso}(V, \psi)$, or $\mathfrak{fsp}(V, \vartheta)$, where $\psi$ and $\vartheta$ are nondegenerate symmetric and skew-symmetric bilinear forms on $V$.*

Definition 2.3. [3] Let $B$ be a subspace of $L$. Then $B$ is called

1) inner ideal if $[B, [B, L]] \subseteq B$,
2) abelian inner ideal if $B$ is inner ideal with $[B, B] = 0$,
3) Jordan-Lie inner ideal if $B$ is inner ideal of $L = A^{(i)}$ such that $B^2 = 0$.

We have the following well-known results.

Lemma 2.4. *Let $M$ be a subalgebra of $L$, $P$ be an ideal of $L$ and $B$ be an inner ideal of $L$. Then*

1) *$B \cap M$ is inner ideal.*
2) *$(B + P)/P$ is inner ideal.*

In [2], Baranov, Mudrov and Shlaka showed that if $A$ is left Artinian ring, then every minimal non-nilpotent left ideal $I$ of $A$ can be written as $I = Ae$ for some idempotent $e \in I$. The following results summarize relation between inner ideals and idempotents.

LEMMA 2.5. [3] *Let $A$ be a ring with centre $Z_A$. Let $e$ and $f$ be idempotents in $A$ such that $fe = 0$. Then*

1) $eAf \bigcap Z_A = 0$;
2) $B = eAf \bigcap A^{(k)}$ *is an inner ideal of* $A^{(k)}$ *for all* $k \geq 0$;
3) $eAf$ *is an inner ideal of* $A^{(-)}$ *and of* $[A, A]$;
4) *There is an idempotent $g$ in $A$ satisfying* $eg = ge = 0$ *such that* $eAf = eAg$.

## 3. The Lie Structure of the Finite Dimensional Simple Associative Algebras

We denote by $M_n(\mathbb{F})$ and $\mathfrak{sl}_n(\mathbb{F}) = [M_n(\mathbb{F}), M_n(\mathbb{F})]$ the associative algebra consisting of all $n \times n$-matrices and its Lie subalgebra which consists of all zero trace matrices of $M_n(\mathbb{F})$, respectively. Recall that a perfect Lie algebra is a Lie algebra $L$ with the property $[L, L] = L$.

DEFINITION 3.1. [3] A perfect Lie algebra $L$ is call ed quasi-simple if $L/Z_L$ is simple.

PROPOSITION 3.2. *Suppose that $A$ is simple and finite dimensional and $p \neq 2$. Then $A^{(1)}$ is a quasi-simple Lie algebra. In particular, $A^{(n)} = A^{(1)}$ for all $n = 2, 3, \ldots, \infty$.*

PROOF. Since $A$ is simple and $\mathbb{F}$ is algebraically closed, $A \cong M_n(\mathbb{F})$ for some $n$. If $n = 1$, then $[A, A] = 0$. Suppose that $n \geq 2$. Then $[A, A] = \mathfrak{sl}_n(\mathbb{F})$, so it is a perfect Lie algebra. It remains to show that $A^{(1)}/Z(A^{(1)}$ is simple. We need to consider two cases depending on $p$. Suppose f irst that $p = 0$, then $[A, A] = \mathfrak{sl}_n(\mathbb{F})$ is simple Lie algebra, and $Z_{\mathfrak{sl}_n(\mathbb{F})} = 0$, so $\mathfrak{sl}_n(\mathbb{F})/Z_{\mathfrak{sl}_n(\mathbb{F})}$ is simple. Suppose now that $p > 0$. Then either $p$ divides $n$ or not. If $p$ does not divide $n$, then this is similar to the case when $p = 0$ above. Suppose that $p$ divides $n$, then $\mathfrak{sl}(\mathbb{F})$ is not simple because $Z_{\mathfrak{sl}_n(\mathbb{F})} \subseteq \mathfrak{sl}_n(\mathbb{F})$, so $\mathfrak{sl}_n(\mathbb{F})/Z_{\mathfrak{sl}_n(\mathbb{F})}$ is simple. Therefore, $[A, A]/Z_{[A,A]}$ is simple, as required. □

THEOREM 3.3. *Suppose that $A$ is simple ring of dimensional more than 4 over its centre $Z_A$ and of characteristic $\neq 2$.*

1) [12] *for any Lie ideal $U$ of $A$ we have $U \supseteq A^{(1)}$ or $U \subseteq Z_A$.*
2) [12] $A^{(1)}/Z_{A^{(1)}}$ *is a simple Lie ring.*
3) $A^{(1)}$ *is perfect.*
4) $A^{(1)}$ *is quasi simple.*
5) *If $A$ is Artinian of characteristic $\neq 3$, then*
   a) [6] *If $B$ is an inner ideal of $A^{(1)}/Z_{A^{(1)}}$, then $B = eAf$, where $e$ and $f$ are idempotents in $A$ such that $fe = 0$.*
   b) *If $B$ is an inner ideal of $A^{(1)}/Z_{A^{(1)}}$, then $B = eAf$, where $e$ and $f$ are idempotents in $A$ such that $fe = ef = 0$.*
   c) *If $B$ is a Jordan-Lie inner ideal of $A^{(1)}$, then $B = eAf$, where $e$ and $f$ are idempotents in $A$ with $ef = fe = 0$.*

PROOF. Part (1.) and part (2.) are proved in [12, Theorems 2 and 4].

3. We have $[A^{(1)}, A^{(1)}] \subseteq A^{(1)}$ is an ideal of $A^{(1)}$. Since $Z_A$ does not contains $A^{(1)}$, by (1.), $A^{(1)} \subseteq [A^{(1)}, A^{(1)}]$. Therefore, $A^{(1)} = [A^{(1)}, A^{(1)}]$, or $A^{(1)}$ is perfect.

4. We have $Z_{A^{(1)}} = Z_A \bigcap A^{(1)}$. By (2.), $A^{(1)}/Z_{A^{(1)}}$ is a simple Lie ring. Since $A^{(1)}$ is perfect (by (3.)), we get that $A^{(1)}$ is quasi simple.

5. Part (a) is proved in [5, Theorem 5]. Part (b) follows from Lemma 2.5 (4).

(c) Let $\bar{B}$ be the image of $B$ in $A^{(1)}/Z_{A^{(1)}}$. Then by Lemma 2.4(2), $\bar{B}$ is an inner ideal of $A^{(1)}/Z_{A^{(1)}}$, so by (5(a)), $\bar{B} = eAf$ for some idempotents $e$ and $f$ in $A$ with $fe = 0$. firstly, we need to show that $B \subseteq eAf$. Let $b \in B$. Then there is $x \in A$ and $z \in Z_A$ such that $b = exf + z$. As $B^2 = 0$ and $fe = 0$,

$$0 = b^2 = (exf + z)(ef + z) = exfz + zexf + z^2 = e(2xz)f + z^2.$$

Hence, $z^2 = e(-2xz)f \in eAf \cap Z_A = 0$ (Lemma 2.5(1)), so $z = 0$. Therefore, $b = exf \in eAf$.

Conversely, we need to show that $eAf \subseteq B$. Let $eyf \in eAf$. Then there is $z \in Z$ such that $eyf + z \in B$. As above, it is easy to show that $z = 0$. Thus, $eyf \in B$. Therefore, $B = eAf$. It remains to show that $B = eAf$ for some idempotents $e$ and $f$ in $A$ such that $ef = fe = 0$. Since $fe = 0$, by Lemma 2.5(4), there exists $g$ in $A$ with $g^2 = g$ satisfying the property $ge = eg = 0$ such that $B = eAf = eAg$, as required. □

The exception is an exception indeed, as in the example below

Example 3.4. Suppose that $A = M_2(\mathbb{F})$ and $p = 2$. Consider the set of all matrices

$$M = \{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \mid a, b \in \mathbb{F}\}.$$

Then $M$ is Lie ideal, but $A^{(1)} = \mathfrak{sl}_2(\mathbb{F}) \nsubseteq M$ and $Z_A$ does not contain $M$.

## 4. The Lie Structure of the Finitary Special Linear Lie Algebra

Recall that the annihilator of $\Pi \subseteq V^*$ is the subspace of $V$ defined by

$$Ann(\Pi) = \{v \in V \mid \alpha v = 0, \quad \text{for all} \quad \alpha \in \Pi\}.$$

Definition 4.1. A subspace $\Pi$ of $V^*$ is said to be total if $Ann(\Pi) = 0$.

We denote by $\mathfrak{F}(V, \Pi)$ to be the algebra over $\mathbb{F}$ defined by

$$\mathfrak{F}(V, \Pi) := \{a \in \operatorname{End} V \mid a(v) = v_1(\delta_1 v) + \cdots + v_n(\delta_n v), \ \ v \in V\},$$

where $n$ is an integer, $v_1, \ldots, v_n \in V$ and $\delta_1, \ldots, \delta_n \in \Pi$. The finitary general linear algebra $\mathfrak{fgl}(V, \Pi)$ is the algebra $\mathfrak{F}(V, \Pi)$ over $\mathbb{F}$ under the Lie commutator defined as in (1). The finitary special linear Algebra $\mathfrak{fsl}(V, \Pi)$ is the algebra of all $x \in \mathfrak{fgl}(V, \Pi)$ with $tr(x) \in [\mathbb{F}, \mathbb{F}]$, where $tr(x)$ is the trace of $x$ (See the definition below).

Definition 4.2. If we choose $V^*$ instead of $\Pi$, then we get the algebra $\mathfrak{F}(V, V^*)$ of all finite rank transformations of $V$ over $\mathbb{F}$. for each transformation $x \in \mathfrak{F}(V, V^*)$, the trace $tr(x) \in \mathbb{F}$ of $x$ is defined to be the trace of the finite dimensional subspace $xV$ over $\mathbb{F}$.

Remark 4.3. $\Pi = V^*$ when the dimension of $V$ is finite. In this case, we have $\mathfrak{fgl}(V, V^*) = \mathfrak{gl}_n(\mathbb{F}) = M_n(\mathbb{F})$ and $\mathfrak{fsl}(V, V^*) = \mathfrak{sl}_n(\mathbb{F}) = [M_n(\mathbb{F}), M_n(\mathbb{F})]$, where $\dim V = n$.

Every $n \times n$-matrix $M_n(\mathbb{F})$ can be extended to an $(n+1) \times (n+1)$-matrix $M_{n+1}(\mathbb{F})$ by placing $M_n(\mathbb{F})$ in the upper left hand corner by bordering the last column and row by zeros. We denote by $M_\infty(\mathbb{F})$ the algebra of infinite matrices with finite numbers of non-zero entries (See [18, Example 2.4] for more details), that is, $M_\infty(\mathbb{F}) = \cup_{n=1}^{\infty} M_n(\mathbb{F})$. This gives the embedding

$$\mathfrak{sl}_2(\mathbb{F}) \to \mathfrak{sl}_3(\mathbb{F}) \to \cdots \to \mathfrak{sl}_n(\mathbb{F}) \to \cdots . \tag{2}$$

The stable special linear Lie algebra $\mathfrak{sl}_\infty(\mathbb{F})$ is the union of the algebras in (2). Note that $\mathfrak{sl}_\infty(\mathbb{F})$ is of countable dimensional. if $\mathbb{F} = \mathbb{C}$, then one can construct the finitary special linear Lie algebra as follows: $\mathfrak{sl}_\infty(\mathbb{F}) = \{X \in M_\infty(\mathbb{F}) \mid tr(X) = 0\}$. Suppose that the dimension of $V$ is countable and $E = \{e_1, e_2, \ldots\}$ is a basis of $V$. Let $\Pi$ be a subspace of $V^*$ which is the span of the the dual basis $E^* = \{e_1^*, e_2^*, \ldots\}$. Then we have the following result.

Proposition 4.4. [1, Proposition 6.2] *$\mathfrak{fsl}(V,\Pi) \cong \mathfrak{sl}_\infty(\mathbb{F})$ if and only if $\mathfrak{fsl}(V,\Pi)$ has a countable dimension.*

Definition 4.5. [15] An inner ideal $B$ of $L$ is called principal if $B = \mathrm{ad}_x^2(L)$ for some $x \in B$, where $\mathrm{ad}_x$ is the adjoint mapping defined by $\mathrm{ad}_x(y) = [x, y]$.

Suppose that $V$ (resp. $W$) is a left (resp. right) vector space over $\mathbb{F}$ and there exists a non-degenerate bilinear form $\psi : V \times W \to \mathbb{F}$. Then $V = (V, W, \psi)$ is said to be a pair of dual vector spaces [15]. Note that from every vector space $V$ we can construct a canonical pair $(V, V^*, \psi)$ for some non-degenerate bilinear form $\psi : V \times V^* \to \mathbb{F}$ defined by $\psi(v, \alpha) = \alpha(v)$ for all $v \in V$ and $\alpha \in V^*$. Let

$$\mathfrak{L}(V) := \{a \in \mathrm{End}(V) \mid \psi(av, w) = \psi(v, a^\# w),\ \ a^\# \in \mathrm{End}(V^*)\},$$

be the algebra over $\mathbb{F}$ consisting of all linear transformations $a : V \to V$ that satisfies the property $\psi(av, w) = \psi(v, a^\# w)$ for all $v \in V$ and $w \in W$, where $a^\# : W \to W$ is a unique transformation on $W$ that satisfies the property.

Remark 4.6. Note that $a^\#$ is not necessarily be existed for all linear transformations $a : V \to V$. However, if we consider the canonical pair $(V, V^*, \psi)$, then by using the relation $a^\# \alpha = \alpha a$ for all $\alpha \in V^*$, we can find $a^\# \in \mathrm{End}(V^*)$ for every $a \in \mathrm{End}(V)$.

We denote by $\mathfrak{f}(V)$ the ideal of $\mathfrak{L}(V)$ of all finitary transformations on V.

Definition 4.7. Let $V = (V, W, \psi)$ be a pair of dual spaces and let $X \subseteq V$ and $Y \subseteq W$ be two subspaces. Then $Y^*X := span\{y^*x \mid x \in X,\ y \in Y\}$, where $y^*x$ is the linear transformation that defined as follows $y^*x(v) = \psi(v, y)x$ for all $v \in V$.

Note that every transformation $a \in \mathfrak{F}(V)$ can be written as $a = y^*x$ for some rank one transformation.

Definition 4.8. Let $V = (V, W, \psi)$ be a pair of dual spaces. Then $\mathfrak{gl}(V)$=$\mathfrak{L}(V)$ is the general linear algebra, $\mathfrak{fgl}(V) = \mathfrak{F}(V)$ is the finitary general linear algebra and $\mathfrak{sl}(V) = [\mathfrak{fgl}(V), \mathfrak{fgl}(V)]$ is the finitary special linear algebra.

Proposition 4.9. [15] *Suppose that $V$ is infinite dimensional. if $p = 0$, then $\mathfrak{fsl}(V)$ is a finitary simple algebra.*

THEOREM 4.10. *Let $V = (V, W, \psi)$ be a pair of dual spaces over $\Delta$, where $\Delta$ is a division algebra. Suppose that $dimV > 1$. Let $V_1 \subseteq V$ and $W_1 \subseteq W$ are subspaces with $\psi(V_1, W_1) = 0$. Then*

1) [15] $W_1^* V_1 \subseteq \mathfrak{gl}(V)$ *is inner ideal of* $\mathfrak{gl}(V)$.
2) [15] $W_1^* V_1 \subseteq \mathfrak{fsl}(V)$ *is inner ideal of* $\mathfrak{fsl}(V)$.
3) [15] $W_1^* V_1 \subseteq \mathfrak{fsl}(V)$ *is principal of* $\mathfrak{fsl}(V)$ *if and only if* $V$ *and* $W$ *are finite dimensional and* $dimV = dimW$.
4) *if* $B \subseteq \mathfrak{fsl}(V)$ *is inner ideal, then the following are equivalents*
   a) [15] $B = e\mathfrak{F}(V)f$ *for some* $e, f \in \mathfrak{F}(V)$ *with* $e^2 = e$, $f^2 = f$ *and* $fe = 0$.
   b) [15] $B = W_2^* V_2$ *for some subspaces* $V_2 \subseteq V$ *and* $W_2 \subseteq W$ *with* $\psi(V_2, W_2) = 0$.
   c) $B = e\mathfrak{F}(V)f$ *for some orthogonal idempotents in* $\mathfrak{F}(V)$.
5) *Suppose that* $\Delta$ *is finite dimensional and central over* $\mathbb{F}$ *with* $p = 0$. *Then*
   a) [15] *if* $B \subseteq \mathfrak{fsl}(V)$ *is inner ideal, then* $B = W_2^* V_2$ *for some subspaces* $V_2 \subseteq V$ *and* $W_2 \subseteq W$ *with* $\psi(V_2, W_2) = 0$.
   b) *Every inner ideal of* $\mathfrak{fsl}(V)$ *is Jordan-Lie.*
   c) *Every inner ideal of* $\mathfrak{fsl}(V)$ *is abelian.*

PROOF. Parts (1.), (2.) and (3.) are proved in [15].

4) (a) $\iff$ (c) This is proved in [15].
   (b) $\Rightarrow$ (a) This is obvious.
   (a) $\Rightarrow$ (b) Let $g = ef - f \in \mathfrak{F}(V)$. Then $g^2 = g$, $ge = 0$; $eg = 0$, $gf = g$ and $fg = f$. Thus, $g$ is idempotent with $eg = ge = 0$. Since $e\mathfrak{F}(V)f = e\mathfrak{F}(V)fg \subseteq e\mathfrak{F}(V)g$ and $e\mathfrak{F}(V)g = e\mathfrak{F}(V)gf \subseteq e\mathfrak{F}(V)f$, we get that $e\mathfrak{F}(V)f = e\mathfrak{F}(V)g$, as required.
5. Parts (a) and (d) are proved in [15].
   (b) Let $B \subseteq \mathfrak{fsl}(V)$ be inner ideal. By (a), $B = W_2^* V_2$ for some subspaces $V_2 \subseteq V$ and $W_2 \subseteq W$ with $\psi(V_2, W_2) = 0$. Hence, by (4.), $B = e\mathfrak{F}(V)f$ for some idempotents $e$ and $f$ in $\mathfrak{F}(V)$ with $fe = 0$. It remains show that $B^2 = 0$. Let $b, c \in B = e\mathfrak{F}(V)f$. Then there exist $x, y \in \mathfrak{F}(V)$ such that $b = exf$ and $c = eyf$. Since $bc = (exf)(eyf) = ex(fe)yf = ex0yf = 0$, $B^2 = 0$. Therefore, $B$ is Jordan-Lie, as required.
   (c) Let $B \subseteq \mathfrak{fsl}(V)$ be inner ideal. Then by (b), $B$ is Jordan-Lie, so $B^2 = 0$. Thus, $[B, B] \subseteq B^2 = 0$. Therefore, $B$ is abelian.

□

## References


1. A. A. Baranov, *Finitary simple Lie algebras*, J. Algebra **219** (1) (1999) 299–329.
2. A. A. Baranov, A. Mudrov and H. M. Shlaka, *Wedderburn–Malcev decomposition of one-sided ideals of finite dimensional algebras*, Commun. Algebra **46** (8) (2018) 3605–3607.
3. A. A. Baranov and H. M. Shlaka, *Jordan-Lie inner ideals of the finite dimensional associative algebras,* J. Pure Appl. Algebra **224** (5) (2020) 106189.
4. A. A. Baranov and H. Strade, *Finitary Lie algebras*, J. Algebra **254** (2002) 173–211.
5. W. E. Baxter, *Lie simplicity of a special class of associative rings II,* Trans. Amer. Math. Soc. **87** (1958) 63–75.
6. G. Benkart, *The Lie inner ideal structure of associative rings,* J. Algebra 43 (1976) 561–584.

7. G. Benkart, *On inner ideals and ad-nilpotent elements of Lie algebras*, Trans. Amer. Math. Soc. **232** (1977) 61–81
8. I. N. Herstein, *On the Lie ring of a division ring*, J. Ann. Math. (1954) 571–75.
9. I. N. Herstein, *On the Lie ring of a simple ring*, Proceedings of the National Academy of Sciences of the United States of America, **40** (5) (1954) 305–306.
10. I. N. Herstein, *On the Lie and Jordan rings of a simple associative ring,* Amer. J. Math. **77** (2) (1955) 279–285.
11. I. N. Herstein, *Lie and Jordan system in simple rings with involution,* Amer. J. Math. **78** (6) (1956) 629–649.
12. I. N. Herstein, *Lie and Jordan structures in simple associative rings*, J. Bull. Amer. Math. Soc. **67** (6) (1961) 517–531.
13. I. N. Herstein, *On the Lie structure of an associative ring,* J. Algebra **14** (4) (1970) 561-571.
14. I. N. Herstein, *Certain submodules of simple rings with involution II*, Can. J. Math. **27** (3) (1975) 629–635.
15. A. Fernández López, E. García and M. Gómez Lozano, *Inner ideals of finitary simple Lie algebras*, J. Lie Theory **16** (1) (2006) 97–114.
16. A. Fernández López, E. García and M. Gómez Lozano, *An Artinian theory for Lie algebras*, J. Algebra **319** (3) (2008) 938–951.
17. W. S. Martindale and C. R. Miers, *Herstein's Lie theory revisited,* J. Algebra **98** (1986) 14–37.
18. H. M. Shlaka, *Locally finite associative algebras and their Lie subalgebras*, J. Phys.: Conf. Ser. **1591** (2020) 012058.

E-mail: hasan.shlaka@uokuf a.edu.iq

# Projective Dimension over Regular Local Rings

Farzaneh Vahdanipour*
Faculty of Sciences, Department of Mathematics, University of Mohaghegh Ardabili, Ardabil, Iran

Abstract. Let $(R, \mathfrak{m}, k)$ be a regular local ring and $M$ be a finitely generated $R$-module. We prove some homological results using some basic properties of homomorphisms between injective modules. Assume that $n \geq 1$ is an integer such that $\operatorname{Tor}_n^R(M, k) \simeq k$. It is shown that $\operatorname{pd}_R M = n$.
**Keywords:** Complete local ring, Flat resolution, Free resolution, Noetherian ring, Injective resolution, Regular ring.
**AMS Mathematical Subject Classification [2010]:** 13E05, 13D05.

## 1. Introduction

Throughout, let $(R, \mathfrak{m})$ denote a commutative Noetherian local ring with identity. In this paper, for any $R$-module $M$ we denote the injective envelope of $M$ by $E_R(M)$. Also, we denote the injective dimension of $M$ by $\operatorname{id}_R M$. Finally, we denote the projective dimension and the flat dimension of $M$ by $\operatorname{pd}_R M$ and $\operatorname{fd}_R M$, respectively.

One of the important and hard problems in local algebra is to determine the homological dimensions of finitely generated modules over local rings. Concerning this topic there are a lot of results in the literature. In this paper we shall prove some results concerning the homomorphisms between injective modules. Then, as our main result, we shall prove the following theorem:

Theorem 1.1. *Let $(R, \mathfrak{m}, k)$ be a regular local ring and $M$ be a non-zero finitely generated $R$-module. Let $n \geq 1$ be an integer such that $\operatorname{Tor}_n^R(M, k) \simeq k$. Then $\operatorname{pd}_R M = n$.*

For any unexplained notation and terminology we refer the reader to [**1**, **2**] and [**3**].

## 2. Main Results

We start this section with the following auxiliary lemmas which are needed in the proof of Theorem 2.8.

Lemma 2.1. [**2**, Exercise 18.6] *Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local ring and $M$ be an $R$-module. If $M$ is faithful $R$-module and is an essential extension of $k$, then $M \simeq E_R(k)$.*

Lemma 2.2. *Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local domain. If $E$ is non-zero injective $R$-module and $f : E \longrightarrow E_R(k)$ is a non-zero $R$-homomorphism, then $f$ is an epimorphism.*

*Speaker

Proof. Set $M := \operatorname{im} f$. By Lemma 2.1, it is enough to show that $M$ is faithful. Assume the opposite which means there is an element $0 \neq x \in \mathfrak{m}$ such that $xM = 0$. Let $\widetilde{f} : E \longrightarrow M$ denote the map induced by $f$. Applying the functor $- \otimes_R R/xR$ to the exact sequence

$$E \xrightarrow{\widetilde{f}} M \longrightarrow 0,$$

we get the exact sequence

$$E/xE \longrightarrow M \longrightarrow 0,$$

which implies that $E/xE \neq 0$. On the other hand, by applying the exact functor $\operatorname{Hom}_R(-, E)$ to the exact sequence

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow R/xR \longrightarrow 0,$$

we get the exact sequence

$$E \xrightarrow{x} E \longrightarrow 0,$$

which implies that $E/xE = 0$, a contradiction. □

Proposition 2.3. *Let $(R, \mathfrak{m}, k)$ be a complete local regular ring and $E$ be a non-zero injective $R$-module. If $f : E \longrightarrow E_R(k)$ is a non-zero $R$-homomorphism, then $f$ is an epimorphism.*

Proof. As any regular ring is domain, the assertion follows by Lemma 2.2. □

Proposition 2.4. *Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local domain and $M$ be a non-zero $R$-module. Suppose that*

$$0 \longrightarrow M \xrightarrow{\varepsilon} E_0 \xrightarrow{f_0} E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} \cdots$$

*is an injective resolution of $M$ and $t \geq 1$ is an integer such that $E_t \simeq E_R(k)$. Then $\operatorname{injdim}_R M \leq t$.*

Proof. As the $R$-homomorphism $f_{t-1} : E_{t-1} \longrightarrow E_t$ is non-zero and $E_t \simeq E_R(k)$, it follows that $f_{t-1}$ ia an epimorphism by Lemma 2.2. Hence, the exact sequence

$$0 \longrightarrow M \xrightarrow{\varepsilon} E_0 \xrightarrow{f_0} E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} \cdots \longrightarrow E_{t-1} \xrightarrow{f_{t-1}} E_t \longrightarrow 0,$$

is an injective resolution of $M$. Therefore, by the definition, $\operatorname{id}_R M \leq t$. □

Proposition 2.5. *Let $(R, \mathfrak{m}, k)$ be a complete regular local ring and $M$ be a non-zero $R$-module. Suppose that*

$$0 \longrightarrow M \xrightarrow{\varepsilon} E_0 \xrightarrow{f_0} E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} \cdots$$

*is a minimal injective resolution of $M$ and $t \geq 1$ is an integer such that $E_t \simeq E_R(k)$. Then $\operatorname{id}_R M = t$.*

Proof. The assertion follows by Proposition 2.4. □

Proposition 2.6. *Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local domain and $M$ be a non-zero $R$-module. Suppose that*

$$\cdots \longrightarrow Q_2 \xrightarrow{g_1} Q_1 \xrightarrow{g_0} Q_0 \xrightarrow{\pi} M \longrightarrow 0,$$

*is a flat resolution of $M$ and $t \geq 1$ is an integer such that $Q_t \simeq R$. Then $\operatorname{fd}_R M \leq t$.*

PROOF. It is enough to prove that the map $g_{t-1}$ is a monomorphism. Let $D(-)$ denote the Matlis dual functor $\mathrm{Hom}_R(-, E_R(k))$. By applying the exact functor $D(-)$ to the exact sequence

$$0 \longrightarrow \ker g_{t-1} \longrightarrow Q_t \xrightarrow{g_{t-1}} Q_{t-1} \longrightarrow \cdots \longrightarrow Q_2 \xrightarrow{g_1} Q_1 \xrightarrow{g_0} Q_0 \xrightarrow{\pi} M \longrightarrow 0,$$

we get an exact sequence

$$0 \longrightarrow D(M) \xrightarrow{\varepsilon} E_0 \xrightarrow{f_0} E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} \cdots \longrightarrow E_{t-1} \xrightarrow{f_{t-1}} E_t \longrightarrow D(\ker g_{t-1}) \longrightarrow 0,$$

where, for each $0 \leq i \leq t$, the $R$-module $E_i := \mathrm{Hom}_R(Q_i, E_R(k))$ is an injective $R$-module and $E_t \simeq E_R(k)$. Thus, by Lemma 2.2, the map $f_{t-1}$ is an epimorphism so that $D(\ker g_{t-1}) = 0$. Therefore, $\ker g_{t-1} = 0$. □

LEMMA 2.7. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, $M$ be a finitely generated $R$-module and*

$$\cdots \longrightarrow L_2 \xrightarrow{h_1} L_1 \xrightarrow{h_0} L_0 \xrightarrow{\pi} M \longrightarrow 0,$$

*be a minimal free resolution of $M$. Then the following statements hold.*

i) $\dim_k \mathrm{Tor}_i^R(M, k) = \mathrm{rank}\, L_i$ *for each* $i \geq 0$.

ii) $\mathrm{pd}_R M = \sup\{i \in \mathbb{N}_0 \ : \ \mathrm{Tor}_i^R(M, k) \neq 0\}$.

PROOF. See [2, §7, Lemma 1]. □

THEOREM 2.8. *Let $(R, \mathfrak{m}, k)$ be a regular local ring and $M$ be a non-zero finitely generated $R$-module. Suppose that $n \geq 1$ is an integer such that $\mathrm{Tor}_n^R(M, k) \simeq k$. Then $\mathrm{pd}_R M = n$.*

PROOF. Using the fact that $\widehat{R}$ is a faithfully flat $R$-algebra and considering the Lemma 2.7(ii), without loss of generality, we may assume that $R$ is a complete regular local ring. Let

$$\cdots \longrightarrow L_2 \xrightarrow{h_1} L_1 \xrightarrow{h_0} L_0 \xrightarrow{\pi} M \longrightarrow 0,$$

be a minimal free resolution of $M$. Then, by hypothesis and Lemma 2.7(i), it follows that $L_n \simeq R$. Now, the assertion follows by Proposition 2.6 and Lemma 2.7(ii). □

## References

1. W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, Cambridge 1998.
2. H. Matsumura, *Commutative Ring Theory*, Cambridge University Press, Cambridge 1986.
3. J. Rotman, *Introduction to Homological Algebra*, 2nd ed., Springer, New York 2009.

E-mail: farzaneh.vahdani@gmail.com

# The Quasi-Frobenius Elements of Simplicial Affine Semigroups

Raheleh Jafari
Mosaheb Institute of Mathematics, Kharazmi University, Tehran, Iran
and Marjan Yaghmaei*
Faculty of Mathematical Sciences and Computer, Kharazmi University, Tehran, Iran

ABSTRACT. The quasi-Frobenius elements of simplicial affine semigroups are introduced as a generalization of pseudo-Frobenius numbers of numerical semigroups.
**Keywords:** Simplicial affine semigroup, Pseudo-Frobenius element, Apéry set.
**AMS Mathematical Subject Classification [2010]:** 20M14,13D02.

## 1. Introduction

Let $S$ be an affine semigroup in $\mathbb{N}^d$, where $\mathbb{N}$ denotes the set of non-negative integers. The affine semigroup ring $\mathbb{K}[S]$, over a field $\mathbb{K}$, is defined as the subring $\oplus_{\mathbf{a}\in S}\mathbb{K}\mathbf{x}^{\mathbf{a}}$ of the polynomial ring $\mathbb{K}[\mathbf{x}] = \mathbb{K}[x_1, \ldots, x_d]$. If $d = 1$, then $S$ is a submonoid of $\mathbb{N}$. Let $h$ be the greatest common divisor of non-zero elements in $S$. Dividing all elements of $S$ by $h$, we obtain an isomorphic semigroup in $\mathbb{N}$. A submonoid $S$ of $\mathbb{N}$ such that $\gcd(s; s \in S) = 1$ is called a *numerical semigroup*. In other words, the study of affine semigroups in $\mathbb{N}$ reduces to the study of numerical semigroups. The condition $\gcd(s; s \in S) = 1$ is equivalent to say that $\mathbb{N} \setminus S$ is a finite set, [7, Lemma 2.1]. Consider the natural partial ordering $\preceq_S$ on $\mathbb{N}$ where, for all elements $x, y \in \mathbb{N}$, $x \preceq_S y$ if $y - x \in S$. The maximal elements of $\mathbb{N} \setminus S$ with respect to $\preceq_S$ are called *pseudo-Frobenius numbers*. Fröberg, Gottlieb and Häggkvist [4], defined the type of the numerical semigroup $S$ as the cardinality of the set of its pseudo-Frobenius numbers. This notion of type coincides with the Cohen-Macaulay type of the numerical semigroup ring $\mathbb{K}[S]$, see [8] for a detailed proof.

By analogy, García-García, Ojeda, Rosales and Vingneron-Tenorio, define a pseudo-Frobenius element of $S$ to be an element $\mathbf{a} \in \mathbb{N}^d \setminus S$ such that $\mathbf{a}+S\setminus\{0\} \subseteq S$, in [5]. They show that the set of pseudo-Frobenius elements of $S$, $\mathrm{PF}(S)$, is not empty, precisely when $\operatorname{depth} \mathbb{K}[S] = 1$. Thus, when $d > 1$ and $\mathbb{K}[S]$ is a Cohen-Macaulay ring, the set of pseudo-Frobenius elements of $S$ is empty and express noting about the Cohen-Macaulay type of the semigroup ring.

In this paper, we present another generalization of pseudo-Frobenius numbers, called *quasi Frobenius* elements. The number of quasi Frobenius elements determines the Cohen-Macaulay type of the semigroup ring $\mathbb{K}[S]$, under the assumption that the affine semigroup $S \subset \mathbb{N}^d$ is simplicial, i.e. the rational polyhedral cone

*Speaker

spanned by $S$ has $d$ extremal rays. All affine semigroups in $\mathbb{N}^d$, for $d = 1, 2$, are simplicial.

## 2. Quasi Frobenius Elements

Throughout this section, $\mathbb{K}$ is a field and $S \subseteq \mathbb{N}^d$ is an affine semigroup minimally generated by $\operatorname{mgs}(S) = \{\mathbf{a}_1, \ldots, \mathbf{a}_e\}$. The semigroup ring $\mathbb{K}[S] = \mathbb{K}[\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_e}]$ has a unique maximal monomial ideal $\mathfrak{m} = (\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_e})$. The affine semigroup $S$ is called *simplicial* if there exist $d$ elements $\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_d} \in \operatorname{mgs}(S)$ such that they are linearly independent over the field of rational numbers $\mathbb{Q}$ (equivalently, over the field of real numbers $\mathbb{R}$), and

$$S \subseteq \sum_{j=1}^{d} \mathbb{Q}_{\geq 0} \mathbf{a}_{i_j}.$$

Let $\operatorname{cone}(S)$ denote the rational polyhedral cone spanned by $S$. Then $\operatorname{cone}(S)$ is the intersection of finitely many closed linear half-spaces in $\mathbb{R}^d$, each of whose bounding hyperplanes contains the origin. These half-spaces are called *support hyperplanes*. The integral vectors in each support hyperplanes, is a face of $S$, and all maximal faces (called facets) are in this form. The intersection of any two adjacent support hyperplane is a one-dimensional vector space, which is called an *extremal ray*. The $\operatorname{cone}(S)$ has at least $d$ facets and at least $d$ extremal rays. It has $d$ facets (equivalently, it has $d$ extremal rays), precisely when $S$ is simplicial.

On each extremal ray of $\operatorname{cone}(S)$, the componentwise smallest element from $S$, is called an *extremal ray* for $S$. Assume that $S$ is simplicial and denote by $\mathbf{a}_1, \ldots, \mathbf{a}_d$ the extremal rays for $S$. Then for each $\mathbf{a} \in S$, we have $n\mathbf{a} \in \mathbb{N}\mathbf{a}_{i_1} + \cdots + \mathbb{N}\mathbf{a}_{i_d}$, for some positive integer $n$.

Let $E = \{\mathbf{a}_1, \ldots, \mathbf{s}_d\}$ and

$$\operatorname{Ap}(S, E) = \{\mathbf{a} \in S \ ; \ \mathbf{a} - \mathbf{a}_i \notin S, \text{ for } i = 1, \ldots, d\}.$$

Definition 2.1. The element $\mathbf{b} - \sum_{i=1}^{d} \mathbf{a}_i$, where $\mathbf{b} \in \operatorname{Max}_{\preceq_S} \operatorname{Ap}(S, E)$, is called a *quasi-Frobenius* element. The set of quasi-Frobenius elements of $S$ is denoted by $\operatorname{QF}(S)$.

Remark 2.2. Let $d > 1$. If $\mathbf{f} \in \operatorname{QF}(S) \cap \operatorname{PF}(S)$, then $f + \mathbf{a}_1 = \mathbf{m} - \sum_{i=2}^{d} \mathbf{a}_i$, where $\mathbf{m} \in \operatorname{Max}_{\preceq_S} \operatorname{Ap}(S, E)$. Since $\mathbf{f} \in \operatorname{PF}(S)$, this follows $\mathbf{f} + \mathbf{a}_1 \in S$, which contradicts $\mathbf{m} - \sum_{i=2}^{d} \mathbf{a}_i \notin S$.

The type of a $d$-dimensional Cohen-Macaulay local ring $(R, \mathfrak{m})$ is $\operatorname{type}(R) = \dim_{R/\mathfrak{m}} \operatorname{Ext}_R^d(R/\mathfrak{m}, R)$. For a Cohen-Macaulay ring $R$, the type is defined as the maximum of $\operatorname{type}(R_\mathfrak{p})$, where $\mathfrak{p}$ ranges in the set of maximal ideals of $R$.

The ring $\mathbb{K}[S]$ is $\mathbb{N}$-graded by setting $\deg(\mathbf{x}^{\mathbf{a}}) = |\mathbf{a}|$, for all $\mathbf{a} \in S$, where $|(a_1, \ldots, a_d)| = \sum_{i=1}^{d} a_i$, denotes the total degree. Therefore,

$$\operatorname{type}(\mathbb{K}[S]) = \operatorname{type}(\mathbb{K}[S]_\mathfrak{m}),$$

by [1, Theorem].

Theorem 2.3. *If $\mathbb{K}[S]$ is a Cohen-Macaulay ring, then*

$$|\operatorname{QF}(S)| = \operatorname{type}(\mathbb{K}[S])_\mathfrak{m} = \operatorname{type}(\mathbb{K}[\![S]\!]).$$

Proof. The ring map $\mathbb{K}[S]_{\mathfrak{m}} \longrightarrow \mathbb{K}[\![S]\!]$ is flat and has only one trivial fiber which is the field $\mathbb{K}$. Thus, $\mathbb{K}[\![S]\!]$ is Cohen-Macaulay and

$$\operatorname{type}(\mathbb{K}[S])_{\mathfrak{m}} = \operatorname{type}(\mathbb{K}[\![S]\!]),$$

by [2, Proposition 1.2.16]. Let $R = \mathbb{K}[\![S]\!]$. Then $R$ is a local ring with maximal ideal $\mathfrak{m} = (\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_{d+r}})$. Note that $\mathfrak{q} = (\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_d})$ is a parameter ideal of $R$, since $S$ is simplicial. As $R$ is Cohen-Macaulay, $\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_d}$ provides a maximal $R$-regular sequence. By [2, Lemma 1.2.19],

$$\operatorname{type}(R) = \dim_{R/\mathfrak{m}}(\operatorname{Hom}_R(R/\mathfrak{m}, R/\mathfrak{q}).$$

Since $\operatorname{Hom}_R(R/\mathfrak{m}, R/\mathfrak{q}) \cong (0 :_{R/\mathfrak{q}} \mathfrak{m}) = \{r \in R/\mathfrak{q} \ ; \ r\mathfrak{m} = 0\}$, it is enough to show that $(0 :_{R/\mathfrak{q}} \mathfrak{m})$ is the $R/\mathfrak{m}$-vector space generated by residue classes of $\mathbf{x}^{\mathbf{s}}$, where $\mathbf{s} \in \operatorname{Max}_{\preceq_S} \operatorname{Ap}(S, E)$. For an element, $\mathbf{f} \in R$, the residue of $\mathbf{f}$ in $R/\mathfrak{q}$ is equal to the residue of $\sum_{i \geq 1} r_i \mathbf{x}^{\mathbf{s}_i}$, for some $r_i \in \mathbb{K}$ and $\mathbf{s}_i \in \operatorname{Ap}(S, E)$. If the residue of $\mathbf{f}$ in $R/\mathfrak{q}$, belongs to $(0 :_{R/\mathfrak{q}} \mathfrak{m})$, then we derive $\mathbf{x}^{\mathbf{s}_i + \mathbf{a}_j} \in \mathfrak{q}$, for $i \geq 1$ and $d+1 \leq j \leq d+r$ which implies $\mathbf{s}_i \in \operatorname{Max}_{\preceq_S} \operatorname{Ap}(S, E)$. Conversely, let $\mathbf{s} \in \operatorname{Max}_{\preceq_S} \operatorname{Ap}(S, E)$. Since $\mathbf{s} + \mathbf{a}_i \notin \operatorname{Ap}(S, E)$, for $i = d+1, \dots, d+r$, we get $\mathbf{x}^{\mathbf{s}+\mathbf{a}_i} \in \mathfrak{q}R$. □

Recall that a Cohen-Macaulay ring is Gorenstein precisely when its Cohen-Macaulay type is one. As an immediate consequence of Proposition 2.3, we derive the following:

Corollary 2.4. [6, 4.6, 4.8] $\mathbb{K}[S]$ *is a Gorenstein ring if and only if it is Cohen-Macaulay and* $\operatorname{Ap}(S, E)$ *has a single maximal element with respect to* $\preceq_S$.

The following example shows that $|\operatorname{QF}(S)|$ might be arbitrary large for a simplicial affine semigroup $S \subset \mathbb{N}^2$, independently of its embedding dimension.

Example 2.5. For an integer $a \geq 3$, let $S$ be the affine semigroup generated by $\mathbf{a}_1 = (a^2, 0)$, $\mathbf{a}_2 = (0, a^2)$, $\mathbf{a}_3 = (a^2 - a, a^2 - a)$, $\mathbf{a}_4 = (a^2 - a + 1, a^2 - a + 1)$, $\mathbf{a}_5 = (a^2 - 1, a^2 - 1)$. Then $S$ is simplicial with extremal rays $\mathbf{a}_1$, $\mathbf{a}_2$. Let $T$ be the numerical semigroup generated by $\{a^2 - a, a^2 - a + 1, a^2 - 1, a^2\}$. Then

$$\operatorname{Ap}(S, E) = \operatorname{Ap}(S, \mathbf{a}_1 + \mathbf{a}_2) = \{(s, s) \ ; s \in \operatorname{Ap}(T, a^2)\}.$$

Therefore, $|\operatorname{QF}(S)| = \operatorname{type}(T) = 2a - 4$, by [3, (3.4)Proposition].

## References


1. Y. Aoyama and S. Goto, *On the type of graded Cohen-Macaulay rings*, J. Math. Kyoto Univ. **15** (1975) 19–23.
2. W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Mathematics, Vol. 39, Cambridge University Press, Cambridge, 1993.
3. M. P. Cavaliere and G. Niesi, *On monomial curves and Cohen-Macaulay type*, Manuscripta Math. **42** (2-3) (1983) 147–159.
4. R. Fröberg, C. Gottlieb and R. Häggkvist, *On numerical semigroups*, Semigroup Forum **35** (1) (1987) 63–83.
5. J. I. García-García, I. Ojeda, J. C. Rosales and A. Vigneron-Tenorio, *On pseudo-Frobenius elements of submonoids of* $\mathbb{N}^d$, Collect. Math. **71** (1) (2020) 189–204.
6. J. C. Rosales and P. A. García-Sánchez, On Cohen-Macaulay and Gorenstein simplicial affine semigroups, Proc. Edinburgh Math. Soc. **41** (3) (1998) 517–537.
7. J. C. Rosales and P. A. García-Sánchez, *Numerical Semigroups,* Vol. 20, Developments in Mathematics, Springer, New York, 2009.

8. D. I. Stamate, *Betti numbers for numerical semigroup rings*, in: Multigraded Algebra and Applications, V. Ene, E. Miller (eds.), NSA 2016, Springer Proc. Math. Stat., 238, Springer, Cham, (2018) pp. 133–157.

E-mail: rjafari@ipm.ir
E-mail: hasti.tmu83@yahoo.com

# On $n$-Centralizer CA-Groups

Mohammad Ali Iranmanesh
Department of Mathematics, Yazd University, Yazd, I. R. Iran
and Mohammad Hossein Zareian*
Department of Mathematics, Yazd University, Yazd, I .R. Iran

ABSTRACT. Let $G$ be a finite non-abelian group and $m = \frac{|G|}{|Z(G)|}$. In this paper we prove that if $G$ is a finite non-abelian $m$-centralizer CA-group, then there exists an integer $1 < r$ such that $m = 2^r$. It is also prove that if $|G'| = 2$, then $G$ is an $m$-centralizer group.
**Keywords:** $m$-Centralizer group, CA-group.
**AMS Mathematical Subject Classification [2010]:** 20C15, 20D15.

## 1. Introduction

Throughout the paper all groups are assumed to be finite. Let $G$ be a group. Then by $Z(G), G', |G|, C_G(x), \mathrm{Cent}(G)$ and $x^G$ we denote the center of $G$, the order of $G$, the derived subgroup of $G$, the centralizer of $x \in G$, the set of centralizers of $G$ and the conjugacy class of $x \in G$ respectively. We consider two equivalence relations on $G$ namely $\sim_1$ and $\sim_2$. We say $x \sim_1 y$ if and only if $C_G(x) = C_G(y)$. Also $x \sim_2 y$ if and only if $xZ(G) = yZ(G)$. The equivalence class including $x$ by is denoted by $[x]_\sim$. The number of equivalence classes of $\sim_1$ and $\sim_2$ on $G$ are equal to $|\mathrm{Cent}(G)|$ and $\frac{|G|}{|Z(G)|}$, respectively.

A group $G$ is called $m$-centralizer if $|\mathrm{Cent}(G)| = m$. The influence of $|\mathrm{Cent}(G)|$ on $G$ has been investigated in [**2, 3, 4**]. It is clear by definition that a group $G$ is 1-centralizer if and only if it is abelian. There is no finite $m$-centralizer groups for $m \in \{2, 3\}$. A non-abelian group $G$ is called a CA-group if $C_G(x)$ is abelian for all $x \in G \setminus Z(G)$.

The main purpose of this paper is to study $m$-centralizer CA-groups where $m = \frac{|G|}{|Z(G)|}$ and $m \neq 2, 3$. We show that a non-abelian group $G$ is $m$-centralizer if and only if $[x]_{\sim_1} = [x]_{\sim_2}$ for all $x \in G$. Also if $G$ is an $m$-centralizer CA-group, then there exists an integer $r > 1$ such that $m = 2^r$. Conversely for an arbitrary integer $r > 1$, there exists an $m$-centralizer CA-group where $m = 2^r$. It is also prove that if $|G'| = 2$, then $G$ is an $m$-centralizer group.

## 2. Preliminaries

LEMMA 2.1. [**1**, Lemma 3.6] *Let $G$ be a group. Then the following are equivalent:*

1) *$G$ is a* CA-*group.*

*Speaker

2) *If* $[x, y] = 1$ *then* $C_G(x) = C_G(y)$ *where* $x, y \in G \setminus Z(G)$.
3) *If* $[x, y] = [x, z] = 1$ *then* $[y, z] = 1$ *where* $x \in G \setminus Z(G)$.
4) *If* $A, B \leq G, Z \lneqq C_G(A) \leqslant C_G(B) \lneqq G$, *then* $C_G(A) = C_G(B)$.

Lemma 2.2. [1, Proposition 2.6] *Let* $G$ *be a finite non-Abelian group and* $\Gamma(G)$ *be a regular graph. Then* $G$ *is nilpotent of class at most* 3 *and* $G = A \times P$, *where* $A$ *is an Abelian group and* $P$ *is a* $p$*-group (*$p$ *is a prime) and furthermore* $\Gamma(P)$ *is a regular graph.*

Lemma 2.3. [6, Theorems A, B]

1) *Every finite* 2*-group* $G$ *of class two with cyclic center, either has the central decomposition:* $G \cong Q(n_1, r_1) \ldots Q(n_\alpha, r_\alpha)Q(l, l)^{\epsilon_l} \ldots Q(1, 1)^{\epsilon_1}$, *where* $\alpha \geq 0, \epsilon_i \geq 0, i = 1, \ldots, l, n_1 > \cdots > n_\alpha \geq l \geq 1,\ n_\alpha > r_1 > \cdots > r_\alpha \geq 0,\ 1 < n_1 - r_1 < \cdots < n_\alpha - r_\alpha$, *or else it has the central decomposition:*
$$G \cong R(n)Q(l, l)^{\epsilon_l} \ldots Q(1, 1)^{\epsilon_1},$$
*where* $n \geq l \geq 1, \epsilon_i \geq 0, i = 1, \ldots, l$.
2) *The above canonical decomposition is unique up to isomorphism.*

Lemma 2.4. [7, Lemma 4] *Let* $p$ *be a prime number and let* $r > 1$ *be a integer. Then there exists a non-Abelian* $p$*-group* $P$ *of order* $p^{2r}$ *such that:*

1) $|Z(P)| = p^r$;
2) $P/Z(P)$ *is an elementary Abelian* $p$*-group;*
3) *for every noncentral element* $x$ *of* $P$, $C_P(x) = Z(P)\langle x\rangle$.

Lemma 2.5. [5, Theorem 2.1] *Let* $G$ *be a non-Abelian group and* $|\mathrm{Cent}(G)| = |G|/|Z(G)|$. *Then* $G/Z(G)$ *is an elementary Abelian 2-group.*

## 3. Main Results

In this section we give main results.

Lemma 3.1. *A non-abelian group* $G$ *is said to be an* $m$*-centralizer group where* $m = \frac{|G|}{|Z(G)|}$ *if and only if* $[x]_{\sim_1} = [x]_{\sim_2}$, *for all* $x \in G$.

Lemma 3.2. *Let* $G$ *be a non-abelian group. Then the following are equivalent:*

a) *If* $[x, y] = 1$, *then* $[x]_{\sim_2} = [y]_{\sim_2}$, *where* $x, y \in G \setminus Z(G)$.
b) $G$ *is a* CA*-group and* $[x]_{\sim_1} = [x]_{\sim_2}$, *for all* $x \in G$.
c) *If* $[x, y] = 1$ *and* $[x, w] = 1$, *then* $[y]_{\sim_2} = [w]_{\sim_2}$, *where* $x, y, w \in G \setminus Z(G)$.

Lemma 3.3. *Let* $G$ *be a non-abelian group. Let* $[x]_{\sim_1}$ *and* $[y]_{\sim_1}$ *be two different classes of relation* $\sim_1$. *If* $[x_0, y_0] \neq 1$ *where* $x_0 \in [x]_{\sim_1}$ *and* $y_0 \in [y]_{\sim_1}$, *then* $[u, v] \neq 1$ *for all* $u \in [x]_{\sim_1}$ *and* $v \in [y]_{\sim_1}$. *Also* $[x_1, x_2] = 1$ *for all* $x_1, x_2 \in [x]_{\sim_1}$.

Lemma 3.4. *Let* $G$ *be a non-Abelian group. Then* $[x]_{\sim_2} \subseteq [x]_{\sim_1}$, *for all* $x \in G$.

Lemma 3.5. *Let* $G$ *be a non-Abelian group. Then the following are equivalent:*

1) *If* $[x, y] = 1$, *then* $xZ(G) = yZ(G)$ *where* $x, y \in G \setminus Z(G)$.
2) $G$ *is a* CA*-group and* $[x]_{\sim_1} = [x]_{\sim_2}$, *for all* $x \in G \setminus Z(G)$.
3) $[x, y] = 1$ *and* $[x, w] = 1$ *imply that* $yZ(G) = wZ(G)$ *where* $x, y, w \in G \setminus Z(G)$.

LEMMA 3.6. *Let $G$ be a non-Abelian group. Then $C_G(x) = Z(G) \cup xZ(G)$, for all $x \in G \setminus Z(G)$ if and only if $G$ is a* CA-*group and $[x]_{\sim_1} = [x]_{\sim_2}$, for all $x \in G$.*

LEMMA 3.7. *Let $G$ be a non-Abelian group. Then the following are equivalent:*

1) $|G| = 4|Z(G)|$.
2) *$G$ is a CA-group, $[x]_{\sim_1} = [x]_{\sim_2}$ for all $x \in G \setminus Z(G)$ and $|G'| = 2$.*
3) *$G = A \times P$, where $A$ is an Abelian group, $P$ is both a 2-group and a* CA-*group and $|P'| = 2$.*

THEOREM 3.8. *Let $G$ be a non-abelian group and $|G^{'}| = 2$. Then $G$ is an $m$-centralizer group where $m = \frac{|G|}{|Z(G)|}$.*

THEOREM 3.9. *Let $G$ be a non-abelian group. Then $C_G(x) = Z(G) \cup xZ(G)$ for all $x \in G \setminus Z(G)$ if and only if $G$ is an $m$-centralizer* CA-*group where $m = \frac{|G|}{|Z(G)|}$.*

EXAMPLE 3.10. The dihedral group $D_8$ is a CA-group and $C_{D_8}(x) = Z(D_8) \cup xZ(D_8)$ for all $x \in D_8 \setminus Z(D_8)$.

PROPOSITION 3.11. *Let $G$ be a* CA-*group and $|Z(G)| = p$ where $p$ is a prime. Then $[x]_1 = [x]_2$ for all $x \in G$ if and only if $G \cong D_8$ or $Q_8$.*

PROOF. Let $[x]_{\sim_1} = [x]_{\sim_2}$. By Lemma 2.5 $G/Z(G)$ and $G'$ are both elementary Abelian 2-group. Therefore $|Z(G)| = |G'| = 2$ and $G$ is an extra-special 2-group. Since $G$ is a CA-group and $[x]_{\sim_1} = [x]_{\sim_2}$ we find by Lemma 3.7, that $|G| = 4|Z(G)$. Hence $|G| = 8$. Since the extra-special groups of order 8 are $D_8$ and $Q_8$, we have $G \cong D_8$ or $Q_8$.

Conversely $D_8$ and $Q_8$ are CA-groups and $[x]_{\sim_1} = [x]_{\sim_2}$ for all $x \in D_8, Q_8$. □

LEMMA 3.12. *Let $G_1$ and $G_2$ be two groups. Let $[g_1]_{\sim_1} = [g_1]_{\sim_2}$ for all $g_1 \in G_1$ and $[g_2]_{\sim_1} = [g_2]_{\sim_2}$ for all $g_2 \in G_2$. Then $[X]_{\sim_1} = [X]_{\sim_2}$, for all $X \in G_1 \times G_2$.*

PROOF. Let $[g_1]_{\sim_1} = [g_1]_{\sim_2}$ for all $g_1 \in G_1$ and $[g_2]_{\sim_1} = [g_2]_{\sim_2}$ for all $g_2 \in G_2$. Let $Y \in [X]_{\sim_1}$ where $X, Y \in G_1 \times G_2$. So there exist $a_1, b_1 \in G_1$ and $a_2, b_2 \in G_2$ such that $X = (a_1, a_2)$ and $Y = (b_1, b_2)$. Then $C_{G_1 \times G_2}(a_1, a_2) = C_{G_1 \times G_2}(b_1, b_2)$. Therefore $C_{G_1}(a_1) \times C_{G_2}(a_2) = C_{G_1}(b_1) \times C_{G_2}(b_2)$. Moreover $C_{G_1}(a_1) = C_{G_1}(b_1)$ and $C_{G_2}(a_2) = C_{G_2}(b_2)$. By assumption $a_1 Z(G_1) = b_1 Z(G_1)$ and $a_2 Z(G_2) = b_2 Z(G_2)$. So $a_1 Z(G_1) \times a_2 Z(G_2) = b_1 Z(G_1) \times b_2 Z(G_2)$. Therefore $(a_1, a_2)(Z(G_1) \times Z(G_2)) = (b_1, b_2)(Z(G_1) \times Z(G_2))$ and $Y \in [X]_{\sim_2}$. Hence $[X]_{\sim_1} \subseteq [X]_{\sim_2}$. Now by Lemma 3.4, $[X]_{\sim_1} = [X]_{\sim_2}$ for all $X \in G_1 \times G_2$. □

PROPOSITION 3.13. *Let $G$ be a non-Abelian group. Then $G$ is a* CA-*group and $[x]_{\sim_1} = [x]_{\sim_2}$, for all $x \in G$ if and only if $|G| = \frac{2|Z(G)|^2}{(3|Z(G)|-k)}$, where $k$ is the number of conjugacy classes of $G$. In particular if $G$ is a* CA-*group and $[x]_{\sim_1} = [x]_{\sim_2}$, for all $x \in G$, then $|G| \leq 2|Z(G)|^2$.*

PROOF. Let $G$ be a CA-group and $[x]_{\sim_1} = [x]_{\sim_2}$, for all $x \in G$. So $|E(\Gamma(G))| = (2^{\frac{|G|}{|Z(G)|}-1})|Z(G)|^2$. On the other hand by [1, Lemma 3.27], $|E(\Gamma(G))| = \frac{|G|^2 - k|G|}{2}$. Therefore $|G| = \frac{2|Z(G)|^2}{3|Z(G)|-k}$. Conversely let $|G| = \frac{2|Z(G)|^2}{(3|Z(G)|-k)}$. So $|G| = |Z(G)| + (k - |Z(G)|)\frac{|G|}{2|Z(G)|}$. On the other hand for all $x \in G \setminus Z(G), |x^G| \leq \frac{|G|}{2|Z(G)}$. Hence $|x^G| = \frac{|G|}{2|Z(G)|}$, for all $x \in G \setminus Z(G)$. So $|C_G(x)| = 2|Z(G)|$, for all $x \in G \setminus Z(G)$. By Lemma 3.6 $G$ is a CA-group and $[x]_{\sim_1} = [x]_{\sim_2}$. □

Proposition 3.14. *Let $G$ be a* CA*-group and $[x]_{\sim_1} = [x]_{\sim_2}$, for all $x \in G$. Then for all $x, y \in G \setminus Z(G)$ such that $[x, y] \neq 1$, there exists one and only one $[w]_{\sim_2}$ so that $[w]_{\sim_2} \neq [y]_{\sim_2}$ and $x^y = x^w$.*

Proof. Let $G$ be a CA-group and $[x]_{\sim_1} = [x]_{\sim_2}$, for all $x \in G$. Then $x^y = x^{xy}$. On the other hand if $x^y = x^w$ where $wZ(G) \neq yZ(G)$ then $wZ(G) = xyZ(G)$. □

In the following we show that if $G$ is a CA-group and $|\mathrm{Cent}(G)| = |G|/|Z(G)|$, then there exists integer $r > 1$ so that $|\mathrm{Cent}(G)| = 2^r$.

Proposition 3.15. *Let $G$ be a* CA*-group and $[x]_{\sim_1} = [x]_{\sim_2}$, for all $x \in G$. Then there exist subgroups $H_i \trianglelefteq G, i = 1, \dots, r$ such that $Z(G) \leq H_1 \leq \cdots \leq H_r = G, |H_i| = 2^i|Z(G)|, i = 1, \dots, r; Z(H_i) = Z(G), i \geq 2$ and $[h]_{\sim_1} = [h]_{\sim_2}$, for all $h \in H_i, i = 2, \dots, r$.*

Proof. Since $G$ is non-Abelian, so there exists $x, y \in G$ such that $[x, y] \neq 1$. Let $H_1 = Z(G)\langle x\rangle$. Then $H_1 \trianglelefteq G$, is Abelian and $|H_1| = 2|Z(G)|$. Let $H_2 = H_1\langle y\rangle$. Then $H_2 \trianglelefteq G, Z(H_2) = Z(G), |H_2| = 2^2|Z(G)|$ and $[x]_{\sim_1} = [x]_{\sim_2}$, for all $x \in H_2$. If $G \neq H_2$, then there exist $u \in G \setminus H_2$. Let $H_3 = H_2\langle u\rangle$. $H_3 \trianglelefteq G, Z(H_3) = Z(G), |H_3| = 2^3|Z(G)|$ and $[x]_{\sim_1} = [x]_{\sim_2}$, for all $x \in H_3$. Since $G$ is finite, there exists a positive integer $r$ such that $H_r = G, |G| = 2^r|Z(G)|$. Hence $|\mathrm{Cent}(G)| = |G|/|Z(G)| = 2^r$. □

Theorem 3.16. *Let $G$ be a non-abelian group. Then the following are equivalent:*

(a) *$G$ is an $m$-centralizer* CA*-group where $m = \frac{|G|}{|Z(G)|}$.*
(b) *$G = A \times P$ where $A$ is an abelian group and $P$ is a $2$-group,* CA*-group and $m$-centralizer where $m = \frac{|G|}{|Z(G)|}$.*
(c) *$G = A \times P$ where $A$ is an abelian group and $P$ is a $p$-group and $C_P(x) = Z(P) \cup xZ(P)$ for all $x \in P \setminus Z(P)$.*

Theorem 3.17. *Let $G$ be an $m$-centralizer* CA*-group where $m = \frac{|G|}{|Z(G)|}$. Then there exists an integer $r > 1$ such that $m = 2^r$. Conversely for an arbitrary integer $r > 1$, there exists an $m$-centralizer* CA*-group where $m = \frac{|G|}{|Z(G)|} = 2^r$.*

## References

1. A. Abdollahi, S. Akbari and H. R. Maimani, *Non commuting graph of group*, J. Algebra **28** (2006) 468–492.
2. A. Abdollahi, S. M. Jafarian Amiri and A. M. Hassanabadi, *Groups with specific number of centralizers*, Houston J. Math. **33** (1) (2007) 43–57.
3. A. R. Ashrafi, *On finite groups with a given number of centralizers*, Algebra Colloq. **7** (2) (2000) 139–146.
4. S. M. Belcastro and G. J. Sherman, *Counting centralizers in finite groups*, Math. Mag. **67** (5) (1994) 366–374.
5. S. M. Jafarian Amiri, H. Madadi and H. Rostami, *Finite groups with certain number of centralizers*, 3rd Biennial Int. Group Theory Conf., Mashhad, 2015.
6. Y. K. Leong, *Finite 2-groups of class two with cyclic center*, J. Austral. Math. Soc. Ser. A **27** (1979) 125–140.
7. A. R. Moghaddamfar, *About noncommuting graphs*, Sib. Math. J. **47** (5) (2006) 911–914

E-mail: iranmanesh@yazd.ac.ir
E-mail: m.zareiyan@stu.yazd.ac.ir

# Contributed Talks

## Analysis

# On the $GG$-Orthogonality in Normed Linear Spaces

Mohammed Yahya Abed*
College of Education for Pure Sciences, University of Kerbala, Kerbala, Iraq

ABSTRACT. The main aim of this paper is to study the relation between the $gg$-orthogonality and semi-inner product orthogonality in the real normed linear spaces. We also define the concept of $gg$-quasi inner product space and some results relative to this new notion are investigated.
**Keywords:** Inner product space, Semi-inner product space, Quasi-inner product space, $gg$-Orthogonality.
**AMS Mathematical Subject Classification [2010]:** 46B20, 47B99, 46C50, 46C99.

## 1. Introduction

Let $(X, \|\cdot\|)$ be a real normed linear space with dimension not less than 2. Miličić in [6] defined the mappings $\rho : X \times X \to \mathbb{R}$ as follows:

$$\rho(x, y) := \frac{\rho_-(x, y) + \rho_+(x, y)}{2},$$

such that $\rho_\pm : X \times X \to \mathbb{R}$ are norm derivatives

$$\rho_\pm(x, y) := \|x\| \lim_{t \to 0^\pm} \frac{\|x + ty\| - \|x\|}{t}.$$

Let $x, y \in X$. Norm derivative orthogonality relations were defined in [2, 3] as follows:

$$x \perp_{\rho_\pm} y \Leftrightarrow \rho_\pm(x, y) = 0, \quad x \perp_\rho y \Leftrightarrow \rho(x, y) = 0.$$

Some new norm derivative orthogonality relations were defined in [8]. Also, it is known that a semi-inner product space is a mapping from $[\cdot|\cdot]_s : X \times X \to \mathbb{R}$ satisfying the following conditions for each $x, y, z \in X$, and all $\alpha, \beta \in \mathbb{R}$:

(1) $[\alpha x + \beta y|z]_s = \alpha[x|z]_s + \beta[y|z]_s$,
(2) $[x|x]_s = \|x\|^2$,
(3) $[x|y]_s \leq \|x\|\|y\|$.

A vector $x \in X$ is called s.i.p-orthogonal to $y \in X$ if $[x|y]_s = 0$ [4]. Recently, the mapping $[\cdot, \cdot]_{gg} : X \times X \to \mathbb{R}$ was defined in [9] by

$$[x, y]_{gg} := \sqrt{|\rho(x, y)|\,|\rho(y, x)|}, \quad (x, y \in X).$$

A vector $x \in X$ is called $gg$-orthogonal to a vector $y \in X$, denoted $x \perp_{gg} y$, if $[x, y]_{gg} = 0$.

We need the following known results from [1] and [9].

LEMMA 1.1. [1, Remark 2.1.1] *Let $(X, \|\cdot\|)$ be a normed linear space. The following conditions are equivalent:*

*Speaker

(1) $\rho_-(x, y) = \rho_+(x, y)$ *for all* $x, y \in X$.
(2) $X$ *is smooth.*

Theorem 1.2. [9, Proposition 2.4] *The gg-orthogonality satisfies the following property:*

(a) *If* $x \perp_{gg} y$ *then* $y \perp_{gg} x$.
(b) *For every* $x, y \in X$ *there exists* $\alpha \in \mathbb{R}$ *such that* $x \perp_{gg} (\alpha x + y)$.

Theorem 1.3. [5, Theorem (Ficken, 1946)] *A normed linear spaces* $X$ *is an i.p.s if and only if for all* $x, y \in X$ *with* $\|x\| = \|y\|$ *and for all scalars* $a$ *and* $b$, $\|ax + by\| = \|ay + bx\|$.

## 2. Main Results

We start this section by a proposition that explains the relation between the *gg*-orthogonality and the s.i.p-orthogonality in real normed linear spaces.

Proposition 2.1. *Let* $(X, \|\cdot\|)$ *be a normed linear spaces. Then the following conditions are equivalent:*

i) $\perp_{gg} \subset \perp_s$.
ii) $\perp_{gg} = \perp_s$.
iii) $[\cdot, \cdot]_{gg} = [\cdot|\cdot]_s$.

Proof. (i)$\Rightarrow$ (ii) and (iii)$\Rightarrow$ (i) are obviously true.
(ii)$\Rightarrow$ (iii). Note that $x \perp \frac{-g(x,y)}{\|x\|^2} x + y$ and $x \perp \frac{-g(y,x)}{\|x\|^2} x + y$, for all $x, y \in X$ with $x \neq 0$. Thus, $[\frac{-g(x,y)}{\|x\|^2} x + y|x]_s = 0$ implies that $\frac{-g(x,y)}{\|x\|^2}\|x\|^2 + [y, x]_s = 0$ and so $g(x, y) = [y|x]_s$. Furthermore, $g(y, x) = [y|x]_s$ and hence $[x, y]_{gg} = [y|x]_s$. $\square$

The next theorem is a generalization of [6, Theorem 1] for *gg*-orthogonality.

Theorem 2.2. *Let* $(X, \|\cdot\|)$ *be a real normed linear space and let*

$$\|x + y\|^4 - \|x - y\|^4 = 8\left(\|x\|^2[x, y]_{gg} + \|y\|^2[y, x]_{gg}\right), \tag{1}$$

*for all* $x, y \in X$. *Then* $X$ *is smooth.*

Proof. Let $x, y \in X$. Since the functionals $\rho_\pm$ and $g$ are continuous at the first variable, we have

$$\lim_{t\to 0^\pm} [tx + y, y]_{gg} = [x, y]_{gg} = [y, x]_{gg}.$$

On the other hand, we have

$$\begin{aligned}
\rho_+(x, y) &= \|x\| \lim_{t\to 0^+} \frac{\|x + ty\| - \|x\|}{t} \frac{(\|x + ty\| + \|x\|)(\|x + ty\|^2 + \|x\|^2)}{(\|x + ty\| + \|x\|)(\|x + ty\|^2 + \|x\|^2)} \\
&= \|x\| \lim_{t\to 0^+} \frac{(\|x + ty\|^4 - \|x\|^4)}{t(\|x + ty\| + \|x\|)(\|x + ty\|^2 + \|x\|^2)} \\
&= \|x\| \lim_{t\to 0^+} \frac{(\|x + \frac{t}{2}y + \frac{t}{2}y\|^4 - \|x + \frac{t}{2}y - \frac{t}{2}y\|^4)}{t(\|x + ty\| + \|x\|)(\|x + ty\|^2 + \|x\|^2)} \\
&= \|x\| \lim_{t\to 0^+} \frac{8\Big(\|x + \frac{t}{2}y\|^2[x + \frac{t}{2}y, \frac{t}{2}y]_{gg} + (\|\frac{t}{2}y\|^2)[\frac{t}{2}y, x - \frac{t}{2}y]_{gg}\Big)}{t(\|x + ty\| + \|x\|)(\|x + ty\|^2 + \|x\|^2)}
\end{aligned}$$

$$= \|x\| \lim_{t\to 0^+} \frac{8\frac{t}{2}\Big(\|x+\frac{t}{2}y\|^2[x+\frac{t}{2}y,y]_{gg} + \frac{t}{2}\|y\|^2[\frac{t}{2}y, x-\frac{t}{2}y]_{gg}\Big)}{t(\|x+ty\|+\|x\|)(\|x+ty\|^2+\|x\|^2)}$$

$$= \|x\| \lim_{t\to 0^+} \frac{4\Big(\|x+\frac{t}{2}y\|^2[x+\frac{t}{2}y,y]_{gg} + \frac{t}{2}\|y\|^2[\frac{t}{2}y, x-\frac{t}{2}y]_{gg}\Big)}{(\|x+ty\|+\|x\|)(\|x+ty\|^2+\|x\|^2)}$$

$$= \|x\| \frac{4\|x\|^2[x,y]_{gg}}{(\|x\|+\|x\|)(\|x\|^2+\|x\|^2)} = [x,y]_{gg}.$$

Therefore, $\rho_+(x,y) = [x,y]_{gg}$ for all $x, y \in X$. By a similar argument, we can prove that $\rho_-(x,y) = [x,y]_{gg}$. Therefore, $\rho_+(x,y) = \rho_-(x,y)$ for all $x, y \in X$, and so $X$ is smooth. □

Remark 2.3. If Eq. (1) holds for all $x, y \in X$, then we say that $X$ is $gg$-quasi inner product space ($gg$-q.i.p.s).

The following proposition shows the resemblance between $gg$-i.p.s and i.p.s. To prove, use some ideas of [7].

Proposition 2.4. *Let $(X, \|\cdot\|)$ be a gg-q.i.p.s, and let the points $(0, x, y, x+y)$ be the vertices of a parallelogram. Then the following statements are holds:*

i) *In Eq. (1), $\|x+y\| = \|x-y\|$ if and only if $x \perp_{gg} y$.*
ii) *If $\|x\| = \|y\|$, then the diagonals $\|x\|$ and $\|y\|$ are $gg-orthogonal$, i.e. $x+y \perp_{gg} x-y$.*
iii) *In Eq. (1), $x \perp_{gg} y$ if and only if $\|x+y\| = \|x-y\|$ and $x+y \perp_{gg} x-y$.*
iv) $\Big[x + \frac{\|x\|}{\|y\|}y, x + \frac{\|x\|}{\|y\|}y\Big]_{gg} = 0$ *for all non-zero $x, y \in X$.*

Proof. (i) is obvious.
(ii) In Eq. (1), we replace $x$ by $x+y$ and $y$ by $x-y$ to get

$$\|2x\|^4 - \|2y\|^4 = 8\Big(\|x+y\|^2[x+y,x-y]_{gg} + \|x-y\|^2[x-y,x+y]_{gg}\Big),$$
$$0 = \frac{1}{2}[x+y,x-y]_{gg}\Big(\|x+y\|^2+\|x-y\|^2\Big).$$

We know that $\Big(\|x+y\|^2+\|x-y\|^2\Big) > 0$. Therefore $[x+y,x-y]_{gg} = 0$.
(iii) In Eq. (1), if $x \perp_{gg} y$, then $\|x+y\| = \|x-y\|$ by $(ii)$, $\|x\| = \|y\| \Longrightarrow x+y \perp_{gg} x-y$.
By $\|x+y\| = \|x-y\|$ and Eq. (1), we have $[x,y]_{gg} = 0$ and since $x+y \perp_{gg} x-y$, $\|x\| = \|y\|$.
(iv) In by (ii) (1), if we replace $x$ by $x + \frac{\|x\|}{\|y\|}y$ and $y$ by $x - \frac{\|x\|}{\|y\|}y$ then we obtain the result. □

The following proposition determines the relation between a $gg$-q.i.p.s and i.p.s.

Proposition 2.5. *A gg-q.i.p.s $X$ is an i.p.s if and only if*

$$\|x+y\| = \|x-y\| \quad \textit{if and only if} \quad [x,y]_{gg} = 0. \tag{2}$$

Proof. If $X$ is an i.p.s, then $[x,y]_{gg} = \langle x,y\rangle$, and so (1) and (2) are holds. Conversely, we assume that (1) and (2) are hold. From (1), we have

$$\|x+\lambda y\|^4 - \|x-\lambda y\|^4 = 8\left(\|x\|^2[x,y]_{gg} + \lambda^2\|y\|^2[y,x]_{gg}\right), \tag{3}$$

where $(\lambda \in \mathbb{R}; x,y \in X)$. Now suppose that $\|x+y\| = \|x-y\|$. Then from (2) and (3) we have $[x,y]_{gg} = 0$, and therefore $\|x+\lambda y\| = \|x-\lambda y\|$ for all $\lambda \in \mathbb{R}$. This implies that

$$\|x+y\| = \|x-y\| \quad \text{implies} \quad \|x+\lambda y\| = \|x-\lambda y\| \quad (\lambda \in \mathbb{R}).$$

Now, it follows from Theorem 1.3 that $X$ is an i.p.s. □

## References

1. C. Alsina, J. Sikorska and M. S. Tomás, *Norm Derivatives and Characterizations of Inner Product Spaces*, World Scientific Publishing Co. Pte. Ltd., Hackensack, 2010.
2. J. Chmieliński and P. Wójcik, *On a $\rho$-orthogonality*, Aequationes Math. **80** (2010) 45–55.
3. J. Chmieliński and P. Wójcik, *$\rho$-orthogonality and its preservation revisited*, Polish Acad. Sci. Inst. Math., Warsaw (2013) 17–30.
4. S. S. Dragomir, *Semi-inner Products and Applications*, Nova Science Publishers, Inc., Hauppauge, New York, 2004.
5. V. I. Istrăţescu, *Inner Products Structure: Theory and Applications*, Mathematics and its Applications, Vol. 25, D. Reidel Publishing Co., Dordrecht, 1987.
6. P. M. Miličić, *A generalization of the parallelogram equality in normed spaces*, J. Math. Kyoto Univ. **38** (1) (1998) 71–75.
7. P. M. Miličić,*On the quasi-inner product spaces*, Mat. Bilten (Skopje) **22** (XLVIII) (1998) 19–30.
8. M. Dehghani, M. Abed and R. Jahanipur, *A generalized orthogonality relation via norm derivatives in real normed linear spaces*, Aequationes Math. **93** (4) (2019) 651–667.
9. M. Nur and H. Gunawan, *A new orthogonality and angle in a normed space*, Aequationes Math. **93** (3) (2019) 547–555.

E-mail: mohammed.y@uokerbala.edu.iq

# A Note on the $p$-Operator Space Structure of the $p$-Analog of the Fourier-Stieltjes Algebra

Mohammad Ali Ahmadpoor*
Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran
and Marzieh Shams Yousefi
Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran

Abstract. In this paper one of the possible $p$-operator space structures of the $p$-analog of the Fourier-Stieltjes algebra will be introduced, and to some extend will be studied. This special sort of operator structure will be given from the predual of this Fourier type algebra, that is the algebra of universal $p$-pseudofunctions. Furthermore, some applicable and expected results will be proven.

Current paper can be considered as a new gate into the collection of problems around the $p$-analog of the Fourier-Stieltjes algebra, in the $p$-operator space structure point of view.

**Keywords:** $p$-Operator spaces, $p$-Analog of the Fourier-Stieltjes algebras, $QSL_p$-Spaces, Universal representation.
**AMS Mathematical Subject Classification [2010]:** 46L07, 43A30, 43A15.

## 1. Introduction

Operator space structure on the Eymard's Fourier-Stieltjes algebra, $B(G)$, was firstly investigated in [2], and fully described in [3]. The natural operator space structure on Fourier-Stieltjes algebra comes from its predual $C^*$-algebra. On the other hand, due to the fact that the Fourier algebra, $A(G)$, is the predual of the von Neumann algebra $VN(G)$, generated by the left regular representation $(\lambda_{2,G}, (L_2(G))$ of the locally compact group $G$, the natural operator space structure is induced on $A(G)$. Many authors benefited from this approach to investigate various problems on the Fourier and Fourier-Stieltjes algebras.

In the next stage, by the advent of Figà-Talamanca-Herz algebras $A_p(G)$, $(1 < p < \infty)$, in [4] and [6], the notion of $p$-operator space (the $p$-analog of the operator space) has been generalized in [1], based on the ideas of studies done by Pisier [9] and Le Merdy [8]. Introduced approach of $p$-operator space has been extensively utilized to turn $A_p(G)$ into a $p$-operator space, and many other properties have been studied. As a fruit of this structure on $A_p(G)$, which is a dual $p$-operator structure, Ilie has studied $p$-completely contractive homomorphisms on such algebras [7].

There has been a variety of approaches of defining the $p$-analog of the Fourier-Stieltjes algebras, and the most appropriate one is brought by Runde in [10],

*Speaker

which is denoted by $B_p(G)$. In this case, there can be found a need for a suitable $p$-operator space structure to be imposed. Herewith, we introduce a $p$-operator space structure on the $p$-analog of the Fourier-Stieltjes algebras, by considering it as the dual space of the $p$-operator space $UPF_p(G)$, the algebra of universal $p$-pseudofunctions.

Definition 1.1.

1) A representation of a locally compact group $G$ is a pair of homomorphism $\pi$ and a Banach space $E$ for which it corresponds each element of $G$ to an invertible isometric operator on $E$. Precisely, $\pi : G \to \mathcal{B}(E)$, so that $\pi(x)$ is an invertible operator with the inverse $\pi(x^{-1})$ that is isometric map.
2) A lift of a representation of the locally compact group $G$ to the group algebra $L_1(G)$ is a contractive homomorphism $\pi : L_1(G) \to \mathcal{B}(E)$ defined through

$$\langle \pi(f)\xi, \eta\rangle = \int_G f(x)\langle \pi(x)\xi, \eta\rangle dx, \quad \xi \in E,\ \eta \in E^*.$$

Furthermore, this operator is continuous with respect to original topology on $G$ and the strong operator topology on $\mathcal{B}(E)$.
3) A representation $(\pi, E)$ is called cyclic with the cyclic vector $\xi$, if the norm closure of the space $\pi(L_1(G))\xi$ is dense in $E$. In this case, we may denote $(\pi, E)$ by $(\pi_\xi, E_\xi)$, and call $E_\xi$ a cyclic space.

Definition 1.2.

1) We say that a Banach space $E$ is an $L_p$-space, if it is of the form of $L_p(X, \mu)$ for a measure space $(X, \mu)$.
2) A Banach space $E$ is called a $QSL_p$-space, if it can be identified with a quotient of a subspace of an $L_p$-space.

Next definition is going to provide a relation between representations.

Definition 1.3. Let $(\pi, E)$ and $(\rho, F)$ be two representations of a locally compact group $G$. Then

1) the representations $(\pi, E)$ and $(\rho, F)$ are equivalent, if there exists an invertible isometry $T : E \to F$ such that $\rho(f) \circ T = T \circ \pi(f)$, for every $f \in L_1(G)$. In this case we write $(\pi, F) \sim (\rho, F)$,
2) the representation $(\rho, F)$ is called a subrepresentation of $(\pi, E)$, if $F$ a closed subspace of $E$, and $\pi(f) = \pi(f)|_F$, for every $f \in L_1(G)$,
3) the representation $(\rho, F)$ is said to be contained in $(\pi, E)$, and write $(\rho, F) \subset (\pi, E)$, if $(\rho, F)$ is equivalent to a subrepresentation of $(\pi, E)$.

**Notation.** Throughout of this paper, the collection of all (classes of) representations of a locally compact group $G$ on a $QSL_p$-space is denoted by $\mathrm{Rep}_p(G)$. Moreover, the set of all cyclic representations is denoted by $\mathrm{Cyc}_p(G)$, as well as for a representation $(\pi, E) \in \mathrm{Rep}_p(G)$, the set of all its cyclic subrepresentations is denoted by $\mathrm{Cyc}_{p,\pi}(G)$.

Definition 1.4. A representation $(\pi, E) \in \mathrm{Rep}_p(G)$ is called a $p$-universal representation, if it contains all cyclic representations in $\mathrm{Cyc}_p(G)$.

Now it is the time for introducing a type of algebras that plays a pivotal role in this paper.

DEFINITION 1.5. Let $(\pi, E) \in \mathrm{Rep}_p(G)$. Then

1) if we define
$$\|f\|_\pi = \|\pi(f)\|,$$
then $\|\cdot\|_\pi$ is an algebra semi-norm on $L_1(G)$,
2) the algebra of $p$-pseudofunctions associated with $(\pi, E)$ is defined to be the operator norm closure of $\pi(L_1(G))$ in $\mathcal{B}(E)$, and is denoted by $PF_{p,\pi}(G)$. Moreover, if $(\pi, E)$ is a $p$-universal representation, then it is called the algebra of universal $p$-pseudofunctions, and we write $UPF_p(G)$ instead.

In the following we define $QSL_p$-operator algebras, similar to [5].

DEFINITION 1.6. Let $\mathcal{A}$ be a Banach algebra.

1) A representation of $\mathcal{A}$ (on a $QSL_p$-space $\mathcal{E}$) is a contractive homomorphism $\Pi : \mathcal{A} \to \mathcal{B}(\mathcal{E})$.
2) We say that the representation $\Pi : \mathcal{A} \to \mathcal{B}(\mathcal{E})$ is non-degenerate (essential), if linear space of $\{\Pi(a)\xi \ : a \in \mathcal{A}, \ \xi \in \mathcal{E}\}$ is dense in $\mathcal{E}$.
3) An essential representation $\Pi : \mathcal{A} \to \mathcal{B}(\mathcal{E})$ of $\mathcal{A}$ (on a $QSL_p$-space $\mathcal{E}$) is called faithful, if it is injective.
4) A Banach algebra $\mathcal{A}$ is said to be $QSL_p$-operator algebra, if there exists a $QSL_p$-space $\mathcal{E}$ and an isometric homomorphism $\Pi : \mathcal{A} \to \mathcal{B}(\mathcal{E})$. In this case one may equivalently expect that there exists a faithful isometric representation of $\mathcal{A}$ on a $QSL_p$-space $\mathcal{E}$.

Now it is the time for the $p$-analog of the Fourier-Stieltjes algebras to be defined (we have swapped indexes $p$ and its conjugate number $p'$).

DEFINITION 1.7. For $p \in (1, \infty)$, the space of all coefficient functions of representations in $\mathrm{Rep}_p(G)$ is called $p$-analog of the Fourier-Stieltjes algebras and denoted by $B_p(G)$.

REMARK 1.8. From [10], the norm of an element $u \in B_p(G)$ defined to be as following:
$$\|u\|_{B_p(G)} = \inf\{\sum_k \|\xi_k\|\|\eta_k\| \ : \ u(x) = \sum_k \langle \pi_k(x)\xi_k, \eta_k\rangle, \ x \in G\},$$
where the infimum is taken over all possible representations of $u$ as a coefficient function of $l_p$-direct sum of cyclic representations. Equipped with this norm of functions and pointwise multiplication, the space $B_p(G)$ is a commutative unital Banach algebra. For more details, a curious reader would be referred to [10].

Two of the most crucial facts about $B_p(G)$ are Lemma 6.5 and Theorem 6.6. in [10], which are gathered in the following theorem.

THEOREM 1.9. *Let $G$ be a locally compact group and $p \in (1, \infty)$, and let $(\pi, E) \in Rep_p(G)$.*

1) [10, Lemma 6.5] *Then for each* $\phi \in PF_{p,\pi}(G)^*$ *there exists a unique* $u \in B_p(G)$ *with* $\|u\|_{B_p(G)} \leq \|\phi\|_{op}$ *such that*

$$\langle \pi(f), \phi \rangle = \int_G f(x)u(x)dx, \quad f \in L_1(G).$$

*Moreover, if* $(\pi, E)$ *is a* $p$*-universal representation we have* $\|u\|_{B_p(G)} = \|\phi\|_{op}$.

2) [10, Theorem 6.6]
   i) *The dual space* $PF_{p,\pi}(G)^*$ *embeds into the algebra* $B_p(G)$ *contractively.*
   ii) *The embedding* $UPF_p(G)^*$ *into* $B_p(G)$ *is an isometric isomorphism.*

Now it is the turn for the definition of the $p$-operator space structure to be revealed. Our references on this topic are [1], and [8].

Definition 1.10.

1) A concrete $p$-operator space is a closed subspace of $\mathcal{B}(E)$, for some $QSL_p$-space $E$.

   In this case for each $n \in \mathbb{N}$ one can define a norm $\|\cdot\|_n$ on $\mathbb{M}_n(X) = \mathbb{M}_n \otimes X$, by identifying $\mathbb{M}_n(X)$ with a subspace of $\mathcal{B}(l_p^n \otimes_p E)$, where $\mathbb{M}_n$ is the space $\mathcal{B}(l_p^n)$. So, we have the family of norms $\Big(\|\cdot\|_n\Big)_{n\in\mathbb{N}}$ satisfying:

   i) [$\mathcal{D}_\infty$ :] For $u \in \mathbb{M}_n(X)$ and $v \in \mathbb{M}_m(X)$, we have that $\|u \oplus v\|_{n+m} = \max\{\|u\|_n, \|v\|_m\}$. Here $u \oplus v \in \mathbb{M}_{n+m}(X)$, has block representation $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$.

   ii) [$\mathcal{M}_p$ :] For every $u \in \mathbb{M}_m(X)$ and $\alpha \in \mathbb{M}_{n,m}$, $\beta \in \mathbb{M}_{m,n}$, we have that

$$\|\alpha u \beta\|_n \leq \|\alpha\|_{\mathcal{B}(l_p^m, l_p^n)} \|u\|_m \|\beta\|_{\mathcal{B}(l_p^n, l_p^m)}.$$

2) An abstract $p$-operator space is a Banach space $X$ equipped with the family of norms $\big(\|\cdot\|_n\big)$ defined by $\mathbb{M}_n(X)$ which satisfy two axioms above.

One of the most famous application of such $p$-operator space is done by Daws for the Figà-Talamanca-Herz algebras, $A_p(G)$. In the following we state some of the results in [1].

Definition 1.11. Let $X$ and $Y$ be two $p$-operator spaces, and $\Phi : X \to Y$ be a linear map. The $(n)$-fold of the map $\Phi$ can be define naturally through:

$$\Phi^{(n)} : \mathbb{M}_n(X) \to \mathbb{M}_n(Y), \quad \Phi^{(n)}([x_{ij}]) = [\Phi(x_{ij})],$$

and its $p$-complete norm is $\|\Phi\|_{p-cb} = \sup_{n\in\mathbb{N}} \|\Phi^{(n)}\|$. Moreover, we say that $\Phi$ is $p$-completely bounded, when $\|\Phi\|_{p-cb} < \infty$, and $p$-completely contractive, if $\|\Phi\|_{p-cb} \leq 1$. Finally, it is called a $p$-completely isometric map, if for each $n \in \mathbb{N}$, the map $\Phi^{(n)}$ is an isometric map.

Theorem 1.12. [1, Theorem 4.3] *Let* $X$ *be a* $p$*-operator space. There exists a* $p$*-completely isometry* $\Phi : X^* \to \mathcal{B}(l_p(I))$ *for some index set* $I$.

## 2. $p$-Operator Space Structure of $B_p(G)$

In this section, it is obtained that $p$-operator space structure on $B_p(G)$ is well-defined.

PROPOSITION 2.1. *Let* $(\pi, E) \in Rep_p(G)$. *Then*

1) *the Banach algebra* $PF_{p,\pi}(G)$ *of p-pseudofunctions associated with* $(\pi, E)$, *is a* $QSL_p$-*operator algebra. More precisely, for* $r \in \mathbb{N}$, *and* $g \in L_1(G)$ *there exists a cyclic subrepresentation* $(\pi_{g,r}, E_{g,r})$ *of* $(\pi, E)$ *with cyclic vector* $\xi_{g,r}$ *such that*

$$\|\pi_{g,r}(g)\xi_{g,r}\| > \|\pi(g)\| - \frac{1}{r}, \quad \|\xi_{g,r}\| \leq 1,$$

*and by considering* $\mathcal{E} = l_p\text{-}\oplus_{g,r} E_{g,r}$ *and* $\Pi = l_p\text{-}\oplus_{g,r} \pi_{g,r}$ *the pair* $(\Pi, \mathcal{E})$ *is a faithful isometric representation of* $PF_{p,\pi}(G)$,

2) *the matrix representation* $(\Pi^{(n)}, \mathcal{E}^{(n)})$ *is an isometric map from* $\mathbb{M}_n(PF_{p,\pi}(G))$ *onto a closed subspace of* $\mathbb{M}_n(\mathcal{B}(\mathcal{E})) = \mathcal{B}(\mathcal{E}^{(n)})$.

THEOREM 2.2. *For a representation* $(\pi, E) \in Rep_p(G)$, *the algebra of p-pseudo functions* $PF_{p,\pi}(G)$ *is a p-operator space.*

PROPOSITION 2.3.

1) *If two representations* $(\pi, E), (\rho, F) \in Rep_p(G)$ *are equivalent, then two algebras* $PF_{p,\pi}(G)$ *and* $PF_{p,\rho}(G)$ *are p-completely isometrically isomorphism.*
2) *If* $(\rho, F)$ *is a subrepresentation of* $(\pi, E)$, *then the algebra* $PF_{p,\rho}(G)$ *embeds into the algebra* $PF_{p,\pi}(G)$ *contractively.*

THEOREM 2.4. *The algebra of universal p-pseudofunctions* $UPF_p(G)$ *is an abstract p-operator space and is independent of choosing specific universal representation.*

As a consequence of previous theorem, we give an immensely important theorem below.

THEOREM 2.5. *For* $p \in (1, \infty)$, *the Banach algebra* $B_p(G)$ *is a p-operator space.*

Next proposition is the output of all materials mentioned before.

PROPOSITION 2.6. *For a locally compact group G, and a complex number* $p \in (1, \infty)$, *the identification* $B_p(G) = UPF_p(G)^*$ *is p-completely isometric isomorphism.*

## References


1. M. Daws, *p-Operator spaces and Figà-Talamanca-Herz algebras*, J. Operator Theory **63** (1) (2010) 47–83.
2. E. G. Effros and Z. -J. Ruan, *A new approach to operator spaces*, Canad. J. Math. **34** (1991) 329–337.
3. E. G. Effros and Z. -J. Ruan, *Operator Spaces*, London Math. Soc. Monogr. (N.S.), Vol. 23, The Clarendon Press, Oxford University Press, New York, 2000.

4. A. Figà-Talamanca, *Translation invariant operators in $L_p$*, Duke Math. J. **32** (1965) 495–501.
5. E. Gardella, *A modern look at algebras of operators on $L_p$-spaces*, Expositiones Math. (2020) inpress. `DOI: 10.1016/j.exmath.2020.10.003`
6. C. Herz, *The theory of $p$-spaces with an application to convolution operators, its second dual*, Trans. Amer. Math. Soc. **154** (1971) 69–82.
7. M. Ilie, *A note on $p$-completely bounded homomorphisms of the Figà-Talamanca-Herz algebras*, J. Math. Anal. Appl. **419** (2014) 273–284
8. C. Le Merdy, *Factorization of $p$-completely bounded multilinear maps*, Pacic J. Math. **172** (1996) 187–213.
9. G. Pisier, *Similarity Problems and Completely Bounded Maps*, Second, expanded edition, Lecture Notes in Math., Vol. 1618, Springer-Verlag, Berlin, 2001.
10. V. Runde, *Representations of locally compact groups on $QSL_p$-spaces and a $p$-analog of the Fourier-Stieltjes algebra*, Pacific J. Math. **221** (2) (2005) 379–397.

E-mail: m-a-ahmadpoor@phd.guilan.ac.ir
E-mail: m.shams@guilan.ac.ir

# Some Results About Generalized Inverse for Modular Operators Based on its Components

Javad Farokhi-Ostad
Department of Basic Sciences, Birjand University of Technology, Birjand, Iran
Mahdi Mohammadzadeh Karizaki*
Department of Computer Engineering, University of Torbat Heydarieh Torbat Heydarieh, Iran
and Mahdi Ali-Akbari
Department of Computer Engineering, University of Torbat Heydarieh Torbat Heydarieh, Iran

Abstract. In this paper, we investigate the generalized inverse of a modular operator, where it is considered as the sum or product of several other operators. Let $T$ be a modular operator that is the sum or product of several other operators. We express its generalized inverse in terms of its components.
**Keywords:** Hilbert $C^*$-module, Generalized inverse, Moore-Penrose inverse.
**AMS Mathematical Subject Classification [2010]:** 47A05, 46L08, 15A09.

## 1. Introduction and Preliminaries

Hilbert $C^*$-modules are generalization of Hilbert spaces. Here, there is a function similar to inner product whose values come from $C^*$-algebra. However, some well known properties of Hilbert spaces like Pythagoras' equality, self-duality, and even decomposition into orthogonal complements do not hold in the framework of Hilbert modules. This concept was introduced by I. Kaplansky [6] and then studied more in the work of W. L. Paschke [11]. Currently, one of the good reference in this field is [7] or [8], but a brief and useful source can also be the [10]. Let us quickly recall the definition of a Hilbert $C^*$-module. Let $\mathcal{A}$ be an arbitrary $C^*$-algebra. An $\mathcal{A}$-module inner-product is a $\mathcal{A}$-module $\mathcal{X}$ with a map $(x, y) \mapsto \langle x, y \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ which is satisfied the following conditions, for any $x, y, z \in \mathcal{X}$, $\alpha, \beta \in \mathbb{C}$ and $a \in \mathcal{A}$:

(i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$;
(ii) $\langle x, ya\rangle = \langle x, y\rangle a$;
(iii) $\langle y, x\rangle = \langle x, y\rangle^*$;
(iv) $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0 \Longleftrightarrow x = 0$.

If $\mathcal{X}$ be complete with respect to the induced norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$ for any $x \in \mathcal{X}$, then it is called Hilbert $C^*$-module. Note that every Hilbert space is a Hilbert $\mathbb{C}$-module and every C*-algebra $\mathcal{A}$ can be regarded as a Hilbert $\mathcal{A}$-module via $\langle a, b\rangle = a^*b$ when $a, b \in \mathcal{A}$. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are two Hilbert $C^*$-modules,

*Speaker

the set of all operators $T : \mathcal{X} \to \mathcal{Y}$ for which there is an operator $T^* : \mathcal{Y} \to \mathcal{X}$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \text{ for any } x \in \mathcal{X} \text{ and } y \in \mathcal{Y},$$

is denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$. It is known that any element $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ must be a bounded linear operator, which is also $\mathcal{A}$-linear in the sense that $T(xa) = (Tx)a$, for $x \in \mathcal{X}$ and $a \in \mathcal{A}$. In the case $\mathcal{X} = \mathcal{Y}$, $\mathcal{L}(\mathcal{X}, \mathcal{X})$ which is abbreviated to $\mathcal{L}(\mathcal{X})$, is a $C^*$-algebra. For any $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, the null and the range space of $T$ are denoted by $\ker(T)$ and ran(T), respectively.

Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. The Moore–Penrose inverse $T^\dagger$ of $T$ (if it exists) is an element $X \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ which satisfies

(1) $TXT = T$,
(2) $XTX = X$,
(3) $(TX)^* = TX$,
(4) $(XT)^* = XT$.

If $X$ only satisfied in (1) it is called inner inverse of $T$, if only (2) be true, $X$ is called outer inverse. We say general inverse, if both conditions hold. Also, for $\mathfrak{S} \subseteq \{1, 2, 3, 4\}$, if $X$ established in $\mathfrak{S}$ conditions, it is called $\mathfrak{S}$-inverse of $T$ and denote it by $T^{\mathfrak{S}}$, for example $T^{\{1\}}$ namely $T$ satisfied in (1). In particular, $\{1, 2, 3, 4\}$-inverse of $T$ must be its Moore-Penrose inverse ($T^\dagger$). The interested reader can refer to [1] or [2] and the references in them for more information.

Let $\mathcal{X}$ and $\mathcal{Y}$ have a given decompositions $\mathcal{X} = \mathcal{M} \oplus \mathcal{M}^\perp$ , $\mathcal{Y} = \mathcal{N} \oplus \mathcal{N}^\perp$, respectively, then for each operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ we can consider the matrix representation of $T$ as following $2 \times 2$ matrix:

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix},$$

where, $T_1 = P_{\mathcal{N}} T P_{\mathcal{M}} \in \mathcal{L}(\mathcal{M}, \mathcal{N})$, $T_2 = P_{\mathcal{N}} T(1 - P_{\mathcal{M}}) \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N})$, $T_3 = (1-P_{\mathcal{N}})TP_{\mathcal{M}} \in \mathcal{L}(\mathcal{M}, \mathcal{N}^\perp)$ and $T_4 = (1-P_{\mathcal{N}})T(1-P_{\mathcal{M}}) \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N}^\perp)$ and $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ denote the projections corresponding to $\mathcal{M}$ and $\mathcal{N}$, respectively.
The following Lemma was appeared in many papers, such as [9] and [12].

Lemma 1.1. *Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ have a closed range. Then according to the type of orthogonal decompositions of closed submodules of $\mathcal{X}$ and $\mathcal{Y}$, the matrix representation of $T$ and $T^\dagger$ is determined.*

(*a*) *If* $\mathcal{X} = \mathrm{ran}(T^*) \oplus \ker(T)$ *and* $\mathcal{Y} = \mathrm{ran}(T) \oplus \ker(T^*)$, *then:*

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(T^*) \\ \ker(T) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(T) \\ \ker(T^*) \end{bmatrix},$$

*where $T_1$ is invertible. Moreover,*

$$T^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(T) \\ \ker(T^*) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(T^*) \\ \ker(T) \end{bmatrix}.$$

(*b*) *If* $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ *and* $\mathcal{Y} = \mathrm{ran}(T) \oplus \ker(T^*)$, *then:*

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(T) \\ \ker(T^*) \end{bmatrix},$$

*where* $D = T_1T_1^* + T_2T_2^* \in \mathcal{L}(\mathrm{ran(T)})$ *is positive and invertible. Moreover,*

$$T^\dagger = \begin{bmatrix} T_1^*D^{-1} & 0 \\ T_2^*D^{-1} & 0 \end{bmatrix}.$$

(*c*) *If* $\mathcal{X} = \mathrm{ran(T^*)} \oplus \mathrm{ker(T)}$ *and* $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$*, then:*

$$T = \begin{bmatrix} T_1 & 0 \\ T_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(T^*)} \\ \ker(T) \end{bmatrix} \to \begin{bmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_2 \end{bmatrix},$$

*where* $\mathfrak{D} = T_1^*T_1 + T_2^*T_2 \in \mathcal{L}(\mathrm{ran(T)})$ *is positive and invertible. Moreover,*

$$T^\dagger = \begin{bmatrix} \mathfrak{D}^{-1}T_1^* & \mathfrak{D}^{-1}T_2^* \\ 0 & 0 \end{bmatrix}.$$

## 2. Main Results

In this section, provides results for generalized inverse of a modular operator, when it is considered as the sum or product of several other operators. These results can have many applications in finding the slotion of a operator equations. To find a solution of operator equation $TX = S$, when we used the matrix representation of the operators, it is important to pay attention to matrix decompositions. The many type of decompositions are used to implement efficient matrix algorithms. To solve a system of linear equations $Ax = b$, the matrix $A$ can be decomposed via the $LU$ or $QR$ or other decompositions. This decompositions factorizes a matrix into simple matrices, and so the systems can be solved easier ($L(Ux) = b$ and $Ux = L^{-1}b$ require fewer additions and multiplications to solve, compared with the original system $Ax = b$). Here, when we consider $T$ in terms of the sum or product of several operators, we express its generalized inverse in terms of its components.

Theorem 2.1. *Let* $\mathcal{X}$ *be Hilbert* $C^*$*-module and* $T \in \mathcal{L}(\mathcal{X})$ *has the factorization* $T = AB$*, such that* $T$*,* $A$ *and* $B$ *have closed ranges.Then* $X = B^{\{1\}}A^\dagger$ *is a* $\{1, 2, 3\}$*-inverse of* $T$ *and* $Y = B^\dagger A^{\{1\}}$ *is a* $\{1, 2, 4\}$*-inverse of* $T$*.*

Theorem 2.2. *Let* $\mathcal{X}$ *be Hilbert* $C^*$*-module and* $T \in \mathcal{L}(\mathcal{X})$ *be an idempotent operator.Then* $T \in T^{\{1,2\}}$*.*

Theorem 2.3. *Let* $\mathcal{X}$ *be Hilbert* $C^*$*-module and* $T \in \mathcal{L}(\mathcal{X})$ *has the factorization* $T = ABC$*, such that* $T$*,* $A$*,* $B$ *and* $C$ *have closed ranges.Then*

1) $A = TC^\dagger B^\dagger$,
2) $\mathrm{ran(T)} \subseteq \mathrm{ran(A)}$,
3) $\mathrm{ran(T^*)} \subseteq \mathrm{ran(C^*)}$,
4) $\ker(C) \subseteq \ker(T)$,
5) $\ker(A^*) \subseteq \ker(T^*)$,
6) $\mathrm{ran(A)} \subseteq \mathrm{ran(B^*)}$,
7) $\mathrm{ran(T)} \subseteq \mathrm{ran(B^*)}$.

Theorem 2.4. *Let* $T = ABC$ *be a modular operator in* $\mathcal{L}(\mathcal{X})$*. Also, consider the matrix representation of any operators* $T, A, B$ *and* $C$*. Then* $C_1(B_1A_2 + B_2A_4) = 0$*.*

Theorem 2.5. *Let $\mathcal{X}$ be Hilbert $C^*$-module and $A, B \in \mathcal{L}(X)$ have closed ranges, whit* $\operatorname{ran}(B) \subseteq \operatorname{ran}(A)$. *If $A^\dagger B$ is invertible, then $(A+B)^\dagger = (I + A^\dagger B)^{-1} A^\dagger$.*

Theorem 2.6. *Let $A, B, C$ and $D$ in $\mathcal{L}(\mathcal{X})$ have closed ranges. If $T = A + B + C + D$, $S = (A+B)T^\dagger(C+D)$, $V = (A+B)^\dagger B$ and $(C+D)^\dagger D$, then*

$$(A+C)T^\dagger(B+D) - AV - CW = (V-W)^* S (V-W).$$

**Acknowledgement**

I sincerely thank the organizers of the 51st Annual Iranian Mathematics Conference.

**References**

1. A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, 2nd ed., Springer, New York, 2003.
2. R. H. Bouldin, *The pseudo-inverse of a product*, SIAM J. Appl. Math. **25** (1973) 489–495.
3. J. Farokhi-Ostad and A. Janfada, *Two equal range operators on Hilbert $C^*$-modules*, Sahand Commun. Math. Anal. (SCMA), to appear.
4. S. Izumino, *The product of operators with closed range and an extension of the reverse order law*, Tohoku Math. J. **34** (1982) 43–52.
5. M. Jalaeian, M. Mohammadzadeh Karizaki and M. Hassani, *Conditions that the product of operators is an EP operator in Hilbert $C^*$-module*, Linear Multilinear Algebra **68** (10) (2020) 657–673.
6. I. Kaplansky, *Module over operator algebra*, Amer. J. Math. **75** (1953) 839–858.
7. E. C. Lance, *Hilbert $C^*$-Modules*, LMS Lecture Note Series 210, Cambridge Univ. Press, Cambridge, 1995.
8. V. M. Manuilov and E. V. Troitsky, *Hilbert $C^*$-Modules*, Translations of Mathematical Monographs, Vol. 226, Amer. Math. Society, Providence, RI, 2005.
9. M. Mohammadzadeh Karizaki and J. Farokhi-Ostad, *New identities of Moore–Penrose inverses of operator products and their reverse order law*, submitted.
10. M. Moslehian, K. Sharifi, M. Forough and M. Chakoshi, *Moore-Penrose inverse of Gram operator on Hilbert $C^*$-modules*, Proceedings of the First Workshop on $C^*$-algebras, Ferdowsi Univ., Mashhad-Iran (2001) pp. 29–38.
11. W. L. Paschke, *Inner product modules over $B^*$-algebras*, Trans Amer. Math. Soc. **182** (1973) 443–468.
12. K. Sharifi, *The product of operators with closed range in Hilbert $C^*$-modules*, Linear Algebra Appl. **435** (2011) 1122–1130.

E-mail: j.farokhi@birjandut.ac.ir
E-mail: m.mohammadzadeh@torbath.ac.ir
E-mail: mahdialiakbari@torbath.ac.ir

# On Hypercyclicity and Local Spectrum

Meysam Asadipour*
Department of Mathematics, College of Sciences, Yasouj University, Yasouj, Iran

Abstract. Let $X$ be a complex Banach space, and $L(X)$ be the space of bounded operators on $X$. Given $T \in L(X)$ and $x \in X$ denote by $\sigma_T(x)$ the local Spectrum of $T$ at $x$. And the operator $T$ is called hypercyclic, if $\overline{orb(T,x)} = X$. In this paper, we will introduce a relationship between the local spectrum and hypercyclicity.

**Keywords:** Spectrum, Local spectrum, Hypercyclicity.
**AMS Mathematical Subject Classification [2010]:** 47A10, 47A16.

## 1. Introduction

This section will be divided into two general sections which each of them introduces a concept separate from the other. We first bring up a brief introduction on spectral theory, then we will explain about the hypercyclicity.

Let $X$ be a Banach algebra with a unit element $e$ and $x \in X$, then the spectrum of $x$ is denoted by $\sigma(x)$ and;

$$\sigma(x) = \{\lambda \in \mathbb{C};\ x - \lambda e \textit{ is not invertible in } X\}.$$

It is well known that, the spectrum of $x$ is non-empty compact subset of $\mathbb{C}$, [4], so the set $\{|\lambda|;\ \lambda \in \sigma(x)\}$ has a maximum member which is called the spectral radius of $x$. And the set $\mathbb{C} \setminus \sigma(x)$ is called the resolvent set of $x$. It is worthwhile to mention that, if $T$ is an operator on finite-dimensional Banach space, then $\sigma(T)$ consists of eigenvalues of $T$ which is denoted by $\sigma_p(T)$ (is called the point spectrum of $T$) and since the eigenvalues of an operator on finite-dimensional Banach space are precisely the roots of its characteristic polynomial, the non-emptiness of $\sigma(T)$ is equivalent to the fundamental theorem of algebra that every polynomial has a root in $\mathbb{C}$. However throughout this paper, we focus on infinite-dimensional separable complex Banach space $X$ and $L(X)$ denotes the algebra of all bounded linear operators on $X$.

Definition 1.1. For an operator $T \in L(X)$ and a vector $x \in X$, the local resolvent set of the operator $T$ at $x$ is the union of all open subsets $U$ of $\mathbb{C}$ for which there is an analytic function $\phi : U \longrightarrow X$ satisfying $(T - zI)\phi(z) = x$ for every $z \in U$. Its complement is called the local spectrum of $T$ at $x$ and denoted by $\sigma_T(x)$.

It is well known that, the local spectrum, $\sigma_T(x)$, is a compact subset of $\sigma(T)$, [8]. Although the spectrum of every operator $T$ is always nonempty, but with an example in the next section, we show that $\sigma_T(x)$ can be an empty subset of $\mathbb{C}$. New and interesting results can be seen in [1] and [5].

*Speaker

For $T \in L(X)$, $x \in X$, and $\Omega$ a non-empty subset of the complex plane $\mathbb{C}$, we denote

$$Orb(T,\ \Omega x) = \{\omega T^n x;\ \ \omega \in \Omega,\ n = 0, 1, 2, \ldots\}.$$

If the set $\Omega \subseteq \mathbb{C}$ reduces to a single nonzero point $\{\omega_0\}$ such that the orbit $\overline{orb(T, \Omega x)} = X$, then $\omega_0 x$ is said to be a hypercyclic vector for hypercyclic operator $T$. In this case, $HC(T)$ denotes the set of all hypercyclic vectors for the operator $T$. Of course, hypercyclic operators cannot exist in non separable Banach space. On the other hand, every separable infinite-dimensional Banach space supports a hypercyclic operator, [6]. Now consider $T$ be an operator on $X$ with continuous inverse $T^{-1}$, then it is well known that the operator $T$ is hypercyclic if, and only if, its inverse is.

There is a well known link between spectral theory and hypercyclicity. In fact, for any hypercyclic operators $T$;

i) The point spectrum of its adjoint is empty: $\sigma_p(T^*) = \phi$.
ii) The spectrum of $T$ meets the unit circle: $\sigma(T) \cap \mathbb{T} \neq \phi$.

In above, if $\Omega = \mathbb{C}$ and $\overline{orb(T, \Omega x)} = X$, then $T$ is called supercyclic operator. In [7] The class of supercyclic operators is divided into the following two classes;

i) Supercyclic operators $T$ for which the point spectrum of its adjoint is empty, $\sigma_p(T^*) = \phi$.
ii) For any nonzero complex number $\xi$ there exists a supercyclic operator $T$ with $\sigma_p(T^*) = \{\xi\}$.

Some other connections between them can be seen in [3].

In this paper, we want to express a relationship between the local spectrum and the orbit of an invertible operator and based on that, we will present two interesting suggestions for researchers.

## 2. Main Results

As we mentioned in the previous section, the next example shows that sometimes the local spectrum is empty.

Example 2.1. Let $H$ be a Hilbert space with an orthonormal basis $\{e_i | i \geq 0\}$. Consider $S \in L(H)$ as the unilateral forward shift ($Se_i = e_{i+1}$) and let $S^*$ be its adjoint,

$$S^* e_0 = 0, \quad S^* e_i = e_{i-1}, \quad i \in \mathbb{N}.$$

Obviously, $S^* S = I$ and if $x = \sum_{i=0}^{\infty} 2^{-i} e_i$, then

$$S^* x = S^*(\sum_{i=0}^{\infty} 2^{-i} e_i) = \sum_{i=1}^{\infty} 2^{-i} e_{i-1} = \sum_{i=0}^{\infty} 2^{-i-1} e_i = \frac{x}{2}.$$

Let $|z| < 1$, then $g(z) = \sum_{i=0}^{\infty} S^{i+1}(x) z^i$ is convergent and

$$(S^* - z) g(z) = \sum_{i=0}^{\infty} S^i(x) z^i - \sum_{i=0}^{\infty} S^{i+1}(x) z^{i+1} = x.$$

And when $|z| > \frac{1}{2}$, consider $f(z) = -\sum_{i=0}^{\infty} \frac{S^{*i}x}{z^{i+1}}$. Thus

$$(S^* - z)f(z) = -\sum_{i=0}^{\infty} \frac{S^{*\,i+1}x}{z^{i+1}} + \sum_{i=0}^{\infty} \frac{S^{*\,i}x}{z^i} = x.$$

Therefore in the definition of local resolvent set of $x$ under $S^*$, $U = \mathbb{C}$ or equivalently $\sigma_{S^*}(x) = \emptyset$.

In the next theorem a relationship is expressed between the local spectrum and the orbit of a vector under an invertible operator.

THEOREM 2.2. *Let $T \in L(X)$ be an invertible operator and $x \in X$ be a hypercyclic vector for $T^{-1}$. The local spectrum $\sigma_T(x)$ does not contain the number zero, if and only if, the orbit of $x$ under $T^{-1}$ has following property;*

$$\sup_{n\in\mathbb{N}} ||T^{-n}x||^{\frac{1}{n}} < \infty.$$

PROOF. Let there exists a neighborhood $U \subset \mathbb{C}$ of zero, for which there exists an analytic function $f : U \longrightarrow X$ satisfying $(T - zI)f(z) = x$ for every $z \in U$, so we can consider $f(z) = \sum_{n=0}^{\infty} x_{n+1}z^n$ as the Taylor expansion of $f$ in $U$, then

$$(T - z)f(z) = Tx_1 + \sum_{n=1}^{\infty} z^n(Tx_{n+1} - x_n) = x,$$

and

$$\sup_{n\geq 1} ||x_n||^{\frac{1}{n}} < \infty.$$

Consequently $Tx_1 = x$ and $Tx_{n+1} = x_n$ for every $n \in \mathbb{N}$.

Therefore $0 \notin \sigma_T(x)$ if and only if the orbit $orb(T^{-1}, x)$ has the desired property and the proof is completed. □

As we mentioned above, $\sigma_p(T^*) = \phi$ for every hypercyclic operator, so we want to know that, what is the relationship between local spectrum and point spectrum? The following theorem partially responds to this curiosity.

THEOREM 2.3. *Assume that $T$ is an operator on $X$ and $0 \in \sigma_T(x)$ for any $x \in X$, then $\sigma_p(T) = \phi$.*

PROOF. Suppose that $x_0 \in X$ and $\lambda$ is a nonzero complex number such that $Tx_0 = \lambda x_0$. For every $n \in \mathbb{N}$ if $\lambda x_n = x_{n-1}$, then $Tx_n = x_{n-1}$ and

$$\sup_{n\geq 1} ||x_n||^{\frac{1}{n}} = \sup_{n\geq 1} ||\frac{1}{\lambda^n}x_0||^{\frac{1}{n}} < \infty.$$

Note that $f(z) = \sum_{n=0}^{\infty} x_{n+1}z^n$ is convergent in the radius of convergence of this power series and

$$(T - z)f(z) = Tx_1 + \sum_{n=0}^{\infty} z^n(Tx_{n+1} - x_n) = x_0.$$

Thus $0 \notin \sigma_T(x_0)$ when $0 \neq \lambda \in \sigma_p(T)$. Since the case $\lambda = 0$ is trivial, so the proof is completed. □

Theorem 2.2 shows that there exist a relationship between the local spectrum and the orbit of an invertible operator. In addition, for a hypercyclic operator $T$, the point spectrum of its adjoint, $\sigma_p(T^*)$, is empty. Hence it is natural to raise the following question;

QUESTION 2.4. Does every hypercyclic operator have a hypercyclic vector $x$ such that $0 \notin \sigma_T(x)$?

The next theorem can be seen in [2].

THEOREM 2.5. *Let* $\Phi : L(X) \longrightarrow L(X)$ *be an additive map such that* $\sigma_{\Phi(T)}(x) = \sigma_T(x)$. *Then* $\Phi(T) = T$ *for all* $T \in L(X)$.

Now, trying to find a convincing answer to the following question can be interesting.

QUESTION 2.6. Consider $\Phi$ be an additive map preserving hypercyclicity on $L(X)$, i.e.

$$HC(T) \neq \phi \Longleftrightarrow HC(\Phi(T)) \neq \phi.$$

Can we characterize this additive map?

## References


1. Z. Abdelali, A. Bourhim and M. Mabrouk, *Lie product and local spectrum preservers*, Linear Algebra Appl. **553** (2018) 328–361.
2. A. Bourhim and T. Ransford, *Additive maps preserving local spectrum*, Integr. Equ. Oper. Theory **55** (3) (2006) 377–385.
3. S. Charpentier, R. Ernst and Q. Menet, *Gamma-supercyclicity*, J. Funct. Anal. **270** (2016) 4443–4465.
4. M. Einsiedler and T. Ward, *Functional Analysis, Spectral Theory and Applications*, Graduate Texts in Mathematics, Vol. 276, Springer International Publishing, Cham, 2017.
5. A. E. Ghazi and R. Marzouki, *Maps preserving the peripheral local spectrum of some product of operators*, Oper. Matrices **13** (3) (2019) 653–666.
6. K. G. Grosse-Erdmann and A. Peris, *Linear Chaos*, Universitext, Springer-Verlag, London, 2011.
7. D. A. Herrero, *Limits of hypercyclic and supercyclic operators*, J. Funct. Anal. **99** (1) (1991) 179–190.
8. K. B. Laursen and M. M. Neumann, *An Introduction to Local Spectral Theory*, Oxford University Press, New York, 2000.


E-mail: Asadipour@yu.ac.ir

# Coincident and Common Fixed Point of Mappings on Uniform Spaces Generated by a Family of $b$-Pseudometrics

Hamid Faraji*
Department of Mathematics, College of Technical and Engineering, Saveh Branch, Islamic Azad University, Saveh, Iran

ABSTRACT. In this paper, we give some coincidence and common fixed point results for two self mappings defined on a uniform space generated by a family of $b$-pseudometrics which is sequentially complete. Our result generalizes the related results proved by Acharya.
**Keywords:** Uniform space, $b$-Pseudometric, Coincident point, Fixed point.
**AMS Mathematical Subject Classification [2010]:** 47H10, 54H25.

## 1. Introduction

Uniform spaces generated by a family of $b$-pseudometrics are considered in [5] in order to, simultaneously, investigate the fixed points of mappings defined on such spaces and generalize one of the main results in [1]. The aim of this paper is to give some coincident and common fixed point results and generalize [1, Theorem 3.1] using uniform spaces generated by a family of $b$-psedumetrics. For recent progress in uniform fixed point theory and more the reader is referred to [2, 3]. To begin with, we recall some definitions.

A function $p : X \times X \to [0, \infty)$ is called a $b$-pseudometric on a nonempty set $X$ if it satisfies the following for all $x, y, z \in X$:

i) $p(x, x) = 0$,
ii) $p(x, y) = p(y, x)$,
iii) $p(x, y) \leq s[p(x, z) + p(z, y)]$ ($b$-triangular inequality), where $s \in [1, \infty)$.

Then, the pair $(X, p)$ is called a $b$-pseudometric space with parameter $s \geq 1$. If, in addition, $p(x, y) = 0$ implies that $x = y$, for all $x, y \in X$, then $(X, p)$ is called a $b$-metric space. For more information on $b$-metric spaces and fixed point results on them, we refer, e.g., to [4, 6]. A uniformity $\mathcal{U}$ on $X$ is a family of subsets of $X \times X$, that the following conditions hold:

U1) $U \in \mathcal{U}$ implies $\Delta = \{(x, x) \in X \times X : x \in X\} \subset U$;
U2) $U_1, U_2 \in \mathcal{U}$ implies $U_1 \cap U_2 \in \mathcal{U}$;
U3) For each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$;
U4) $U \in \mathcal{U}$ implies that $V^{-1} = \{(x, y) \in X \times X : (y, x) \in V\} \subset U$ for some $V \in \mathcal{U}$ ;
U5) If $U \in \mathcal{U}$ and $U \subseteq V$ imply $V \in \mathcal{U}$.

*Speaker

Then, the pair $(X,\mathcal{U})$ is called a uniform space. A sequence $\{x_n\}$ in uniform space $(X,\mathcal{U})$ is called convergent to a point $x \in X$, if for each $U \in \mathcal{U}$ there exists $n_0 \in \mathbb{N}$ such that $(x_n, x) \in U$, for all $n \geq n_0$. It is called Cauchy sequence in uniform space $(X,\mathcal{U})$, if for all $U \in \mathcal{U}$, there exists $n_0 \in \mathbb{N}$ such that $(x_n, x_m) \in U$, for all $n, m \geq n_0$. The uniform space $X$ is called sequentially complete, if every Cauchy sequence in $X$ is convergent in $X$.

Let $\mathcal{U}$ be the uniformity generated by a nonempty family $\mathcal{F}$ of $b$-pseudometrics with the same parameter $s \geq 1$. Define

$$V_{(p,r)} = \{(x, y) \in X : p(x, y) < r\},$$

where $p \in \mathcal{F}$ and $r > 0$ and let $\mathcal{V}$ be the family of all sets of the form

$$\bigcap_{i=1}^{k} V_{(p_i,r_i)},$$

where $k$ is a positive integer, $p_i \in \mathcal{F}$ and $r_i > 0$ for $i = 1, \ldots, k$. Then $\mathcal{V}$ is a base for the uniformity $\mathcal{U}$ and for $V = \bigcap_{i=1}^{k} V_{(p_i,r_i)} \in \mathcal{V}$ and $\alpha > 0$, we have

$$\alpha V = \bigcap_{i=1}^{k} V_{(p_i,\alpha r_i)} \in \mathcal{V}.$$

Also, if $Y \subseteq X$, then

$$\mathcal{U}_Y = \{U \cap (Y \times Y) | U \in \mathcal{U}\},$$

is a uniformity on $Y$ and $\mathcal{V}_Y = \{V \cap (Y \times Y) | V \in \mathcal{V}\}$ is a base for $\mathcal{U}_Y$ (See, e.g., [7]).

We shall need the following lemma of Acharya [1].

LEMMA 1.1. [1] *Let $(X,\mathcal{U})$ be a uniform space.*

i) *Suppose that $p$ is a $b$-pseudometric on $X$ with parameter $s \geq 1$ and $\alpha, \beta$ are any two positive numbers. Then*

$$(x, y) \in \alpha V_{(p,r_1)} \circ \beta V_{(p,r_2)},$$

*implies that*

$$p(x, y) < s(\alpha r_1 + \beta r_2).$$

ii) *For any $V$ in $\mathcal{V}$, there is a $b$-pseudometric $p$ (so-called the Minkowski's $b$-pseudometric of $V$) on $X$ such that*

$$V = V_{(p,1)}.$$

## 2. Main Results

Hereafter, suppose that $(X,\mathcal{U})$ is a Hausdorff uniform space whose uniformity $\mathcal{U}$ is generated by a family $\mathcal{F}$ of $b$-pseudometrics with the same parameter $s \geq 1$ on $X$ and $\mathcal{V}$ is the family of all sets of the form

$$\bigcap_{i=1}^{k} \Big\{(x, y) \in X \times X : p_i(x, y) < r_i\Big\},$$

where $k$ is a positive integer, $p_i \in \mathcal{F}$ and $r_i > 0$ for $i = 1, \ldots, k$.

THEOREM 2.1. *Let* $(X,\mathcal{U})$ *be a sequentially complete Hausdorff uniform space and* $f$ *and* $g$ *be self-mappings on* $X$ *which satisfy*

$$(f(x), g(y)) \in \alpha V, \quad \text{if} \quad (x, y) \in V, \tag{1}$$

*for all* $x, y \in X$ *and* $V \in \mathcal{V}$, *where* $0 < s\alpha < 1$. *Then* $f$ *and* $g$ *have a unique common fixed point.*

PROOF. Let $x_0 \in X$ be an arbitrary point. Then we construct a sequence $\{x_n\}$ in $X$ such that $x_{2n+1} = f(x_{2n}), x_{2n+2} = g(x_{2n+1})$, for all $n \geq 0$. Let $V$ be any member of $\mathcal{V}$ and $p$ is Minkowski's $b$-pseudometric of $V$. Fix $x, y \in X$ and $p(x, y) = r$. Then $(x, y) \in (r + \varepsilon)V$. By (1), we have $(f(x), g(y)) \in \alpha(r + \varepsilon)V$. Since $\varepsilon > 0$ was arbitrary, we have

$$p(f(x), g(y)) \leq \alpha p(x, y). \tag{2}$$

By (2), we have

$$p(x_{2n+1}, x_{2n}) = p(f(x_{2n}), g(x_{2n-1})) \leq \alpha p(x_{2n}, x_{2n-1}), \tag{3}$$

for all $n \geq 1$. Similarly

$$p(x_{2n+1}, x_{2n+2}) = p(f(x_{2n}), g(x_{2n+1})) \leq \alpha p(x_{2n}, x_{2n+1}), \tag{4}$$

for all $n \geq 0$. From (3) and (4), we get

$$p(x_{n+1}, x_n) \leq \alpha p(x_n, x_{n-1}), \tag{5}$$

for all $n \geq 1$. Using (5), for $m, n \in \mathbb{N}$ and $m \geq n$, we have

$$\begin{aligned} p(x_{2n}, x_{2m}) &\leq sp(x_{2n}, x_{2n+1}) + s^2 p(x_{2n+1}, x_{2n+2}) + \cdots + s^{2m-2n} p(x_{2m-1}, x_{2m}) \\ &\leq s\alpha^{2n} p(x_0, x_1) + s^2 \alpha^{2n+1} p(x_0, x_1) + \cdots + s^{2m-2n} \alpha^{2m-1} p(x_0, x_1) \\ &\leq s\alpha^{2n} p(x_0, x_1)\big(1 + s\alpha + (s\alpha)^2 + \cdots + (s\alpha)^{2m-2n-1}\big) \\ &\leq s\alpha^{2n} p(x_0, x_1) \left(\frac{1}{1 - s\alpha}\right). \end{aligned}$$

Choose $N \in \mathbb{N}$ such that $\dfrac{s\alpha^{2n} p(x_0, x_1)}{1 - s\alpha} < 1$, for all $n \geq N$. Then, we have $p(x_{2n}, x_{2m}) < 1$ for all $m, n \geq N$. So, $(x_{2n}, x_{2m}) \in V$, for all $m, n \geq N$. Since $V$ was arbitrary, then $\{x_{2n}\}$ is Cauchy sequence in $X$. Therefore, there exists $x \in X$ such that $\lim_{n\to\infty} x_{2n} = x$. Let $V$ be arbitrary and $V = V(p, 1)$, where $p$ is the Minkowski's $b$-pseudometric of $V$. Then, we have

$$\begin{aligned} p(x, x_{2n+1}) &\leq s\big(p(x, x_{2n}) + p(x_{2n}, x_{2n+1})\big) \\ &\leq sp(x, x_{2n}) + s\alpha^{2n} p(x_0, x_1), \end{aligned}$$

for all $n \geq 0$. Then, we have $\lim_{n\to\infty} x_{2n+1} = x$. Now, we show that $x$ is a common fixed point of $f$ and $g$. Let $V \in \mathcal{V}$ be arbitrary and $V = V(p, 1)$, where $p$ is the Minkowski's $b$-pseudometric of $V$. Then, we have

$$\begin{aligned} p(f(x), x) &\leq s\big(p(f(x), g(x_{2n-1})) + p(g(x_{2n-1}), x)\big) \\ &\leq s\alpha p(x, x_{2n-1}) + sp(x_{2n}, x), \end{aligned}$$

for all $n \geq 1$. Therefore $f(x) = x$. Again suppose that $V \in \mathcal{V}$ is arbitrary and $V = V(p, 1)$, where $p$ is the Minkowski's $b$-pseudometric of $V$. Then, we have $p(x, g(x)) = p(f(x), g(x)) \leq \alpha p(x, x) = 0$. So $g(x) = x$ and $f(x) = g(x) = x$. To prove the uniqueness of $x$, we assume that $y \in X$ is another common fixed point of $f$ and $g$. Then, we have $p(x,y) = p(f(x), g(y)) \le \alpha p(x,y) < p(x,y)$. So $x = y$. $\square$

For $s = 1$ and $f = g$, Theorem 2.1 reduces to [1, Theorem 3.1].

THEOREM 2.2. *Let $(X, \mathcal{U})$ be a Hausdorff uniform space and $f, g$ be self-mappings on $X$ such that for all $V_1, V_2 \in \mathcal{V}$ and $x, y \in X$*

$$(f(x), f(y)) \in \alpha V_1 \circ \beta V_2 \quad \text{if} \quad (f(x), g(x)) \in V_1,\ (f(y), g(y)) \in V_2, \tag{6}$$

*where $\alpha > 0, \beta > 0$ and $s^2(\alpha + \beta) < 1$. If $f(X) \subseteq g(X)$ and $(g(X), \mathcal{U}_{g(X)})$ is sequentially complete, then $f$ and $g$ have a unique coincident point.*

PROOF. Let $x_0 \in X$ be arbitrarily chosen. Consider the sequence $\{x_n\}$ defined by $y_n = f(x_n) = g(x_{n+1})$, for each $n \ge 0$. Now take $V \in \mathcal{V}$ and suppose that $p$ is the Minkowski's $b$-pseudometric of $V$. Fix $x, y \in X$, $p(f(x), g(x)) = r_1$ and $p(f(y), g(y)) = r_2$. Let $\varepsilon > 0$ be given. Then, we have $(f(x), g(x)) \in (r_1 + \varepsilon)V$ and $(f(y), g(y)) \in (r_2 + \varepsilon)V$. By (6), we have

$$(f(x), f(y)) \in \alpha(r_1 + \varepsilon)V \circ \beta(r_2 + \varepsilon)V.$$

Then from Lemma 1.1, we get

$$p(f(x), f(y)) \le s\alpha(r_1 + \varepsilon) + s\beta(r_2 + \varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, we have

$$p(f(x), f(y)) \le s\alpha p(f(x), g(x)) + s\beta p(f(y), g(y)). \tag{7}$$

Using (7), we get

$$\begin{aligned} p(y_n, y_{n+1}) = p(f(x_n), f(x_{n+1})) &\le s\alpha p(f(x_n), g(x_n)) + s\beta p(f(x_{n+1}), g(x_{n+1})) \\ &= s\alpha p(y_n, y_{n-1}) + s\beta p(y_{n+1}, y_n),\ (n = 1, 2, 3, \dots). \end{aligned}$$

So,

$$p(y_n, y_{n+1}) \le \frac{s\alpha}{1 - s\beta} p(y_{n-1}, y_n),\ (n = 1, 2, 3, \dots).$$

Then, we have

$$p(y_n, y_{n+1}) \le \lambda^n p(y_0, y_1),$$

for all $n = 0, 1, 2, \dots$, where $\lambda = \frac{s\alpha}{1 - s\beta} < 1$. An argument similar to that of give in the proof of Theorem 2.1, it is easy to verify that $\{y_n\}$ is a Cuachy sequence in $g(X)$. So there is $z$ in $X$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_{n+1}) = g(z). \tag{8}$$

We show that $f(z) = g(z)$. Take $V \in \mathcal{V}$ and suppose $p$ is the Minkowskis $b$-pseudometric of $V$. So, by (7), we have

$$\begin{aligned} p(f(x_n), f(z)) &\le s\alpha p(f(x_n), g(x_n)) + s\beta p(f(z), g(z)) \\ &\le s\alpha p(f(x_n), g(x_n)) + s^2\beta p(f(x_n), f(z)) + s^2\beta p(f(x_n), g(z)). \end{aligned}$$

Then

$$p(f(x_n), f(z)) \le \frac{s\alpha}{1 - s^2\beta} p(f(x_n), g(x_n)) + \frac{s^2\beta}{1 - s^2\beta} p(f(x_n), g(z)).$$

By (8), choose $N_2 \in \mathbb{N}$ such that $p(f(x_n), f(z)) < 1$, for all $n \geq N_2$. Then $(f(x_n), f(z)) \in V$ for all $n \geq N_2$. Since $V$ is arbitrary, we get $\lim_{n\to\infty} f(x_n) = f(z)$. Then $f(z) = g(z) = t$. We claim that the coincident point $t$ is unique. Let $f(z_1) = g(z_1) = t_1$. Take any $V \in \mathcal{V}$ and suppose that $p$ is the Minkowski's $b$-pseudometric of $V$. Using (7), we have

$$p(t, t_1) = p(f(z), f(z_1)) \leq s\alpha p(f(z), g(z)) + s\beta p(f(z_1), g(z_1)) = 0.$$

Then $(t, t_1) \in V$. Since $V$ is arbitrary, it follows that $t = t_1$. □

## References


1. S. P. Acharya, *Some results on fixed points in uniform spaces*, Yokohama Math. J. **22** (1974) 105–116.
2. A. Aghanians and K. Nourouzi, *Fixed points for Kannan type contractions in uniform spaces endowed with a graph*, Nonlinear Anal. Model. Control. **21** (1) (2016) 103–113.
3. V. Angelov, *Fixed Points in Uniform Spaces and Applications*, Cluj University Press, 2009.
4. K. Berrah, A. Aliouche and T. e. Oussaeif, *Applications and theorem on common fixed point in complex valued b-metric space*, AIMS Math. **4** (3) (2019) 1019–1033.
5. H. Faraji, K. Nourouzi and D. O'Regan, *A fixed point theorem in uniform spaces generated by a family of b-pseudometrics*, Fixed Point Theory **20** (2019) 177–183.
6. W. Kirk and N. Shahzad, *Fixed Point Theory in Distance Spaces*, Springer, Cham, 2014.
7. K. D. Joshi, *Introduction to General Topology*, John Wiley and Sons Ltd, New Delhi, 1983.


E-mail: faraji@iau-saveh.ac.ir

# Some Preorder on Operators in Semi-Hilbertian Spaces

Farzaneh Gorjizadeh*
Faculty of Mathematical Sciences, University of Shahrekord, Shahrekord, Iran
and Noha Eftekhari
Department of Mathematical Sciences, University of Shahrekord, Shahrekord, Iran

Abstract. Let $A$ be a positive operator in $\mathcal{B}(\mathcal{H})$. Then for $x, y \in \mathcal{H}$, the semi-inner product $\langle x, y\rangle_A = \langle Ax, y\rangle$, and the seminorm $\|x\|_A = \|A^{\frac{1}{2}}x\|$ are defined on complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. The aim of this work is to investigate a preorder on semi-Hilbertian space operators, it is called $A$-majorizarion. In some sense, the $A$-majorizarion is equivalent to Barnes's majorization. Some equivalent Theorems are obtained. The relations between $A$-majorization, range inclusion and $A$-numerical radius are studied.

**Keywords:** Majorization, Semi-Hilbertian space, Semi-Inner product.
**AMS Mathematical Subject Classification [2010]:** 47A05, 46C05, 47B65.

## 1. Introduction

The following assumptions will be needed throughout the paper. Let $\mathcal{B}(\mathcal{H})$ denote the Banach space of all bounded linear operators on complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ with norm $\|\cdot\|$. Let $R(T)$ and $N(T)$ be the range and the null space of $T \in \mathcal{B}(\mathcal{H})$, respectively.

An operator $T \in \mathcal{B}(\mathcal{H})$ is called positive and denoted by $T \geq 0$, if $\langle Tx, x\rangle \geq 0$, for all $x \in \mathcal{H}$. Let $\mathcal{B}(\mathcal{H})^+$ denote the set of all positive operators in $\mathcal{B}(\mathcal{H})$, that is

$$\mathcal{B}(\mathcal{H})^+ = \{T \in \mathcal{B}(\mathcal{H}) : \ \langle Tx, x\rangle \geq 0, \ \forall x \in \mathcal{H}\}.$$

From now on, $A \in \mathcal{B}(\mathcal{H})$ is a positive operator and so $A^{\frac{1}{2}}$ is positive. The positive operator $A \in \mathcal{B}(\mathcal{H})$ defines a positive semidefinite sesquilinear form

$$\langle\cdot,\cdot\rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \qquad \langle x, y\rangle_A = \langle Ax, y\rangle.$$

Note that $\langle\cdot,\cdot\rangle_A$ is a semi-inner product on $\mathcal{H}$ and the induced seminorm defined by

$$(1) \qquad \|x\|_A = \langle x, x\rangle_A^{\frac{1}{2}} = \langle Ax, x\rangle^{\frac{1}{2}} = \langle A^{\frac{1}{2}}x, A^{\frac{1}{2}}x\rangle^{\frac{1}{2}} = \|A^{\frac{1}{2}}x\|,$$

for all $x \in \mathcal{H}$.
The above semi-inner product follows a seminorm on $\mathcal{B}^A(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \ \|T\|_A < \infty\}$, the subspace of $\mathcal{B}(\mathcal{H})$, the set of all $T \in \mathcal{B}(\mathcal{H})$ so that for some $c > 0$ and all $x \in \overline{R(A)}$, we have $\|Tx\|_A \leq c\|x\|_A$. In fact,

$$\|T\|_A = \sup\{\|Tx\|_A : \ \|x\|_A = 1\} = \sup_{x \in \overline{R(A)}, x \neq 0} \frac{\|Tx\|_A}{\|x\|_A} < \infty.$$

*Speaker

In 1966, Douglas proved the next theorem [2].

THEOREM 1.1. [2, Theorem 1] *Let $S, T \in \mathcal{B}(\mathcal{H})$. Then the following three conditions are equivalent.*

1) $R(S) \subseteq R(T)$.
2) *There exists a positive number $\lambda$ such that $\|S^*x\| \leq \lambda\|T^*x\|$, for all $x \in \mathcal{H}$.*
3) *There exists $V \in \mathcal{B}(\mathcal{H})$ such that $TV = S$.*

For $T \in \mathcal{B}(\mathcal{H})$, an operator $S \in \mathcal{B}(\mathcal{H})$ is called an $A$-adjoint of $T$ if $\langle Tx, y\rangle_A = \langle x, Sy\rangle_A$, for all $x, y \in \mathcal{H}$, that is $AS = T^*A$. An operator $T$ is called $A$-selfadjoint if $AT = T^*A$ and is called $A$-positive if $AT$ is positive. By Theorem 1.1, $T \in \mathcal{B}(\mathcal{H})$ admits an $A$-adjoint if and only if $R(T^*A) \subseteq R(A)$. Let $\mathcal{B}_A(\mathcal{H})$ denotes the set of all $T \in \mathcal{B}(\mathcal{H})$ which admit $A$-adjoints, i.e.,

$$\mathcal{B}_A(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \ R(T^*A) \subseteq R(A)\}.$$

For $T \in \mathcal{B}_A(\mathcal{H})$ there exists a distinguished $A$-adjoint operator of $T$, namely, the reduced solution of the equation $AX = T^*A$ denoted by $T^\sharp$.

The $A$-numerical radius and the $A$-Crawford number of $T \in \mathcal{B}(\mathcal{H})$ denoted by $\omega_A(T)$ and $c_A(T)$, respectively and defined by

$$\omega_A(T) = \sup\{|\langle Tx, x\rangle_A| : x \in \mathcal{H}, \|x\|_A = 1\},$$

and

$$c_A(T) = \inf\{|\langle Tx, x\rangle_A| : x \in \mathcal{H}, \|x\|_A = 1\}.$$

Also, the $A$-Davis-Wielandt radius of $T$ defined by

$$d\omega_A(T) = \sup\left\{\sqrt{|\langle Tx, x\rangle_A|^2 + \|Tx\|_A^4} : \ x \in \mathcal{H}, \ \|x\|_A = 1\right\},$$

and the $A$-total cosine of $T$ defined by

$$|\cos_A|T = \inf\left\{\frac{|\langle Tx, x\rangle_A|}{\|Tx\|_A \ \|x\|_A} : \ x \in \mathcal{H}, \ A^{\frac{1}{2}}Tx \neq 0, \ A^{\frac{1}{2}}x \neq 0\right\}.$$

Recently, some results for operators defined on a complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ are extended to $(\mathcal{H}, \langle\cdot,\cdot\rangle_A)$, for example (See, [3, 4]). In [3], M. S. Moslehian, Q. Xu and A. Zamani obtain new upper and lower bounds for the $A$-numerical radius of operators in semi-Hilbertian spaces. In [4], A. Zamani characterized $\omega_A(T)$, the $A$-numerical radius of operator $T$ in semi-Hilbertian space $(\mathcal{H}, \langle\cdot,\cdot\rangle_A)$ and apply it to find upper and lower bounds for $\omega_A(T)$. In the next section, we obtain a majorization on operators in $\mathcal{B}(\mathcal{H})$ and consider the relations between $A$-majorization, range inclusion and $A$-numerical radius.

## 2. Main Results

In this section, we introduce some majorization on $\mathcal{B}(\mathcal{H})$ and consider the relations between $A$-majorization, range inclusion and $A$-numerical radius. Our $A$-majorization and Barnes's majorization are compared.

DEFINITION 2.1. Let $S, T \in \mathcal{B}(\mathcal{H})$. Then $S$ is $A$-majorized by $T$ and denoted by $S \prec_{Am} T$, if there exists $M > 0$ such that for all $x \in \mathcal{H}$, we have

$$\|Sx\|_A \leq M\|Tx\|_A. \tag{2}$$

By (1), $S \prec_{Am} T$ is equivalent to $\|A^{1/2}Sx\| \leq M\|A^{1/2}Tx\|$. The inequality (2) induces $N(A^{1/2}T) \subseteq N(A^{1/2}S)$.

The $A$-majorization is a preordering, i.e. it is reflexive and transitive. Definition 2.1 and Proposition [1, Proposition 3] follow the next Theorem.

THEOREM 2.2. *Let $S, T \in \mathcal{B}(\mathcal{H})$. Then the following statements are equivalent.*

i) $S \prec_{Am} T$,
ii) *There exists* $V \in \mathcal{B}(\overline{R(A^{1/2}T)}, \mathcal{H})$ *such that* $A^{1/2}S = VA^{1/2}T$,
iii) *Whenever* $\{x_n\} \subseteq \mathcal{H}$ *with* $\|Tx_n\|_A \to 0$, *then* $\|Sx_n\|_A \to 0$.

In the next Theorem, we will use the ideas of Theorems 1.1 and 2.2.

THEOREM 2.3. *Let $S, T \in \mathcal{B}(\mathcal{H})$. Then the following statements are equivalent.*

i) $S^* \prec_{Am} T^*$,
ii) $SA^{1/2} = TA^{1/2}U$ *for some* $U \in \mathcal{B}(\mathcal{H})$,
iii) $R(SA^{1/2}) \subseteq R(TA^{1/2})$.

DEFINITION 2.4. Let $S, T \in \mathcal{B}(\mathcal{H})$. Then we say that, $S$ is $A$-strong majorized by $T$ and denoted by $S \prec_{Asm} T$, if there exists $M > 0$ such that for all $x, y \in \mathcal{H}$,

$$|\langle Sx, y\rangle_A| \leq M|\langle Tx, y\rangle_A|. \tag{3}$$

The $A$-strong majorization is a preordering, on $\mathcal{B}(\mathcal{H})$, i.e. it is reflexive and transitive.

PROPOSITION 2.5. *Let $S, T \in \mathcal{B}(\mathcal{H})$. If $S \prec_{Asm} T$, then $S \prec_{Am} T$.*

PROOF. By assumption, there exists $M > 0$ such that for all $x, y \in \mathcal{H}$, we have (3). In (3), put $y = Sx$, then

$$\|Sx\|_A^2 = |\langle Sx, Sx\rangle_A| \leq M|\langle Tx, Sx\rangle_A| \leq M\|Tx\|_A \,\|Sx\|_A,$$

so for all $x \in \mathcal{H}$, we have

$$\|Sx\|_A \leq M\|Tx\|_A.$$

That is $S \prec_{Am} T$. □

REMARK 2.6. Let $S, T \in \mathcal{B}_A(\mathcal{H})$. Then the following statements hold.

i) $S \prec_{Asm} T$ if and only if $S^\sharp \prec_{Asm} T^\sharp$.
ii) If $T$ is $A$-selfadjoint and $S \prec_{Asm} T$, then $S^\sharp \prec_{Asm} T$.

In the next example, we show that $S \prec_{Am} T$ does not imply $S \prec_{Asm} T$. That is the inverse of Proposition 2.5 is not true.

EXAMPLE 2.7. Let $\mathcal{H} = \ell^2 = \{(x_n) : x_n \in \mathbb{C},\ \Sigma_{n=1}^{\infty}|x_n|^2 < \infty\}$. Let $S, T, A \in \mathcal{B}(\mathcal{H})$ for $x = (x_1, x_2, \ldots) \in \mathcal{H}$ are defined by

$$\begin{aligned} S(x_1, x_2, \ldots) &= (x_1, 0, x_2, 0, x_3, 0, \ldots), \\ T(x_1, x_2, \ldots) &= (0, x_1, x_2, \ldots), \\ A(x_1, x_2, \ldots) &= (0, x_2, x_3, \ldots). \end{aligned}$$

Thus

$$AS(x_1, x_2, \dots) = (0, 0, x_2, 0, x_3, 0, x_4, \dots),$$
$$AT(x_1, x_2, \dots) = (0, x_1, x_2, \dots),$$

and so

$$\|Sx\|_A^2 = |\langle Sx, Sx\rangle_A| = |\langle ASx, Sx\rangle| = |x_2|^2 + |x_3|^2 + \cdots ,$$
$$\|Tx\|_A^2 = |\langle Tx, Tx\rangle_A| = |\langle ATx, Tx\rangle| = |x_1|^2 + |x_2|^2 + \cdots .$$

Obviously, $\|Sx\|_A \leq \|Tx\|_A$ and so $S \prec_{Am} T$.
But for $x = (1, 0, 1, 0, 0, 0, \dots)$ and $y = (1, 0, 1, 0, 1, 0, 0, \dots)$ in $\mathcal{H}$, we have

$$|\langle Sx, y\rangle_A| = |\langle ASx, y\rangle| = |x_2\bar{y_3} + x_3\bar{y_5} + x_4\bar{y_7} + \cdots| = 1,$$
$$|\langle Tx, y\rangle_A| = |\langle ATx, y\rangle| = |x_1\bar{y_2} + x_2\bar{y_3} + x_3\bar{y_4} + \cdots| = 0.$$

Therefore $S \not\prec_{Asm} T$.

THEOREM 2.8. *Let $S_1, S_2, S, T \in \mathcal{B}_A(\mathcal{H})$. Then the following statements hold.*

i) *If $S_1 \prec_{Asm} T$ and $S_2 \prec_{Asm} T$, then for $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, we have $\alpha S_1 + \beta S_2 \prec_{Asm} T$.*

ii) *If $S \prec_{Asm} T$ and $T$ is $A$-selfadjoint, then $Re_A(S) \prec_{Asm} T$, $Im_A(S) \prec_{Asm} T$, where $Re_A(S) = \frac{S+S^\sharp}{2}$ and $Im_A(S) = \frac{S-S^\sharp}{2i}$.*

Theorem 2.2 and [1, Proposition 6] follow the next proposition.

PROPOSITION 2.9. *Suppose that $S, T \in \mathcal{B}(\mathcal{H})$ and $S \prec_{Am} T$. Then the following statements are satisfied.*

i) *If $T$ is compact, then $A^{1/2}S$ is compact.*

ii) *If $T$ is weakly compact, then $A^{1/2}S$ is weakly compact.*

iii) *If $T$ is strictly singular, then $A^{1/2}S$ is strictly singular.*

THEOREM 2.10. *Let $S, R, T \in \mathcal{B}_A(\mathcal{H})$ and $S \prec_{Asm} T$. Then*

i) $S^\sharp S \prec_{Asm} S^\sharp T$,

ii) $SS^\sharp \prec_{Asm} ST^\sharp$,

iii) $T^\sharp S \prec_{Asm} T^\sharp T$ *and* $S^\sharp T \prec_{Asm} T^\sharp T$,

iv) $S^\sharp S \prec_{Asm} T^\sharp T$,

v) $R^\sharp SR \prec_{Asm} R^\sharp TR$.

Two elements $x, y \in \mathcal{H}$ are called $A$-orthogonal and denoted by $x \perp_A y$, if $\langle x, y\rangle_A = 0$.

PROPOSITION 2.11. *Let $S, T \in \mathcal{B}(\mathcal{H})$ and $S \prec_{Asm} T$ and $M \subseteq \mathcal{H}$. If $TM \subseteq M^{\perp_A}$, then $SM \subseteq M^{\perp_A}$.*

PROPOSITION 2.12. *Let $S, T \in \mathcal{B}_A(\mathcal{H})$ and $S \prec_{Asm} T$. If $S$ and $T$ are both $A$-selfadjoint, $R(T) \subseteq \overline{R(A)}$ and $R(S) \subseteq \overline{R(A)}$, then $S^n \prec_{Asm} T^n$, where $n = 2^m$, for all $m \in \mathbb{N}$.*

THEOREM 2.13. *Suppose that $S, T \in \mathcal{B}(\mathcal{H})$ and $S \prec_{Asm} T$, i.e. there exists $M > 0$ such that for all $x, y \in \mathcal{H}$,*

$$|\langle Sx, y\rangle_A| \leq M|\langle Tx, y\rangle_A|.$$

*Then the following statements hold.*

i) $d\omega_A(S) \leq M d\omega_A(T)$,
ii) $|\cos_A|S \leq M|\cos_A|T$,
iii) $c_A(S) \leq M c_A(T)$,
iv) $\|S\|_A \leq M\|T\|_A$,
v) $\omega_A(S) \leq M\omega_A(T)$.

## References

1. B. A. Barnes, *Majorization, range inclusion, and factorization for bounded linear operators,* Proc. Amer. Math. Soc. **133** (1) (2005) 155–162.
2. R. G. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space,* Proc. Amer. Math. Soc. **17** (1966) 413–415.
3. M. S. Moslehian, Q. Xu and A. Zamani, *Seminorm and numerical radius inequalities of operators in semi-Hilbertian spaces,* Linear Algebra Appl. **591** (2020) 299–321.
4. A. Zamani, *A-Numerical radius inequalities for semi-Hilbertian space operators,* Linear Algebra Appl. **578** (2019) 159–183.

E-mail: Farzaneanaliz@gmail.com
E-mail: eftekhari-n@sku.ac.ir

# Weak Solutions for a System of Non-Homogeneous Problem

Samira Heidari*
Department of Pure Mathematics, Faculty of Sciences, Imam Khomeini International University, Qazvin, Iran
and Abdolrahman Razani
Department of Pure Mathematics, Faculty of Sciences, Imam Khomeini International University, Qazvin, Iran

Abstract. Using the variational method, the existence of weak solutions is proved.
**Keywords:** Infinitely many solutions, Variational method, Nonhomogeneous operator.
**AMS Mathematical Subject Classification [2010]:** 35J60, 35J50, 34B10.

## 1. Introduction

In this paper we prove the existence of infinitely many weak solutions for the system

$$(1)\qquad \begin{cases} -div(\beta_1(|\nabla u|))\nabla u = \lambda H_u(\sigma, u, v), & in \quad \Omega, \\ -div(\beta_2(|\nabla v|))\nabla v = \lambda H_v(\sigma, u, v), & in \quad \Omega, \\ u = v = 0, & on \quad \partial\Omega. \end{cases}$$

where $\Omega$ is an open bounded subset of $\mathbb{R}^N (N \geq 3)$, with smooth boundary $\partial\Omega$, and $\lambda \in (0, +\infty)$.

Moreover, $H : \bar{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that $H(\cdot, \tau_1, \tau_2)$ is measurable in $\bar{\Omega}$, for each $(\tau_1, \tau_2) \in \mathbb{R} \times \mathbb{R}$ and $H(\sigma, \cdot, \cdot)$ is $C^1$ in $\mathbb{R} \times \mathbb{R}$ for every $\sigma \in \bar{\Omega}$. $H_u$ and $H_v$ denote the partial derivatives of $H$ with respect to $u$ and $v$, respectively.

## 2. Preliminaries

First, some essential concept are recalled (See [2, 3, 5, 6]).

For $i = 1, 2$, assume that $\beta_i : (0, +\infty) \to \mathbb{R}$ are two functions such that the mapping $\varphi_i : \mathbb{R} \to \mathbb{R}$ are defined by

$$\varphi_i(\tau) = \begin{cases} \beta_i(|\tau|)\tau, & \text{for } \tau \neq 0, \\ 0, & \text{for } \tau = 0, \end{cases}$$

are odd, strictly increasing homeomorphisms. For $i = 1, 2$, set $\Phi_i(\tau) := \int_0^\tau \varphi_i(\sigma) d\sigma$ and $\Phi_i^*(\tau) = \int_0^\tau \varphi_i^{-1}(\sigma) d\sigma$, for all $\tau \in \mathbb{R}$. Notice that that $\Phi_i$, $i = 1, 2$, are Young functions (See in [6]).

*Speaker

Also, $\Phi_i$ are called $N$-functions (See in [6]). The functions $\Phi_i^*$, $i = 1,2$, are called the complementary functions of $\Phi_i$ and they satisfy $\Phi_i^*(\tau) = \sup\{\varsigma\tau - \Phi_i(\varsigma); \varsigma \geq 0\}$, for all $\tau \geq 0$. We define the numbers

$$(p_i)_0 := \inf_{\tau>0} \frac{\tau\varphi(\tau)}{\Phi(\tau)}, \quad \text{and} \quad (p_i)^0 := \sup_{\tau>0} \frac{\tau\varphi(\tau)}{\Phi(\tau)}.$$

All over this paper, we presume the following case

$$N < (p_i)_0 \leq \frac{\tau\varphi_i(\tau)}{\Phi_i(\tau)} \leq (p_i)^0 < \infty, \quad \text{for all } \tau > 0. \tag{2}$$

Consider the Orlicz spaces $L_{\Phi_i}(\Omega), i = 1,2$, with the norm

$$\|u\|_{L_{\Phi_i}} := \sup\{\int_\Omega u(\sigma)z(\sigma)d\sigma; \int_\Omega \Phi_i^*(|z(\sigma)|)d\sigma \leq 1\} < \infty.$$

Then $(L_{\Phi_i}(\Omega), \|.\|_{L_{\Phi_i}})$ are Banach spaces that norms are similar to the Luxemburg norm

$$\|u\|_{\Phi_i} := \inf\{k > 0; \int_\Omega \Phi_i(\frac{u(\sigma)}{k})d\sigma \leq 1\}.$$

The Orlicz-Sobolev spaces $W^{1,\Phi_i}(\Omega), i = 1,2$, for problem (1), defined by

$$W^{1,\Phi_i}(\Omega) = \{u \in L_{\Phi_i}(\Omega), \frac{\partial u}{\partial \sigma_j} \in L_{\Phi_i}(\Omega), j = 1,\ldots,N\}.$$

These are Banach spaces with regard to the norms $\|u\|_{1,\Phi_i} := \|u\|_{\Phi_i} + \||\nabla u|\|_{\Phi_i}$ for $i = 1,2$.

For $i = 1,2$ let $W_0^{1,\Phi_i}(\Omega)$ be the Orlicz-Sobolev spaces with corresponding norms $\|u\|_i := \||\nabla u|\|_{\Phi_i}$.

The relation (2) assures that $\Phi_i$ and $\Phi_i^*$, $i = 1,2$, both satisfy the $\Delta_2$-condition, i.e. $\Phi_i(2\tau) \leq k\Phi_i(\tau)$, for all $\tau \geq 0$, where $k$ is a positive steady. Furthermore, we need the following conditions:

$$\text{Functions } \tau \to \Phi_i(\sqrt{\tau}) \text{ are convex for all } \tau \in [0,\infty)\ i = 1,2. \tag{3}$$

Condition $\Delta_2$ for $\Phi_i$ assures that for all $i \in \{1,2\}$, $L_{\Phi_i}(\Omega)$ and $W_0^{1,\Phi_i}(\Omega)$ are separable. $\Delta_2$ condition and (3) assure that $L_{\Phi_i}(\Omega)$ are uniformly convex spaces and hence, reflexive Banach spaces, that implies Orlicz-Sobolev spaces $W_0^{1,\Phi_i}(\Omega)$, $i \in \{1,2\}$ are reflexive Banach spaces also.

Now, one can define the reflexive Banach space $X := W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega)$ endowed with the norm $\|(u,v)\| = \|u\|_1 + \|v\|_2$, where $\|u\|_1 := \||\nabla u|\|_{\Phi_1}$ and $\|v\|_2 := \||\nabla v|\|_{\Phi_2}$.

In [4] is shown that $W_0^{1,\Phi_i}(\Omega), i = 1,2$, are continuously embedded in $W_0^{1,(p_i)_0}(\Omega)$. Beside, since $(p_i)_0 > N$, one can conclude that $W_0^{1,(p_i)_0}(\Omega) \hookrightarrow C^0(\bar{\Omega})$ are compact. Thus the embedding $X \hookrightarrow C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$ is compact.

Proposition 2.1. *Let $u \in W_0^{1,\Phi_i}(\Omega)$, then*

i) $\|u\|_i^{(p_i)_0} \leq \int_\Omega \Phi_i(|\nabla u(\sigma)|)d\sigma \leq \|u\|_i^{(p_i)^0}$ *if* $\|u\|_i > 1$, $i = 1,2$,

ii) $\|u\|_i^{(p_i)^0} \leq \int_\Omega \Phi_i(|\nabla u(\sigma)|)d\sigma \leq \|u\|_i^{(p_i)_0}$ *if* $\|u\|_i < 1$, $i = 1,2$.

We need the following fact from [2, Lemma 2.1].

PROPOSITION 2.2. *Let* $u \in W_0^{1,\Phi_i}(\Omega)$ *and* $\int_\Omega \Phi_i(|\nabla u(\sigma)|)d\sigma \leq r$, *for some* $0 < r < 1$. *So* $\|u\|_i < 1$.

We set

$$\zeta := \max\{ \sup_{u \in W_0^{1,\Phi_1}\setminus\{0\}} \frac{\max_{\sigma\in\bar{\Omega}} |u(\sigma)|^{(p_1)^0}}{\|u\|_1^{(p_1)^0}}, \sup_{v \in W_0^{1,\Phi_2}\setminus\{0\}} \frac{\max_{\sigma\in\bar{\Omega}} |v(\sigma)|^{(p_2)^0}}{\|v\|_2^{(p_2)^0}} \}.$$

For fixed $\sigma_0 \in \Omega$, set $D > 0$ such that $\overline{S(\sigma_0, D)} \subseteq \Omega$, where $S(\sigma_0, D)$ is the ball with center at $\sigma_0$ and radius $D$.

$$\theta_{(p_1)^0} = \frac{\Gamma(1+\frac{N}{2})(\frac{D}{2})^{(p_1)^0}}{\left(\zeta^{\frac{1}{(p_1)^0}} + \zeta^{\frac{1}{(p_2)^0}}\right)^{p^*} \varrho \pi^{\frac{N}{2}}} \left(\frac{2^N}{D^N(2^N-1)}\right),$$

$$\theta_{(p_2)^0} = \frac{\Gamma(1+\frac{N}{2})(\frac{D}{2})^{(p_2)^0}}{\left(\zeta^{\frac{1}{(p_1)^0}} + \zeta^{\frac{1}{(p_2)^0}}\right)^{p^*} \varrho \pi^{\frac{N}{2}}} \left(\frac{2^N}{D^N(2^N-1)}\right).$$

## 3. Multiple Solutions

Here we introduce the suitable hypothesis such that the system (1) has solutions. We set

$$\begin{aligned} \phi(r) &:= \inf_{u\in I^{-1}(-\infty,r)} \frac{\sup_{v\in I^{-1}(-\infty,r)} J(v) - J(u)}{r - I(u)}, \\ \delta &:= \liminf_{r \to (\inf_X I)^+} \phi(r). \end{aligned}$$

We show that $\delta < \infty$, and there is a global minimum of $J$ that is not a local minimum of $g_\lambda$. In this case, there is a sequence $\{u_n\}$ of pairwise distinct critical points which is weakly convergent to a global minimum of $J$.

THEOREM 3.1. *Assume that*

(h1) $H(\sigma, \tau_1, \tau_2) \geq 0$ *for every* $(\sigma, \tau_1, \tau_2) \in \Omega \times (\mathbb{R}^+)^2$.

(h2) $H(\sigma, 0, 0) = 0$ *for every* $\sigma \in \Omega$.

(h3) *There exist* $\sigma_0 \in \Omega$, *and values* $D, \varrho > 0$ *such that* $\overline{S(\sigma_0, D)} \subseteq \Omega$, $\lim_{\sigma\to 0^+} \dfrac{\Phi_i(\sigma)}{\sigma^{(p_i)^0}} < \varrho$, *and* $\mathcal{E} < \theta\mathcal{F}$, *where* $\theta = \min\{\theta_{(p_1)^0}, \theta_{(p_2)^0}\}$ *and*

$$\mathcal{E} := \liminf_{\xi\to 0^+} \frac{\int_\Omega \sup_{|\tau_1|+|\tau_2|\leq \xi} H(\sigma,\tau_1,\tau_2)d\sigma}{\xi^{p^*}}, \ \mathcal{F} := \limsup_{\tau_1,\tau_2\to 0^+} \frac{\int_{S(\sigma_0,\frac{D}{2})} H(\sigma,\tau_1,\tau_2)d\sigma}{\tau_1^{(p_1)^0} + \tau_2^{(p_2)^0}},$$

*where* $p^* = \max((p_1)^0, (p_2)^0)$. *So, for each* $\lambda \in \Lambda := \frac{1}{\left(\zeta^{\frac{1}{(p_1)^0}} + \zeta^{\frac{1}{(p_2)^0}}\right)^{p^*}} (\frac{1}{\theta\mathcal{F}}, \frac{1}{\mathcal{E}})$, *the system* (1) *admits a sequel of pairwise separate weak solutions which strongly converges to zero in* $X$.

PROOF. We apply Bonanno's theorem [1] and show that $\delta < \infty$. First, we define $g_\lambda : X \to \mathbb{R}$ by

$$g_\lambda(u, v) = I(u, v) - \lambda J(u, v),$$

where

$$I(u,v)=\int_\Omega \Phi_1(|\nabla u|)d\sigma+\int_\Omega \Phi_2(|\nabla v|)d\sigma, \quad \text{and} \quad J(u,v)=\int_\Omega H(\sigma,u,v)d\sigma.$$

It is well known that $I$ and $J$ are Gâteaux differentiable functional and sequentially weakly lower semicontinuous and $I$ is coercive. Let $\{\xi_n\}$ be a sequel of positive numbers such that $\lim_{n\to+\infty}\xi_n=0$ and

$$\begin{aligned}\lim_{n\to+\infty}\frac{\int_\Omega \sup_{|\tau_1|+|\tau_2|\le\xi_n} H(\sigma,\tau_1,\tau_2)d\sigma}{\xi_n^{p^*}} &= \liminf_{\xi\to0^+}\frac{\int_\Omega \sup_{|\tau_1|+|\tau_2|\le\xi} H(\sigma,\tau_1,\tau_2)d\sigma}{\xi^{p^*}}\\ &= \mathcal{E}<+\infty.\end{aligned}$$

Set $r_n=\Big(\frac{\xi_n}{(\zeta)^{\frac{1}{(p_1)^0}}+(\zeta)^{\frac{1}{(p_2)^0}}}\Big)^{p^*}$. For all $n\in\mathbb{N}$ big enough $0<r_n<1$, then

$$|u(\sigma)|+|v(\sigma)|\le(\zeta r_n)^{\frac{1}{(p_1)^0}}+(\zeta r_n)^{\frac{1}{(p_2)^0}}<\Big((\zeta)^{\frac{1}{(p_1)^0}}+(\zeta)^{\frac{1}{(p_2)^0}}\Big)r_n^{\frac{1}{p^*}}=\xi_n.$$

We have

$$\begin{aligned}\delta\le\liminf_{n\to+\infty}\phi(r_n)\le&\Big((\zeta)^{\frac{1}{(p_1)^0}}+(\zeta)^{\frac{1}{(p_2)^0}}\Big)^{p^*}\liminf_{n\to+\infty}\frac{\int_\Omega \sup_{|\tau_1|+|\tau_2|<\xi_n} H(\sigma,\tau_1,\tau_2)d\sigma}{\xi_n^{p^*}}\\ \le&\Big((\zeta)^{\frac{1}{(p_1)^0}}+(\zeta)^{\frac{1}{(p_2)^0}}\Big)^{p^*}\mathcal{E}<+\infty.\end{aligned}$$

So, $\Lambda\subseteq]0,\frac{1}{\delta}[$. For $\lambda\in\Lambda$, we assertion that the functional $g_\lambda$ is unbounded from below. There exist a sequel $\{\tau_n\}$ of positive numbers and $\eta>0$ such that $\tau_n\to0^+$, and

$$\frac{1}{\lambda}<\eta<\theta\Big((\zeta)^{\frac{1}{(p_1)^0}}+(\zeta)^{\frac{1}{(p_2)^0}}\Big)^{p^*}\frac{\int_{S(\sigma_0,\frac{D}{2})}H(\sigma,\tau_n,\tau_n)d\sigma}{\tau_n^{(p_1)^0}+\tau_n^{(p_2)^0}},$$

for any $n\in\mathbb{N}$ big enough. Let $\{\varpi_n\}\subseteq X$ be a sequel defined by

$$\varpi_n(\sigma):=\begin{cases}0, & \sigma\in\bar{\Omega}\setminus S(\sigma_0,D),\\ \dfrac{2\tau_n}{D}\Big(D-\{\Sigma_{i=1}^n(\sigma^i-\sigma_0^i)^2\}^{\frac{1}{2}}\Big), & \sigma\in S(\sigma_0,D)\setminus S(\sigma_0,\frac{D}{2}),\\ \tau_n, & \sigma\in S(\sigma_0,\frac{D}{2}).\end{cases}$$

Since $\lim_{n\to\infty}\frac{2\tau_n}{D}=0$, there exist $\nu>0$ and $n_1,n_2\in\mathbb{N}$ such that $\frac{2\tau_n}{D}\in(0,\nu)$, and $\Phi_1(\frac{2\tau_n}{D})<\varrho(\frac{2}{D})^{(p_1)^0}\tau_n^{(p_1)^0}$ for all $n\ge n_1$, and $\Phi_2(\frac{2\tau_n}{D})<\varrho(\frac{2}{D})^{(p_2)^0}\tau_n^{(p_2)^0}$ for all $n\ge n_2$.

So, for all $n\ge\max\{n_1,n_2\}$, we have

$$\begin{aligned}g_\lambda(\varpi_n,\varpi_n)&=I(\varpi_n,\varpi_n)-\lambda J(\varpi_n,\varpi_n)\\ &\le\frac{1}{\Big((\zeta)^{\frac{1}{(p_1)^0}}+(\zeta)^{\frac{1}{(p_2)^0}}\Big)^{p^*}}\Big(\frac{\tau_n^{(p_1)^0}}{\theta_{(p_1)^0}}+\frac{\tau_n^{(p_2)^0}}{\theta_{(p_2)^0}}\Big)-\lambda\int_{S(\sigma_0,\frac{D}{2})}H(\sigma,\tau_n,\tau_n)d\sigma\\ &<\frac{1-\lambda\eta}{\theta\Big((\zeta)^{\frac{1}{(p_1)^0}}+(\zeta)^{\frac{1}{(p_2)^0}}\Big)^{p^*}}(\tau_n^{(p_1)^0}+\tau_n^{(p_2)^0})<0=g_\lambda(0,0),\end{aligned}$$

for every $n \in \mathbb{N}$ big enough. Then $(0, 0)$ is not a local minimum of $g_\lambda$. Thus Bonanno's theorem prove the existence of the sequel $\{(u_n, v_n)\}$ of pairwise distinct critical points (local minima) of $g_\lambda$ such that $\|(u_n, v_n)\| \to 0$. □

## References


1. G. Bonanno and G. Molica Bisci, *Infinitely many solutions for a boundary value problem with discontinuous nonlinearities*, Bound. Value Probl. **2009** (2009) 1–20.
2. M. Makvand Chaharlang and A. Razani, *Existence of infinitely many solutions for a class of nonlocal problems with Dirichlet boundary condition*, Commun. Korean Math. Soc. **34** (1) (2019) 155–167.
3. M. Makvand Chaharlang and A. Razani, *Two weak solutions for some Kirchhoff-type problem with Neumann boundary condition*, Georgian Math. J. (2020). DOI:10.1515/gmj-2019-2077
4. P. Cle'ment, M. Garcia Huidobro, R. Manasevich and K. Schmitt, *Mountain pass type solutions for quasilinear elliptic equations,* Calc. Var. Partial Differential Equations **11** (2000) 33–62.
5. H. Hudzik, *The problem of separability, duality, reflexivity and of comparison for generalized Orlicz-Sobolev spaces* ($W_M^k(\Omega)$), Comment. Math. **21** (2) (1979) 315–324.
6. M. M. Rao and Z. D. Ren, *Applications of Orlicz Spaces, Monographs and Textbooks in Pure and Applied Mathematics*, Vol. 250, Marcel Dekker Inc., New York, 2002.


E-mail: S.Heidari@edu.ikiu.ac.ir
E-mail: razani@sci.ikiu.ac.ir

# $C$-Norm Inequalities for Special Operator Matrices

Sharifeh Rezagholi
Department of Basic Science, Payame Noor University, Tehran, Iran
and Mahya Hosseini*
Department of Basic Science, Payame Noor University, Tehran, Iran

Abstract. $C$-norm of $2 \times 2$ operator matrices, in the form of $\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ are studied and examples indicate equalities not hold.

**Keywords:** $C$-Norm, Inequality, Operator matrices.
**AMS Mathematical Subject Classification [2010]:** 15A18, 47A30, 15A60.

## 1. Introduction

Let $H$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Also $B(H_1, H_2)$ denote the set of all bounded linear operators from $H_1$ into $H_2$. We use $B(H)$ instead of $B(H, H)$. Let $\mathcal{U}$ be the group of unitary operators in $B(H)$. When $H$ is of finite dimension $n$, we use $M_n$ instead of $B(H)$ and $\mathcal{U}_n$ instead of $\mathcal{U}$. Let $C, A \in M_n$. Recall that the $C$-norm of an operator $A$ is defined by

$$\|A\|_C = \max\{|tr(CUAV)| : U, V \in \mathcal{U}_n\}.$$

Which at first defined by J. von Neuman [2]. If $C = diag(1, 0, \dots, 0), A \in M_n, U = [x_1, x_2, \dots, x_n] \in \mathcal{U}_n$ and $V = [y_1, y_2, \dots, y_n] \in \mathcal{U}_n$, then

$$C(U^*AV) = C \begin{bmatrix} x_1^*Ay_1 & x_1^*Ay_2 & \cdots & x_1^*Ay_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in M_n.$$

So

$$\begin{aligned} \|A\|_C &= \max\{|tr(CU^*AV)| : U, V \in \mathcal{U}_n\} \\ &= \max\{|x^*Ay|, \|x\| = \|y\| = 1\} \\ &= \max_{\|x\|=1} \|Ax\| = \|A\|_2. \end{aligned}$$

This shows that $C$-norm is a generalization of the operator norm. The following Theorem, which states in [2], is useful in calculating $C$-norm of matrices.

Theorem 1.1. *Let $A, C \in M_n$ and $a_1 \geq a_2 \geq \cdots \geq a_n, c_1 \geq c_2 \geq \cdots \geq c_n$, be singular values of $A$ and $C$, respectively. Then*

$$\|A\|_C = \Sigma_{j=1}^n a_j c_j.$$

*Speaker

As a corollary of the above theorem, let $A \in M_n$ with $a_1$ be the largest singular values of $A$. Then $\|A\|_2 = a_1$. In the following theorem we see some norm properties of $C$-norms.

Theorem 1.2. [1, Theorem 3.1] *Let $C \in M_n$ with the largest singular values $c_1$. Then, the following statements hold:*

i) $\|\cdot\|_C$ *is a semi - norm on* $M_n$*;*
ii) $\|\cdot\|_C$ *is a vector norm on* $M_n$ *if and only if* $C \neq 0$*;*
iii) $\|\cdot\|_C$ *is a matrix norm on* $M_n$ *if and only if* $c_1 \geq 0$.

The following proposition, also is useful in $C$-norm calculations.

Proposition 1.3. [1, Corollary 3.2] *Let $0 \neq C, A \in M_n$. Then, the following properties hold:*

i) *If $U \in \mathcal{U}_n$, then* $\|U^*AU\|_C = \|A\|_C = \|A\|_{U^*CU}$*;*
ii) *If $c_1$ is the largest singular values of $C$, then for every $k = 1, 2, \ldots,$*

$$\|A^k\|_C \leq \|A\|_C^k \Leftrightarrow c_1 \geq 1.$$

In this paper, we are using above properties, to show some inequalities for $C$-norm of special $2 \times 2$ operator matrices. Also we have some examples to show that equality cannot hold in general.

## 2. Main Results

We begin with a theorem for $2\times2$ operator matrices. matrices which have operators as their entries.

Theorem 2.1. *Let $A, B \in B(H)$, $C \in M_n$ and $C' = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}$. Then*

$$\left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\|_{C'} = \left\| \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right\|_{C'} \geq \max\{\|A\|_C, \|B\|_C\}.$$

Proof. Let $U = \begin{bmatrix} X_1 & 0 \\ 0 & Y_1 \end{bmatrix}, V = \begin{bmatrix} X_2 & 0 \\ 0 & Y_2 \end{bmatrix}$ where $X_i, Y_i \in \mathcal{U}_n$. Then,

$$\left| tr \left( \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} X_1 & 0 \\ 0 & Y_1 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} X_2 & 0 \\ 0 & Y_2 \end{bmatrix} \right) \right| = \left| tr \left( \begin{bmatrix} CX_1AX_2 & 0 \\ 0 & CY_1AY_2 \end{bmatrix} \right) \right|.$$

So,

$$\begin{aligned} \left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\|_{C'} &\geq \max_{X_i, Y_i \in \mathcal{U}_n} \{|tr(CX_1AX_2) + tr(CY_1AY_2)|\} \\ &\geq \max_{X_i \in \mathcal{U}_n} \{|tr(CX_1AX_2)|\} = |A|_C (i = 1, 2). \end{aligned}$$

Also we have

$$\left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\|_{C'} \geq \|B\|_C.$$

Then,

$$\left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\|_{C'} \geq \max\{\|A\|_C, \|B\|_C\}.$$

Now, let $T_1 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, T_2 = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$. So we have $T_1^*T_1 = \begin{bmatrix} A^*A & 0 \\ 0 & B^*B \end{bmatrix}, T_2^*T_2 = \begin{bmatrix} B^*B & 0 \\ 0 & A^*A \end{bmatrix}$ and $C'C'^* = \begin{bmatrix} CC^* & 0 \\ 0 & CC^* \end{bmatrix}$. This shows that singular values of $T_1$ and $T_2$ are equal. Using Theorem 1.1, one can see easily that $\|T_1\|_{C'} = \|T_2\|_{C'}$. □

In following example we use Theorem 1.1 to show that the equality cannot hold in the above theorem.

Example 2.2. Let $T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and $C' = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}$, where $A = diag(2, 1, \frac{1}{2})$, $B = diag(\frac{3}{2}, \frac{1}{3}, \frac{1}{4})$ and $C = diag(4, 3, 1)$. We can see that singular values of $A$ are $2 \geq 1 \geq \frac{1}{2}$, singular values of $B$ are $\frac{3}{2} \geq \frac{1}{3} \geq \frac{1}{4}$, also singular values of $C$ are $4 \geq 3 \geq 1$, singular values of $C'$ are $4 \geq 4 \geq 3 \geq 3 \geq 1 \geq 1$ and singular values of $T$ are $2 \geq \frac{3}{2} \geq 1 \geq \frac{1}{2} \geq \frac{1}{3} \geq \frac{1}{4}$. We list singular values in this way instead of set way to show repetition of some singular values and to use Theorem 1.1 easily. By Theorem 1.1 we have $\|T\|_{C'} = 19 + \frac{1}{12}$, $\|A\|_C = 11 + \frac{1}{2}$ and $\|B\|_C = 7 + \frac{1}{4}$. So, $\max\{\|A\|_C, \|B\|_C\} = 11 + \frac{1}{2} < 19 + \frac{1}{12} = \|T\|_{C'}$. Also we have $\|A\|_C + \|B\|_C < 19 + \frac{1}{12} = \|T\|_{C'}$.

By the same manner as in the proof of Theorem 2.1, we can see the following proposition.

Proposition 2.3. *Let* $A, C \in M_n, T = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \in M_{2n}$ *and* $C' = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \in M_{2n}$. *Then,*

$$\|T\|_{C'} \geq \|A\|_C.$$

Using Theorem 1.1, one can see that

$$\left\| \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \right\|_{C'} = \left\| \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \right\|_{C'} = \left\| \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix} \right\|_{C'} = \left\| \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \right\|_{C'}.$$

Also if $C' = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix}$ or $\begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$, where the largest singular value of $D$ is less than or equal to the smallest singular value of $C$, then $\left\| \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \right\|_{C'} = \|A\|_C$. In the following example we show that equality cannot be hold in the above proposition.

Example 2.4. Let singular values of $A$ are $a_1 \geq a_2 \geq \cdots \geq a_n$ and singular values of $C$ are $c_1 \geq c_2 \geq \cdots \geq c_n$. So singular values of $T = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$ are $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 \geq 0 \geq \cdots \geq 0$, because of $T^*T = \begin{bmatrix} 0 & 0 \\ 0 & A^*A \end{bmatrix}$. On the other hand, singular values of $C'$ are $c_1 \geq c_1 \geq c_2 \geq c_2 \geq \cdots \geq c_n \geq c_n$. For example if singular values of $A$ are $2 \geq 1 \geq \frac{1}{2}$ and singular values of $C$ are $3 \geq 2 \geq 1$, then singular values of $T$ are $2 \geq 1 \geq \frac{1}{2} \geq 0 \geq 0 \geq \cdots \geq 0$ and singular values of $C'$ are $3 \geq 3 \geq 2 \geq 2 \geq 1 \geq 1$. Therefore, $\left\| \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \right\|_{C'} = 10$. But $\|A\|_C = \frac{17}{2}$.

## References

1. C. K. Li, T. Y. Tam and N. K. Tsing, *The generalized spectral radius, Numerical radius and spectral norm*, Linear Multilinear Algebra **16** (1-4) (1984) 215–237.
2. J. von Neumann, *Some matrix-inequalities and metrization of matrix space*, Tomsk Univ. Rev. **1** (1937) 286–300.

E-mail: rezagholi@pnu.ac.ir
E-mail: m.hosseini.math@gmail.com

# Mean Ergodicity of Multiplication Operators on Besov Spaces

Fakhreddin Falahat
Department of Mathematics, Faculty of Sciences, Shiraz Branch, Islamic Azad University, Shiraz, Iran
and Zahra Kamali*
Department of Mathematics, Faculty of Sciences, Shiraz Branch, Islamic Azad University, Shiraz, Iran

Abstract. In this paper, the power boundedness and mean ergodicity of multiplication operators are investigated on the Besov Space $\mathcal{B}_p$. Let $\mathbb{U}$ be the unit disk in the complex plane $\mathbb{C}$ and $\psi$ be a function in the space of holomorphic functions $H(\mathbb{U})$, our goal is to find out when the multiplication operator $M_\psi$ is power bounded, mean ergodic and uniformly mean ergodic on $\mathcal{B}_p$.

**Keywords:** Multiplication operator, Power bounded, Mean Ergodic operator, Besov spaces.
**AMS Mathematical Subject Classification [2010]:** 47B38, 46E15, 47A35.

## 1. Introduction

The space we dealt with in this paper is the *Besov space* $\mathcal{B}_p(1 < p < \infty)$ which is defined to be the space of holomorphic functions $f$ on $\mathbb{U}$ such that

$$\gamma_f^p = \int_{\mathbb{U}} |f'(z)|^p (1-|z|^2)^{p-2} dA(z) = \int_{\mathbb{U}} |f'(z)|^p (1-|z|^2)^p d\lambda(z) < \infty,$$

where $d\lambda(z)$ is the Möbius invariant measure on U, with definition

$$d\lambda(z) = \frac{dA(z)}{(1-|z|^2)^2}.$$

For $p = 1$ , the Besov space $\mathcal{B}_1$ consists of all holomorphic functions $f$ on $\mathbb{U}$ whose second derivatives are integrable,

$$\mathcal{B}_1 = \{f \in H(\mathbb{U}) : ||f||_{\mathcal{B}_1} = \int_{\mathbb{U}} |f''(z)| dA(z) < \infty\}.$$

For $1 < p < \infty$, it is well-known that $||f||_p = |f(0)| + \gamma_f$ is a norm on $\mathcal{B}_p$ which makes it a Banach space. $\mathcal{B}_p$ is reflexive space (while $\mathcal{B}_1$ is not) and polynomials are dense in it. the following useful lemma determines that norm convergence implies pointwise convergence in the Besov spaces.

*Speaker

Lemma 1.1. [8] *For each* $f \in \mathcal{B}_p$ $(1 < p < \infty)$ *and for every* $z \in \mathbb{U}$, *there is* $C \geq 0$ *(depends only on p) such that*

$$|f(z)| \leq C||f||_p(\log \frac{2}{1-|z|^2})^{1-1/p}.$$

[8] is a perfect reference for studying about Besov spaces.
If $\psi$ is a holomorphic function on $\mathbb{U}$, the *multiplication operator* $M_\psi$ on $H(\mathbb{U})$ is defined by

$$M_\psi(f) = \psi f.$$

We know in Besov spaces $||\psi||_\infty \leq ||M_\psi||$, as $||\psi||_\infty = \sup_{z\in\mathbb{U}} |\psi(z)|$. A function $\psi \in H(\mathbb{U})$ is said to be *multiplier* of $\mathcal{B}_p$ if $M_\psi(\mathcal{B}_p) \subseteq \mathcal{B}_p$. If the space of multipliers on $\mathcal{B}_p$ in to itself represented by $M(\mathcal{B}_p)$, then by Closed Graph theorem $\psi \in M(\mathcal{B}_p)$ if and only if $M_\psi$ is a bounded operator on $\mathcal{B}_p$. Following proposition is an applied result of Zorboska about multiplication operators on the Besov spaces.

Proposition 1.2. [5] *Suppose that* $1 < p < \infty$ *and* $\psi \in H^\infty(\mathbb{U})$.

i) *If* $\psi \in M(\mathcal{B}_p)$ *and* $0 < r < 1$, *then*

$$\sup_{\omega\in D} \int_{D(\omega,r)} (1-|z|^2)^{p-2}|\psi'(z)|^p(\log \frac{2}{1-|z|^2})^{p-1}dA(z) < \infty,$$

*where* $D(\omega, r) = \{z \in \mathbb{U} : \beta(z, \omega) < r\}$ *is the hyperbolic disk with radius* $r$, $\beta(z, \omega) = \log \frac{1+|\psi_z(\omega)|}{1-|\psi_z(\omega)|}$ *and* $\psi_z(\omega) = \frac{z-\omega}{1-\bar{z}\omega}$ *for all* $z, \omega \in \mathbb{U}$.

ii) *If* $\int_{\mathbb{U}}(1-|z|^2)^{p-2}|\psi'(z)|^p(\log \frac{2}{1-|z|^2})^{p-1}dA(z) < \infty$, *then* $\psi \in M(\mathcal{B}_p)$.

Let $L(X)$ be the space of all linear bounded operators from locally convex Hausdorff space $X$ into itself and $T \in L(X)$, the Cesáro means of $T$ is defined by

$$T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m, \ n \in \mathbb{N}.$$

An operator $T$ is (*uniformly*) *mean ergodic* if $\{T_{[n]}\}_{n=0}^\infty$ is a convergent sequence in (norm) strong topology and is called *power bounded* if the sequence $\{T^n\}_{n=0}^\infty$ is bounded in $L(X)$. In this paper, we lookfor conditions under which the multiplication operator $M_\psi$ is power bounded and its Cesáro means is convergent or uniformly convergent on the Besov Space $\mathcal{B}_p$.

Bonet and Ricker [3], characterized the mean ergodicity of multiplication operators in weighted spaces of holomorphic functions and recently Bonet, Jordá and Rodrguez [2] extended the results to the weighted space of continuous functions. For more study on Ergodic Theory one can refer to [1, 6].

## 2. Main Results

Before starting this section, it is necessary to remind that a Banach space $X$ is said to be *mean ergodic* if each power bounded operator is mean ergodic. Lorch by extending the result of Rizes, showing that $L_p$ spaces are mean ergodic, proved that the reflexive spaces are also mean ergodic, see [1]. According to the introduction, for $1 < p < \infty$ Besov Spaces $\mathcal{B}_p$ are reflexive spaces and therefore power boundedness of an operator implies mean ergodicity. In this section we only consider the case $1 < p < \infty$.

THEOREM 2.1. *Suppose $\psi \in H(\mathbb{U})$ and $M_\psi$ is a bounded operator on Besov space $\mathcal{B}_p$. If $M_\psi$ is power bounded, mean ergodic or uniformly mean ergodic operator on $\mathcal{B}_p$, then $||\psi||_\infty \leq 1$.*

PROOF. First suppose $||\psi||_\infty > 1$. Then there is $\alpha > 0$ such that $||\psi||_\infty > \alpha > 1$. Since $||\psi||_\infty \leq ||M_\psi||$, for all $n \in \mathbb{N}$. we have $\alpha^n < ||\psi^n||_\infty \leq ||M_{\psi^n}||$ and therefore $M_\psi$ can not be power bounded on $\mathcal{B}_p$. Now suppose $M_\psi$ is uniformly mean ergodic (or mean ergodic) on $\mathcal{B}_p$, then for all $f \in \mathcal{B}_p$ we have $\lim_{n\to\infty} \frac{1}{n} f.\psi^n = 0$ when $n \to \infty$. Let $f \equiv 1$, then $||\frac{\psi^n}{n}||_p \to 0$. By Lemma 1.1, $|\frac{\psi^n(z)}{n}| \leq C||\frac{\psi^n}{n}||_p(\log \frac{2}{1-|z|^2})^{1-\frac{1}{p}}$ for all $z \in \mathbb{U}$ and some $C \geq 0$. So $|\frac{\psi^n(z)}{n}| \to 0$ when $n \to \infty$ for all $z \in \mathbb{U}$, it forces $|\psi(z)| \leq 1$ for all $z \in \mathbb{U}$ and finally $||\psi||_\infty \leq 1$. □

From now on, we assume that analytic function $\psi$ holds in the following condition:

$$(1) \qquad \int_{\mathbb{U}} (1-|z|^2)^{p-2}|\psi'(z)|^p(\log \frac{2}{1-|z|^2})^{p-1} dA(z) < \infty.$$

THEOREM 2.2. *Suppose that $\psi \in H(\mathbb{U})$ and condition (1) is met. If $M_\psi$ is a bounded operator on the Besov space $\mathcal{B}_p$, then the following statements are equivalent.*

i) $||\psi||_\infty \leq 1$.
ii) *$M_\psi$ is power bounded.*
iii) *$M_\psi$ is mean ergodic.*

PROOF. According to the initial interpretations of the section, it is sufficient to show that (i) and (ii) are equivalent. Let $||\psi||_\infty \leq 1$. If there exist $z \in \mathbb{U}$ such that $|\psi(z)| = 1$, then $\psi(z) = \lambda$, $|\lambda| = 1$ and $\psi' \equiv 0$. So for $f \in \mathcal{B}_p$ and $||f||_p = 1$ we have

$$||M_{\psi^n} f||_p = |\psi^n(0)f(0)| + \gamma_{\psi^n f} \leq |f(0)| + \gamma_f = ||f||_p,$$

and $M_\psi$ is power bounded on $\mathcal{B}_p$, in fact $||M_{\psi^n}|| \leq 1$, for all $n \in \mathbb{N}$. Now suppose $|\psi(z)| < 1$ for all $z \in \mathbb{U}$ and let $f \in \mathcal{B}_p$. In this case, the followings can be deduced;
1) $|\psi^n(0)f(0)| \to 0$, when $n \to \infty$, since $|\psi(0)| < 1$.
2) $\int_{\mathbb{U}} |f'(z)|^p|\psi^n(z)|^p(1-|z|^2)^{p-2}dA(z) \to 0$, as $n \to \infty$, since $|f'(z)\psi^n(z)|^p(1-|z|^2)^{p-2} \leq |f'(z)|^p(1-|z|^2)^{p-2}$, and $f \in \mathcal{B}_p$ gives us that $\int_{\mathbb{U}} |f'(z)|^p(1-|z|^2)^{p-2}dA(z) < \infty$, then $|f'(z)\psi^n(z)|^p(1-|z|^2)^{p-2}$ is integrable for all $n \in \mathbb{N}$. By using Lebesgue Convergence theorem the result is obtained.
3) $\int_{\mathbb{U}} |n\psi'(z)\psi^{n-1}(z)f(z)|^p(1-|z|^2)^{p-2}dA(z) \to 0$, since by Lemma 1.1

$$|n\psi'(z)\psi^{n-1}(z)f(z)|^p(1-|z|^2)^{p-2} \leq n^pC^p||f||_p^p(1-|z|^2)^{p-2}|\psi'(z)|^p(\log \frac{2}{1-|z|^2})^{p-1},$$

by hypothesis the right side of the last inequality is integrable for all $n \in \mathbb{N}$ and so is

$|n\psi'(z)\psi^{n-1}(z)f(z)|^p(1-|z|^2)^{p-2}$. Lebesgue Converges theorem gives the desired result.

Consequently for all $f \in \mathcal{B}_p$, $||M_{\psi^n} f||_p \to 0$ when $n \to \infty$. So $\{M_{\psi^n} f\}$ is bounded sequence for all $f \in \mathcal{B}_p$ and by Principle uniform boundedness $M_\psi$ is power bounded on $\mathcal{B}_p$. □

Note that by $\sigma(T)$ (spectrum of $T$) we mean the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not invertible.

LEMMA 2.3. *Suppose $\psi \in H(\mathbb{U})$ which satisfies condition (1) and $M_\psi$ is a bounded operator on the Besov space $\mathcal{B}_p$, then $\overline{\psi(\mathbb{U})} = \sigma(M_\psi)$, ($\overline{\psi(\mathbb{U})}$ means the norm closure of $\psi(\mathbb{U})$).*

PROOF. First since $M_\psi - \lambda I = M_{\psi-\lambda}$, then $\lambda \in \sigma(M_\psi)$ if and only if $M_{\psi-\lambda}$ is not invertible. If $M_{\psi-\lambda}$ is invertible, then $(M_{\psi-\lambda})^{-1} = M_{(\psi-\lambda)^{-1}} = M_{\frac{1}{\psi-\lambda}}$. So if $\lambda \in \psi(\mathbb{U})$ then there exists $z_0 \in \mathbb{U}$ such that $\psi(z_0) = \lambda$ therefore $\frac{1}{\psi-\lambda} \notin H^\infty(\mathbb{U})$ and $M_{\psi-\lambda}$ is not invertible that means $\lambda \in \sigma(M_\psi)$ and $\psi(\mathbb{U}) \subseteq \sigma(M_\psi)$. But $\sigma(M_\psi)$ is closed so $\overline{\psi(\mathbb{U})} \subseteq \sigma(M_\psi)$. Now assume that (1) holds and $\lambda \notin \overline{\psi(\mathbb{U})}$, hence $\frac{1}{\psi(z)-\lambda} \in H^\infty(\mathbb{U})$. By (1)

$$\int_{\mathbb{U}} \frac{|\psi'(z)|^p}{|\psi(z)-\lambda|^{2p}} \log \frac{2}{1-|z|^2})^{p-1}(1-|z|^2)^{p-2} dA(z) < \infty.$$

Thus by Proposition 1.2, $M_{\frac{1}{\psi-\lambda}}$ is bounded on $\mathcal{B}_p$ and $M_{\psi-\lambda}$ is invertible which means $\lambda \notin \sigma(M_\psi)$. □

The following theorem states the connection between the spectral properties of an operator and its uniform mean ergodicity. See [4, 7].

THEOREM 2.4. (Dunford-Lin) [4, 7] *An operator $T$ on a Banach space $X$ is uniformly mean ergodic if and only if both $\{||T^n||/n\}_n$ converges to $0$ and either $1 \in \mathbb{C} \setminus \sigma(T)$ or $1$ is a pole of order $1$ of the resolvent $R_T : \mathbb{C} \setminus \sigma(T) \to L(X), R_T(\lambda) = (T - \lambda I)^{-1}$. Consequently if $1$ is an accumulation of $\sigma(T)$, then $T$ is not uniformly mean ergodic.*

It's time to set up the final result:

THEOREM 2.5. *Suppose $\psi \in H(\mathbb{U})$ which holds (1) and $M_\psi$ is a bounded operator on the Besov space $\mathcal{B}_p$, then $M_\psi$ is uniformly mean ergodic on $\mathcal{B}_p$ if and only if $||\psi||_\infty \leq 1$ and either $\psi \equiv \xi$ for some $\xi \in \partial\mathbb{U}$ or $\frac{1}{1-\psi} \in H^\infty(\mathbb{U})$.*

PROOF. Let $||\psi||_\infty \leq 1$. Consider that $(M_\psi)_{[n]} f(z) = \frac{f(z)}{n} \sum_{m=1}^n (\psi(z))^n$. So if $\psi \equiv 1$, we can easily see that $||(M_\psi)_{[n]} - I|| \to 0$ when $n \to \infty$, where $I$ is the identity operator on $\mathcal{B}_p$. In the case $\psi \equiv \xi$, where $\xi \neq 1$, we have $(M_\psi)_{[n]} = \frac{\xi+\xi^2+\cdots+\xi^n}{n} f = \frac{f}{n} \frac{\xi(1-\xi^{n+1})}{1-\xi}$ and clearly $||(M_\psi)_{[n]}|| \to 0$. If $\frac{1}{1-\psi} \in H^\infty(\mathbb{U})$, an application of Proposition 1.2 shows that the function $\frac{1}{1-\psi} \in M(\mathcal{B}_p)$ and $M_{\frac{1}{1-\psi}}$ is bounded on $\mathcal{B}_p$, it means that $1 \notin \sigma(M_\psi)$ and since $M_\psi$ is power bounded, Dunford-Lin Theorem guaranties the uniform mean ergodicity of $M_\psi$ on $\mathcal{B}_p$.

Conversely, assume that $M_\psi$ is uniformly mean ergodic on $\mathcal{B}_p$. So by Theorems 2.1 and 2.2 it is power bounded and $||\psi||_\infty \leq 1$. suppose $\psi$ is not uni- modular constant function. By Dunford-Lin Theorem, $1 \notin \sigma(M_\psi)$ so $M_{1-\psi}$ is invertible and $\frac{1}{1-\psi} \in H^\infty(\mathbb{U})$. □

## References

1. A. A. Albanese, J. Bonet and W. Ricker, *On mean ergodic operators in Frèchet spaces*, Ann. Acad. Sci. Fenn. Math. **34** (2009) 401–436.
2. J. Bonet, E. Jordá and A. Rodríguez, *Mean ergodic multiplication operators on weighted space of continuous functions*, Mediterranean J. Math. **15** (3) (2018) 108.
3. J. Bonet and W. J. Ricker, *Mean ergodicity of multiplication operators in weighted spaces of holomorphic functions*, Arch. Math. **92** (2009) 428–437.
4. N. Dunford, *Spectral theory I. Convergence to projections*, Trans. Amer. Math. Soc. **54** (1943) 185–217.
5. P. Galanopoulos, D. Girela and J. Martín, *Besov spaces, multipliers and univalent functions*, Complex Anal. Oper. Theory **7** (2013) 1081–1116.
6. U. Krengel, *Ergodic Theorems*, De Gruyter Studies in Mathematics, Vol. 6, Walter de Gruyter & Co., Berlin, 1985.
7. M. Lin, *On the uniform ergodic theorem*, Proc. Amer. Math. Soc. **43** (1974) 337–340.
8. K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Graduate Texts in Mathematics, Vol. 226, Springer, New York, 2005.

E-mail: ffalahat@gmail.com
E-mail: zkamali@shirazu.ac.ir

# First Hachschild Cohomology Group of Triangular Banach Algebras on Induced Semigroup Algebras

Kianoush Kazemi*
Department of Mathematics, University of Birjand, Birjand, Iran
Mohammad Reza Miri
Department of Mathematics, University of Birjand, Birjand, Iran
and Ebrahim Nasrabadi
Department of Mathematics, University of Birjand, Birjand, Iran

Abstract. Let $S$ be a discrete semigroup with a left multiplier operator $T$ on $S$. A new product on $S$ defined by $T$ related to $S$ and $T$ creates a new induced semigroup $S_T$. Suppose that $T$ is bijective and

$$\mathcal{T}_1 = \begin{bmatrix} \ell^1(S) & \ell^1(S) \\ & \ell^1(S) \end{bmatrix} \quad \text{and} \quad \mathcal{T}_2 = \begin{bmatrix} \ell^1(S_T) & \ell^1(S_T) \\ & \ell^1(S_T) \end{bmatrix}.$$

In this paper, we show that the first cohomology groups $\mathcal{H}^1(\mathcal{T}_1, \mathcal{T}_1^*)$ and $\mathcal{H}^1(\mathcal{T}_2, \mathcal{T}_2^*)$ are equal. Therefore $\mathcal{T}_1$ is weakly amenable if and only if $\mathcal{T}_2$ is weakly amenable.
**Keywords:** Inducted semigroup, Triangular Banach algebra, Cohomology group, Weak ameanability.
**AMS Mathematical Subject Classification [2010]:** 46H25, 16E40.

## 1. Introduction

Let $X$ be a Banach $A$-bimodule, then so is dual space $X^*$, where the actions of $A$ on $X^*$ are defined by

$$(1) \qquad (a \cdot f)(x) = f(x \cdot a) \text{ and } (f \cdot a)(x) = f(a \cdot x) \quad (a \in A, x \in X, f \in X^*).$$

A bounded (continuous) map $D : A \longrightarrow X$ is called a derivation if

$$D(ab) = D(a) \cdot b + a \cdot D(b), \quad (a, b \in A),$$

each $x$ in $X$ defines a derivation $D(a) = \mathbf{ad}_x(a) = a \cdot x - x \cdot a \quad (a \in A)$. These are called inner derivations. If $X$ is a commutative Banach $A$-bimodule, then the inner derivations are zero.

Let $X$ be a Banach $A$-bimodule. We use the notation $\mathcal{Z}^1(A, X)$ for the set of all derivations $D : A \longrightarrow X$ and $\mathcal{B}^1_{\mathfrak{A}}(A, X)$, for those which are inner. The first cohomology group with coefficient in $X$ is denoted by $\mathcal{H}^1(A, X)$ which is the quotient $\mathcal{Z}^1(A, X)/\mathcal{B}^1(A, X)$.

Definition 1.1. The Banach algebra $A$ is called amenable, if for every Banach $A$-bimodule $X$, every derivation $D : A \longrightarrow X^*$ is inner. Indeed $\mathcal{H}^1(A, X^*) = 0$, for each Banach $A$-bimodule $X$. Also $A$ is called weak amenable, if every derivation $D : A \longrightarrow A^*$ is inner, or $\mathcal{H}^1(A, A^*) = 0$.

*Speaker

Let $A$ and $B$ be Banach algebras and $M$ be a Banach $A,B$-module (left $A$-module and right $B$-module) and let

$$\mathcal{T}=\mathrm{Tri}(A,B,M)=\left\{\begin{bmatrix} a & m \\ & b\end{bmatrix}; a\in A, b\in B, m\in M\right\},$$

be equipped with the usual $2\times 2$ matrix addition and formal multiplication and with the norm $\left\|\begin{bmatrix} a & m \\ & b\end{bmatrix}\right\| = \|a\|_{A}+\|b\|_{B}+\|m\|_{M}$. Then it is a Banach algebra. We call this algebra the triangular Banach algebra. Since, as a Banach space, $\mathcal{T}$ is isomorphic to the $\ell^1$-sum of $A$, $B$ and $M$, it is clear that $\mathcal{T}^*\simeq A^*\oplus_{\ell^\infty} B^*\oplus_{\ell^\infty} M^*=\begin{bmatrix} A^* & M^* \\ & B^*\end{bmatrix}$.

Suppose that $\begin{bmatrix} a & m \\ & b\end{bmatrix}\in\mathcal{T}$ and $\begin{bmatrix} \phi & \varphi \\ & \psi\end{bmatrix}\in\mathcal{T}^*$. Then the action of $\mathcal{T}^*$ upon $\mathcal{T}$ is given by:

$$\begin{bmatrix} \phi & \varphi \\ & \psi\end{bmatrix}\left(\begin{bmatrix} a & m \\ & b\end{bmatrix}\right)=\phi(a)+\varphi(m)+\psi(b).$$

Forrest and Marcoux in [1] studied derivations on triangular Banach algebras and in [2] showed that triangular Banach algebra $\mathcal{T}$ is weakly amenable if and only if Banach algebras $A$ and $B$ are weakly amenable.

## 2. Semigroup Algebra of Induced Semigroup

Let $S$ be a discrete semigroup. The map $T : S\longrightarrow S$ is called left multiplier on $S$ if $T(st)=T(s)t$ and right multiplier on $S$ if $T(st)=sT(t)$ for all $s,t\in S$. The class of left multiplier map on $S$ is denoted by $\mathrm{Mul}_l(S)$ and the class of right multiplier map on $S$ is denoted by $\mathrm{Mul}_r(S)$. An operator $T$ is multiplier map if $T\in\mathrm{Mul}_l(S)\cap\mathrm{Mul}_r(S)$. The space of all multiplier operators on $S$ is denoted by $\mathrm{Mul}(S)$. Let $T\in\mathrm{Mul}_l(S)$, we define a new operation "$\circ$" on $S$ as follow $s\circ t:=sT(t)$ for every $s$ and $t$ in $S$. The semigroup $S$ equips the new operation $\circ$, denoted by $S_T$. It's easy to check that $S_T$ is semigroup which called **induced semigroup** by left multiplier $T$.

Mohammadi and Laali proved in [3] the new semigroup $S_T$ have the same underlying set as $S$, and showed if $T$ is bijective then $\ell^1(S)$ is amenable if and only if $\ell^1(S_T)$ is amenable, moreever if $S$ completely regular, $\ell^1(S_T)$ is weakly amenable.

Throughout this paper, we will assume that $S$ is a discrete semigroup, $T\in\mathrm{Mul}(S)$ and $T$ is bijective. The Banach space $\ell^1(S)$ is the set of all complex functions $f : S\longrightarrow\mathbb{C}$ such $f(x)=0$ except at the most countable subset $A$ of $S$, so in general we can consider them as $f=\sum_{x\in S}f(x)\delta_x=\sum_{x\in A}f(x)\delta_x$ that $\delta_x$ is the point math function at point $x$ and $\|f\|_1=\sum_{x\in S}|f(x)|\leq\infty$.

Lemma 2.1. *Let $S$ be a semigroup, $T\in\mathrm{Mul}(S)$ and $T:S\to S$ be bijective, then*

i) $T\in\mathrm{Mul}_l(S)$ *if and only if* $T^{-1}\in\mathrm{Mul}_l(S)$.
ii) *If* $T\in\mathrm{Mul}(S)$, *then* $s\circ T(t)=T(s)\circ t$ *and* $s\circ T^{-1}(t)=T^{-1}(s)\circ t$ *for every* $s,t\in S$.

Proof. It is easy to prove with some calculations. $\square$

For $f, g \in \ell^1(S)$ the convolution product on $\ell^1(S)$ define as fallow

$$(f * g)(s) = \sum_{s=xy} f(x)g(y), \qquad (s \in S).$$

With this convolution product $(\ell^1(S), *)$ became a Banach algebra and is called the semigroup algebra on $S$. We know that the set of point masses $\{\delta_s; s \in S\}$ is dense in $\ell^1(S)$. So from, since module actions and derivations are continuous, we consider point masses as representing elements of semigroup algebras $\ell^1(S)$ and $\ell^1(S_T)$. Thus, semigroup algebra $\ell^1(S_T)$ is a Banach algebra with the different convolution ($\circledast$), as follow

(2) $$\delta_s \circledast \delta_t = \delta_{s \circ t} = \delta_s * \delta_{T(t)} = \delta_{sT(t)} = \delta_{T(s)t}, \qquad (s, t \in S).$$

The following assumes that $S$ is a discrete semigroup, $T \in \mathrm{Mul}(S)$. If

$$\mathcal{T}_1 = \begin{bmatrix} \ell^1(S) & \ell^1(S) \\ & \ell^1(S) \end{bmatrix} \qquad \text{and} \qquad \mathcal{T}_2 = \begin{bmatrix} \ell^1(S_T) & \ell^1(S_T) \\ & \ell^1(S_T) \end{bmatrix},$$

then we show that $\mathcal{H}^1(\mathcal{T}_1, \mathcal{T}_1^*) \simeq \mathcal{H}^1(\mathcal{T}_2, \mathcal{T}_2^*)$, where $T$ is bijective.

## 3. Main Results

LEMMA 3.1. *Let $S$, $S_T$ and $T$ be as above. Then $D : \mathcal{T}_1 \to \mathcal{T}_1^*$ is derivation if and only if $\widetilde{D} : \mathcal{T}_2 \to \mathcal{T}_2^*$ defined as $\widetilde{D}\left(\begin{bmatrix} \delta_x & \delta_y \\ & \delta_z \end{bmatrix}\right) = D\left(\begin{bmatrix} \delta_{T(x)} & \delta_{T(y)} \\ & \delta_{T(z)} \end{bmatrix}\right)$ is derivation. Furthermore, $D$ is inner if and only if $\widetilde{D}$ is inner.*

PROOF. In the first, let $D$ be derivation. Clearly $\widetilde{D}$ is linear. Let $r, s, t \in S_T$, with the help of Lemma 2.1 and by (1) and (2), for $\mathbf{t}_i = \begin{bmatrix} \delta_{x_i} & \delta_{y_i} \\ & \delta_{z_i} \end{bmatrix}$ $(i \in \{1, 2, 3\})$, we have

$$\begin{aligned}
&[\widetilde{D}(\mathbf{t}_1) \cdot \mathbf{t}_2 + \mathbf{t}_1 \cdot \widetilde{D}(\mathbf{t}_2)]\,(\mathbf{t}_3) = [\widetilde{D}(\mathbf{t}_1)]\,(\mathbf{t}_2 \cdot \mathbf{t}_3) + [\widetilde{D}(\mathbf{t}_2)]\,(\mathbf{t}_3 \cdot \mathbf{t}_1) \\
&= \widetilde{D}\left(\begin{bmatrix} \delta_{x_1} & \delta_{y_1} \\ & \delta_{z_1} \end{bmatrix}\right)\left(\begin{bmatrix} \delta_{x_2} & \delta_{y_2} \\ & \delta_{z_2} \end{bmatrix}\begin{bmatrix} \delta_{x_3} & \delta_{y_3} \\ & \delta_{z_3} \end{bmatrix}\right) \\
&+ \widetilde{D}\left(\begin{bmatrix} \delta_{x_2} & \delta_{y_2} \\ & \delta_{z_2} \end{bmatrix}\right)\left(\begin{bmatrix} \delta_{x_3} & \delta_{y_3} \\ & \delta_{z_3} \end{bmatrix}\begin{bmatrix} \delta_{x_1} & \delta_{y_1} \\ & \delta_{z_1} \end{bmatrix}\right) \\
&= D\left(\begin{bmatrix} \delta_{T(x_1)} & \delta_{T(y_1)} \\ & \delta_{T(z_1)} \end{bmatrix}\right)\left(\begin{bmatrix} \delta_{x_2 \circ x_3} & \delta_{x_2 \circ y_3} + \delta_{y_2 \circ z_3} \\ & \delta_{z_2 \circ z_3} \end{bmatrix}\right) \\
&+ D\left(\begin{bmatrix} \delta_{T(x_2)} & \delta_{T(y_2)} \\ & \delta_{T(z_2)} \end{bmatrix}\right)\left(\begin{bmatrix} \delta_{x_3 \circ x_1} & \delta_{x_3 \circ y_1} + \delta_{y_3 \circ z_1} \\ & \delta_{z_3 \circ z_1} \end{bmatrix}\right) \\
&= D\left(\begin{bmatrix} \delta_{T(x_1)} & \delta_{T(y_1)} \\ & \delta_{T(z_1)} \end{bmatrix}\right)\left(\begin{bmatrix} \delta_{T(x_2)x_3} & \delta_{T(x_2)y_3} + \delta_{T(y_2)z_3} \\ & \delta_{T(z_2)z_3} \end{bmatrix}\right) \\
&+ D\left(\begin{bmatrix} \delta_{T(x_2)} & \delta_{T(y_2)} \\ & \delta_{T(z_2)} \end{bmatrix}\right)\left(\begin{bmatrix} \delta_{x_3T(x_1)} & \delta_{x_3T(y_1)} + \delta_{y_3T(z_1)} \\ & \delta_{z_3T(z_1)} \end{bmatrix}\right) \\
&= D\left(\begin{bmatrix} \delta_{T(x_1)} & \delta_{T(y_1)} \\ & \delta_{T(z_1)} \end{bmatrix}\right)\left(\begin{bmatrix} \delta_{T(x_2)} & \delta_{T(y_2)} \\ & \delta_{T(z_2)} \end{bmatrix}\begin{bmatrix} \delta_{x_3} & \delta_{y_3} \\ & \delta_{z_3} \end{bmatrix}\right)
\end{aligned}$$

$$
\begin{aligned}
&+ D\left(\begin{bmatrix}\delta_{T(x_2)} & \delta_{T(y_2)}\\ & \delta_{T(z_2)}\end{bmatrix}\right)\left(\begin{bmatrix}\delta_{x_3} & \delta_{y_3}\\ & \delta_{z_3}\end{bmatrix}\begin{bmatrix}\delta_{T(x_1)} & \delta_{T(y_1)}\\ & \delta_{T(z_1)}\end{bmatrix}\right)\\
&= [D\left(\begin{bmatrix}\delta_{T(x_1)} & \delta_{T(y_1)}\\ & \delta_{T(z_1)}\end{bmatrix}\right)\cdot\begin{bmatrix}\delta_{T(x_2)} & \delta_{T(y_2)}\\ & \delta_{T(z_2)}\end{bmatrix}]\left(\begin{bmatrix}\delta_{x_3} & \delta_{y_3}\\ & \delta_{z_3}\end{bmatrix}\right)\\
&+ [\begin{bmatrix}\delta_{T(x_1)} & \delta_{T(y_1)}\\ & \delta_{T(z_1)}\end{bmatrix}\cdot D\left(\begin{bmatrix}\delta_{T(x_2)} & \delta_{T(y_2)}\\ & \delta_{T(z_2)}\end{bmatrix}\right)]\left(\begin{bmatrix}\delta_{x_3} & \delta_{y_3}\\ & \delta_{z_3}\end{bmatrix}\right)\\
&= D\left(\begin{bmatrix}\delta_{T(x_1)} & \delta_{T(y_1)}\\ & \delta_{T(z_1)}\end{bmatrix}\begin{bmatrix}\delta_{T(x_2)} & \delta_{T(y_2)}\\ & \delta_{T(z_2)}\end{bmatrix}\right)\left(\begin{bmatrix}\delta_{x_3} & \delta_{y_3}\\ & \delta_{z_3}\end{bmatrix}\right)\\
&= D\left(\begin{bmatrix}\delta_{T(x_1)T(x_2)} & \delta_{T(x_1)T(y_2)}+\delta_{T(y_1)T(z_2)}\\ & \delta_{T(z_1)T(z_2)}\end{bmatrix}\right)\left(\begin{bmatrix}\delta_{x_3} & \delta_{y_3}\\ & \delta_{z_3}\end{bmatrix}\right)\\
&= D\left(\begin{bmatrix}\delta_{T(x_1T(x_2))} & \delta_{T(x_1T(y_2))}+\delta_{T(y_1T(z_2))}\\ & \delta_{T(z_1T(z_2))}\end{bmatrix}\right)\left(\begin{bmatrix}\delta_{x_3} & \delta_{y_3}\\ & \delta_{z_3}\end{bmatrix}\right)\\
&= \widetilde{D}\left(\begin{bmatrix}\delta_{x_1T(x_2)} & \delta_{x_1T(y_2)}+\delta_{y_1T(z_2)}\\ & \delta_{z_1T(z_2)}\end{bmatrix}\right)\left(\begin{bmatrix}\delta_{x_3} & \delta_{y_3}\\ & \delta_{z_3}\end{bmatrix}\right)\\
&= \widetilde{D}\left(\begin{bmatrix}\delta_{x_1} & \delta_{y_1}\\ & \delta_{z_1}\end{bmatrix}\cdot\begin{bmatrix}\delta_{x_2} & \delta_{y_2}\\ & \delta_{z_2}\end{bmatrix}\right)\left(\begin{bmatrix}\delta_{x_3} & \delta_{y_3}\\ & \delta_{z_3}\end{bmatrix}\right) = \widetilde{D}\left(\mathbf{t}_1\cdot\mathbf{t}_2\right)\left(\mathbf{t}_3\right).
\end{aligned}
$$

This shows that $\widetilde{D}(\mathbf{t}_1)\cdot\mathbf{t}_2+\mathbf{t}_1\cdot\widetilde{D}(\mathbf{t}_2)=\widetilde{D}\left(\mathbf{t}_1\cdot\mathbf{t}_2\right)$ and so $\widetilde{D}$ is derivation. It is similarly proved $D$ is derivation when $\widetilde{D}$ is derivation. Now we assume $D$ is inner, so exists $\begin{bmatrix}\phi & \varphi\\ & \psi\end{bmatrix}\in\mathcal{T}^*$ such that

$$
D\left(\begin{bmatrix}\delta_x & \delta_y\\ & \delta_z\end{bmatrix}\right)=\begin{bmatrix}\delta_x & \delta_y\\ & \delta_z\end{bmatrix}\cdot\begin{bmatrix}\phi & \varphi\\ & \psi\end{bmatrix}-\begin{bmatrix}\phi & \varphi\\ & \psi\end{bmatrix}\cdot\begin{bmatrix}\delta_x & \delta_y\\ & \delta_z\end{bmatrix}\qquad\left(\begin{bmatrix}\delta_x & \delta_y\\ & \delta_z\end{bmatrix}\in\mathcal{T}_1\right).
$$

Let $\begin{bmatrix}\delta_x & \delta_y\\ & \delta_z\end{bmatrix},\begin{bmatrix}\delta_r & \delta_s\\ & \delta_t\end{bmatrix}\in\mathcal{T}_2$, by (1), (2) and Lemma 2.1, we have

$$
\begin{aligned}
&\widetilde{D}\left(\begin{bmatrix}\delta_x & \delta_y\\ & \delta_z\end{bmatrix}\right)\left(\begin{bmatrix}\delta_r & \delta_s\\ & \delta_t\end{bmatrix}\right)=D\left(\begin{bmatrix}\delta_{T(x)} & \delta_{T(y)}\\ & \delta_{T(z)}\end{bmatrix}\right)\left(\begin{bmatrix}\delta_r & \delta_s\\ & \delta_t\end{bmatrix}\right)\\
&=\left(\begin{bmatrix}\delta_{T(x)} & \delta_{T(y)}\\ & \delta_{T(z)}\end{bmatrix}\cdot\begin{bmatrix}\phi & \varphi\\ & \psi\end{bmatrix}-\begin{bmatrix}\phi & \varphi\\ & \psi\end{bmatrix}\cdot\begin{bmatrix}\delta_{T(x)} & \delta_{T(y)}\\ & \delta_{T(z)}\end{bmatrix}\right)\left(\begin{bmatrix}\delta_r & \delta_s\\ & \delta_t\end{bmatrix}\right)\\
&=\begin{bmatrix}\phi & \varphi\\ & \psi\end{bmatrix}\left(\begin{bmatrix}\delta_r & \delta_s\\ & \delta_t\end{bmatrix}\cdot\begin{bmatrix}\delta_{T(x)} & \delta_{T(y)}\\ & \delta_{T(z)}\end{bmatrix}\right)-\begin{bmatrix}\phi & \varphi\\ & \psi\end{bmatrix}\left(\begin{bmatrix}\delta_{T(x)} & \delta_{T(y)}\\ & \delta_{T(z)}\end{bmatrix}\cdot\begin{bmatrix}\delta_r & \delta_s\\ & \delta_t\end{bmatrix}\right)\\
&=\begin{bmatrix}\phi & \varphi\\ & \psi\end{bmatrix}\left(\begin{bmatrix}\delta_r\cdot\delta_{T(x)} & \delta_r\cdot\delta_{T(y)}+\delta_s\cdot\delta_{T(z)}\\ & \delta_t\cdot\delta_{T(z)}\end{bmatrix}\right)\\
&-\begin{bmatrix}\phi & \varphi\\ & \psi\end{bmatrix}\left(\begin{bmatrix}\delta_{T(x)}\cdot\delta_r & \delta_{T(x)}\cdot\delta_s+\delta_{T(y)}\cdot\delta_t\\ & \delta_{T(z)}\cdot\delta_t\end{bmatrix}\right)
\end{aligned}
$$

$$= \begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} \left( \begin{bmatrix} \delta_{rox} & \delta_{roy} + \delta_{soz} \\ & \delta_{toz} \end{bmatrix} \right) - \begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} \left( \begin{bmatrix} \delta_{xor} & \delta_{xos} + \delta_{yot} \\ & \delta_{zot} \end{bmatrix} \right)$$
$$= \begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} \left( \begin{bmatrix} \delta_r \cdot \delta_x & \delta_r \cdot \delta_y + \delta_s \cdot \delta_z \\ & \delta_t \cdot \delta_z \end{bmatrix} \right) - \begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} \left( \begin{bmatrix} \delta_x \cdot \delta_r & \delta_x \cdot \delta_s + \delta_y \cdot \delta_t \\ & \delta_z \cdot \delta_t \end{bmatrix} \right)$$
$$= \left( \begin{bmatrix} \delta_x & \delta_y \\ & \delta_z \end{bmatrix} \cdot \begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} - \begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} \cdot \begin{bmatrix} \delta_x & \delta_y \\ & \delta_z \end{bmatrix} \right) \left( \begin{bmatrix} \delta_r & \delta_s \\ & \delta_t \end{bmatrix} \right).$$

So $\widetilde{D}$ is inner. With the same way, we can show that $D$ is inner if $\widetilde{D}$ is inner. □

THEOREM 3.2. *Let $S$ be a discrete semigroup and $T$ is a bijective left multiplier operator on $S$. Then $\mathcal{H}^1(\mathcal{T}_1, \mathcal{T}_1^*) \simeq \mathcal{H}^1(\mathcal{T}_2, \mathcal{T}_2^*)$.*

PROOF. Consider the map $\Gamma : \mathcal{Z}^1(\mathcal{T}_1, \mathcal{T}_1^*) \longrightarrow \mathcal{H}^1(\mathcal{T}_2, \mathcal{T}_2^*)$ defined by $\Gamma(D) = \widetilde{D} + \mathcal{B}^1(\mathcal{T}_2, \mathcal{T}_2^*)$, Where $\widetilde{D} : \mathcal{T}_2 \longrightarrow \mathcal{T}_2^*$ defined by

$$\widetilde{D}\left( \begin{bmatrix} \delta_x & \delta_y \\ & \delta_z \end{bmatrix} \right) := D\left( \begin{bmatrix} \delta_{T(x)} & \delta_{T(y)} \\ & \delta_{T(z)} \end{bmatrix} \right).$$

Clearly $\Gamma$ is linear and Lemma 3.1 shows that $\Gamma$ is well-define. For surjectivity $\Gamma$, let $P \in \mathcal{Z}^1(\mathcal{T}_2, \mathcal{T}_2^*)$ and $D : \mathcal{T}_1 \longrightarrow \mathcal{T}_1^*$ defined by

$$D\left( \begin{bmatrix} \delta_x & \delta_y \\ & \delta_z \end{bmatrix} \right) := P\left( \begin{bmatrix} \delta_{T^{-1}(x)} & \delta_{T^{-1}(y)} \\ & \delta_{T^{-1}(z)} \end{bmatrix} \right).$$

Clearly $\Gamma(D) = \widetilde{D} = P$. But $D \in \mathcal{Z}^1(\mathcal{T}_1, \mathcal{T}_1^*)$ by Lemma 3.1. That shows, $\Gamma$ is surjective. On the other hand, Lemma 3.1, also shows that $\ker \Gamma = \mathcal{B}^1(\mathcal{T}_1, \mathcal{T}_1^*)$. But

$$\mathcal{H}^1(\mathcal{T}_1, \mathcal{T}_1^*) = \frac{\mathcal{Z}^1(\mathcal{T}, \mathcal{T}_1^*)}{\mathcal{B}^1(\mathcal{T}_1, \mathcal{T}_1^*)} = \frac{\mathcal{Z}^1(\mathcal{T}_1, \mathcal{T}_1^*)}{\ker \Gamma} \simeq \operatorname{Im} \Gamma = \mathcal{H}^1(\mathcal{T}_2, \mathcal{T}_2^*).$$

□

COROLLARY 3.3. *Let $S$, $T$, $S_T, \mathcal{T}_1$ and $\mathcal{T}_2$ be as above. Then $\mathcal{T}_1$ is weakly amenable if and only if $\mathcal{T}_2$ is weakly amenable.*

## References

1. B. E. Forrest and L. W. Marcoux, *Derivations of triangular Banach algebras*, Indiana Univ. Math. J. **45** (2) (1996) 441–462.
2. B. E. Forrest and L. W. Marcoux, *Weak amenability of triangular Banach algebras*, Trans. Amer. Math. Soc. **354** (4) (2002) 1435–1452.
3. S. M. Mohammadi and J. Laali, *Amenability and weak amenability of semigroup algebra* $\ell^1(S_T)$, Math. Reserches **2** (1) (2016) 33–46.

E-mail: kianoush.kazemi@birjand.ac.ir
E-mail: mrmiri@birjand.ac.ir
E-mail: nasrabadi@birjand.ac.ir

# Schweitzer Integral Inequality for Fuzzy Integrals

Bayaz Daraby
Department of Mathematics, University of Maragheh, Maragheh, Iran
Maryam Keyvani
Department of Mathematics, University of Maragheh, Maragheh, Iran
and Morteza Tahmoresi*
Department of Mathematics, University of Maragheh, Maragheh, Iran

Abstract. Fuzzy integrals are well known aggregation operators. They can be used integrant variety of decision making applications. In this paper, we want to extend the Schweitzer integral inequality for fuzzy case. More precisely, we show that:

$$i)\ \int_{[0,a]}^{\oplus} f dx \oplus \int_{[0,b]}^{\oplus} f^{-1} dx \leq a+b,$$

$$ii)\ 0<m\leq f\leq M \Rightarrow \fint_a^b f d\mu \fint \frac{1}{f} d\mu \leq \frac{(M+n)^2}{4Mm}(b-a)^2,$$

$$\int_{[a,b]}^{\oplus} f dx \odot \int_{[a,b]}^{\oplus} \frac{1}{f} dx \leq \frac{(M+m)^2}{4Mm}(b-a)^2.$$

**Keywords:** Fuzzy integral, Fuzzy measure, Fuzzy integral inequality, Pseudo integral, Pseudo integral inequality.
**AMS Mathematical Subject Classification [2010]:** 03E72, 26E50, 28E10.

## 1. Introduction

In mathematics, fuzzy measure theory considers generalized measures in which the additive properties replaced by the weaker of monotonicity. Sugeno integral is applied in many fields such as management decision-making, medical decision-making, control engineering.

One application of fuzzy integral is to solve the multi-criteria decision question. To solve multi-criteria decision equations the most important part is finding the best integration function so that the whole set of decision-making preference can be applied to the equation.

Now, we will provide some definitions and concept for using in the next section. Throughout this paper, we introduce and prove the fuzzy state of following theorem which is established in the classical state. We let $X$ be a non-empty set and $\Sigma$ be a $\sigma$-algebra of subset of $X$.

In the following, we will express the classic state of inequality.

*Speaker

THEOREM 1.1. [1] (Integral Analogues (Schweitzer)) *If* $f, \frac{1}{f} \in L([a,b])$ *with* $0 < m \leq f \leq M$*, then*

(1) $$\int_a^b f dx \int_a^b \frac{1}{f} dx \leq \frac{(M+m)^2}{4Mm}(b-a)^2.$$

DEFINITION 1.2. [5] A set function $\mu : \Sigma \to [0, +\infty]$ is called a fuzzy measure if the following properties are satisfied:

(1) $\mu(\emptyset) = 0$;
(2) $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$ (monotonicity);
(3) $A_1 \subseteq A_2 \subseteq \cdots \Rightarrow \lim_{i\to\infty} \mu(A_i) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$ (continuity from below);
(4) $A_1 \supseteq A_2 \supseteq \cdots$ and $\mu(A_1) < \infty \Rightarrow \lim_{i\to\infty} \mu(A_i) = \mu\left(\bigcap_{i=1}^{\infty} A_i\right)$ (continuity from above).

When $\mu$ is a fuzzy measure, the triple $(X, \Sigma, \mu)$ is called a fuzzy measure space.

DEFINITION 1.3. [2, 3] Let $\mu$ be a fuzzy measure on $(X, \Sigma)$. If $f \in F^{\mu}(X)$ and $A \in \Sigma$, then the Sugeno integral of $f$ on $A$ is defined by

$$⨍_A f d\mu = \bigvee_{\alpha \geq 0} \left(\alpha \wedge \mu(A \cap F_\alpha)\right),$$

where $\vee$ and $\wedge$ denotes the operations *sup* and *inf* on $[0, \infty]$, respectively and $\mu$ is the Lebesgue measure. If $A = X$, the fuzzy integral may also be denoted by $⨍ f d\mu$.

REMARK 1.4. [4] Consider the distribution function $F$ associated to $f$ on $A$, that is to say,

$$F(\alpha) = \mu(A \cap \{f \geq \alpha\}).$$

Then

$$F(\alpha) = \alpha \Rightarrow ⨍_A f d\mu = \alpha.$$

Thus, from a numerical (or computational) point of view, the Sugeno integral can be calculated by solving the equation $F(\alpha) = \alpha$ (if the solution exists).

## 2. Main Result

THEOREM 2.1. (Integral analogs Schweitzer) *If* $f, \frac{1}{f} \in \mathfrak{F}^{\mu}(X)$ *with* $0 < m \leq f \leq M$*. Then*

$$⨍_a^b f d\mu ⨍_a^b \frac{1}{f} d\mu \leq \frac{(M+m)^2}{4Mm}(b-a)^2.$$

PROOF. The proof relieased in two cases.

Case 1. If $\mu$ is a Lebesgue measure, in this case inequality is hold. Because we have

$$\begin{matrix} ⨍_a^b f d\mu \leq b-a \\ ⨍_a^b f d\mu \leq b-a \end{matrix} \Rightarrow ⨍_a^b f d\mu \cdot ⨍_a^b \frac{1}{f} d\mu \leq (b-a)^2 \leq \frac{(M+m)^2}{4Mm}(b-a)^2.$$

Case 2. If $\mu$ is a orbitrary fuzzy measure, we assume $s = ⨍_A f d\mu$ and $t = ⨍_A \frac{1}{f} d\mu$. Then we have

i) if $st = 0$ so $\frac{(M+m)^2}{4Mm}a^2 > 0$ and the proof is hold,
ii) if $0 \le s, t \le 1$ since the right hand of ineqaulity is non0negative so the ineqaulity istablish i.e.

$$\fint_0^a f d\mu = \sup\left[\alpha \wedge \mu(A \cap F_\alpha)\right],$$
$$F_\alpha = \{x|\ f(x) \ge \alpha\}, \qquad A(\alpha) = \mu\left([0,a] \cap F_\alpha\right),$$
$$G_\alpha = \left\{x|\ \frac{1}{f(x)} \ge \alpha\right\}, \qquad B(\alpha) = \mu\left([0,a] \cap G_\alpha\right).$$

Since we show by translate properties that $m$ is Lebesgue measure, the below inequality is establish:

$$\fint_0^1 A(\alpha)dm \cdot \fint_0^1 B(\alpha)dm \le \frac{(M+m)}{4Mm}(1-0)^2.$$

We have

$$\begin{matrix} \fint_0^1 A(\alpha)dm \le m(A) = 1 \\ \fint_0^1 B(\alpha)dm \le m(A) = 1 \end{matrix} \Rightarrow \fint_0^1 A(\alpha)dm \cdot \fint_0^1 B(\alpha)dm \le 1.$$

□

Example 2.2. If $A = [0,1]$ and $\mu$ is Lebesgue measure. Let $f, \frac{1}{f} \in \mathfrak{F}^\mu(X)$ and $0 < n \le f \le M$. Thus by stright calculus we have:

$$\begin{matrix} \fint_0^1 f d\mu \le \mu(A) = 1 \\ \fint_0^1 \frac{1}{f} d\mu \le \mu(A) = 1 \end{matrix} \Rightarrow \fint_0^1 f d\mu \cdot \fint_0^1 \frac{1}{f} d\mu \le 1,$$

and

$$\fint_0^1 f d\mu \cdot \fint_0^1 \frac{1}{f} d\mu \le 1 \le \frac{(M+m)^2}{4Mm}.$$

Example 2.3. Suppose $f$ is a increasing function. Then $\frac{1}{f}$ is a decreasing function. We assume $a^2 > 4m$ or $\frac{a}{2} > \sqrt{m}$ then we have

$$0 < m \le f \le M \Rightarrow \frac{1}{M} < \frac{1}{f} < \frac{1}{m},$$
$$\fint_0^a f d\mu = p \le f(a-p) \le M,$$
$$\fint_0^a \frac{1}{f} d\mu = q \le \left(\frac{1}{f}\right)(q) \le \frac{1}{m},$$
$$\Rightarrow p \cdot q \le \frac{M}{m} \le \frac{(M+m)^2}{m} \overset{\frac{a}{2} > \sqrt{m}}{\longrightarrow} \le \frac{(M+m)^2}{4mM}(a-0)^2.$$

In the second above relation we use the main theorem of [3].

Proposition 2.4. *Let $f : [a,b] \to [c,d]$ be continues function and $g : [c,d] \to [0,\infty)$ be a continues increasing generator function. Then we have*

$$\int_{[a,b]}^{\oplus} f dx \odot \int_{[a,b]}^{\oplus} \le \frac{(M+m)^2}{4Mm} \odot \mu^2(A),$$

*that $0 < m \le f \le M$ and is $A = [a,b]$.*

Proof.

$$\left.\begin{array}{l} ⨍_A f d\mu \leq \mu(A) \\ ⨍_A \frac{1}{f} d\mu \leq \mu(A) \end{array}\right. \Rightarrow ⨍_A \cdot ⨍_A \frac{1}{f} d\mu \leq \mu(A)\cdot\mu(A) = \mu^2(A) \leq \frac{(M+m)^2}{4Mm}\mu^2(A).$$

□

Proposition 2.5. *Let $f:[a,b]\to[c,d]$ be a continues function and $g:[c,d]\to[0,\infty)$ function be a increasing continues generator function. Then we have*

$$\int_{[0,a]}^{\oplus} f dx \oplus \int_{[0,b]}^{\oplus} \frac{1}{f} dx \leq M(a+b).$$

Proof.

$$\int_{[a,b]}^{\oplus} f dx \odot \int_{[a,b]}^{\oplus} \frac{1}{f} dx = g^{-1}\int_a^b g(f)dx \odot g^{-1}\int_a^b g\left(\frac{1}{f}\right)dx$$

$$\Rightarrow g^{-1}\left(\int_a^b g(f)dx \int_a^b g\left(\frac{1}{f}\right)dx\right) \leq g^{-1}\left(g\left(\frac{(M+m)^2}{4Mm}(b-a)^2\right)\right)$$

$$\Rightarrow \int_{[a,b]}^{\oplus} f dx \odot \int_{[a,b]}^{\oplus} \frac{1}{f} dx \leq \frac{(M+m)^2}{4Mn} \odot \mu^2(A).$$

□

## References


1. P. S. Bullen, *A Dictionary of Inequalities*, Pitman Monographs and Surveys in Pure and Applied. Mathematics 97, Addison Wesley Longman Limited, USA, 1998.
2. B. Daraby, *Generalization of the Stolarsky type inequality for pseudo-integrals*, Fuzzy Sets Syst. **194** (2012) 90–96.
3. H. Román-Flores, A. Flores-Franulič and Y. Chalco-Cano, *The fuzzy integral for monotone functions*, Appl. Math. Comput. **185** (2007) 492–498.
4. M. Sugeno, *Theory of Fuzzy Integrals and its Applications*, Ph.D. Thesis, Tokyo Institute of Technology, 1974.
5. Z. Wang and G. J. Klir, *Fuzzy Measure Theory*, Springer US, New York, 1992.



E-mail: bdaraby@maragheh.ac.ir
E-mail: mortazatahmoras@gmail.com
E-mail: mraghymrym@gmail.com

# Reverse Order Law for Moore-Penrose Inverses of Operators with Acting Involution

Mahdi Mohammadzadeh Karizaki*
Department of Computer Engineering, University of Torbat Heydarieh, Torbat Heydarieh, Iran
Javad Farokhi-Ostad
Department of Basic Sciences, Birjand University of Technology, Birjand, Iran
and Mahdi Ali-Akbari
Department of Computer Engineering, University of Torbat Heydarieh Torbat Heydarieh, Iran

Abstract. We study some relations on operators in Hilbert $C^*$-module setting. New condition are represented which allows to obtain many results for Moore-Penrose operators. Also, we show star can play the role of the Moore-Penrose inverse in the reverse order law.
**Keywords:** Closed range, Moore-Penrose inverse, Star partial ordering, Hilbert $C^*$-module.
**AMS Mathematical Subject Classification [2010]:** 47A62, 15A24, 46L08.

## 1. Introduction

Let $M_{m,n}(\mathbb{C})$ be the algebra of all $m \times n$ matrices, and let $B(\mathcal{H})$ be the algebra of all bounded linear operators on infinite-dimensional complex Hilbert space $\mathcal{H}$. On $M_{m,n}(\mathbb{C})$ a lot of partial orders and their properties, which can not be fully generalized to $B(H)$, were studied. One of such orders is the star partial order, which was defined by Drazin [2] as complex matrices, and Dolinar [1] state the equivalent definition of the star partial order on $B(\mathcal{H})$, by using orthogonal projections.

Drazin [2] introduced two binary relations in the set of complex matrices by combining each of the conditions

$$T^*T = T^*S \qquad \text{and} \qquad TT^* = ST^*, \tag{1}$$

and

$$T^\dagger T = T^\dagger S = S^\dagger T \qquad \text{and} \qquad TT^\dagger = TS^\dagger = ST^\dagger,$$

which (1) defines the star partial ordering that is due to Drazin [2]. Hartwig [3] inspired from Drazin [2] and introduced the plus partial order (or minus partial order).

We study some relations on operators in Hilbert $C^*$-module setting. New condition are represented which allows to obtain many results for operators. Also, we show star can play the role of the Moore-Penrose inverse in the reverse order law.

*Speaker

THEOREM 1.1. [4] *Suppose that* $\mathcal{X}$ *and* $\mathcal{Y}$ *are Hilbert* $\mathcal{A}$*-modules and* $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *has closed range. Then*

i) $\ker(T)$ *is orthogonally complemented in* $\mathcal{X}$*, with complement* $\mathrm{ran}(T^*)$.
ii) $\mathrm{ran}(T)$ *is orthogonally complemented in* $\mathcal{Y}$*, with complement* $\ker(T^*)$.
iii) *The map* $T^* \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ *has closed range.*

Xu and Sheng [6] showed that an adjointable operator between two Hilbert $\mathcal{A}$-modules admits a bounded Moore-Penrose inverse if and only if it has closed range. The Moore-Penrose inverse $T^\dagger$ of $T$ is the unique element in $\mathcal{L}(\mathcal{Y}, \mathcal{X})$ which satisfies the following conditions:

$$TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger, \quad (T^\dagger T)^* = T^\dagger T.$$

From these conditions we obtain that $(T^\dagger)^* = (T^*)^\dagger$, $TT^\dagger$ and $T^\dagger T$ are orthogonal projections, in the sense that they are self-adjoint idempotent operators. Furthermore, we have

$$\mathrm{ran}(T) = \mathrm{ran}(TT^\dagger), \qquad \mathrm{ran}(T^\dagger) = \mathrm{ran}(T^\dagger T) = \mathrm{ran}(T^*),$$

$$\ker(T) = \ker(T^\dagger T), \qquad \ker(T^\dagger) = \ker(TT^\dagger) = \ker(T^*).$$

It is well known, that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is regular if there exists $S \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ such that $TST = T$. Also if $T$ is regular, then $T^\dagger$ exists.

A matrix form of a bounded adjointable operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ can be induced by some natural decompositions of Hilbert $C^*$-modules. Indeed, if $\mathcal{M}$ and $\mathcal{N}$ are closed orthogonally complemented submodules of $\mathcal{X}$ and $\mathcal{Y}$, respectively, and $\mathcal{X} = \mathcal{M} \oplus \mathcal{M}^\perp$, $\mathcal{Y} = \mathcal{N} \oplus \mathcal{N}^\perp$, then $T$ can be written as the following $2 \times 2$ matrix

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix},$$

where, $T_1 = P_\mathcal{N} T P_\mathcal{M} \in \mathcal{L}(\mathcal{M}, \mathcal{M})$, $T_2 = P_\mathcal{N} T(1 - P_\mathcal{M}) \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{M})$, $T_3 = (1 - P_\mathcal{N}) T P_\mathcal{M} \in \mathcal{L}(\mathcal{M}, \mathcal{N}^\perp)$ and $T_4 = (1 - P_\mathcal{N}) T (1 - P_\mathcal{M}) \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N}^\perp)$ and $P_\mathcal{M}$ and $P_\mathcal{N}$ denote the projections corresponding to $\mathcal{M}$ and $\mathcal{N}$, respectively.
The following lemmata can be found or obtained in [5].

LEMMA 1.2. *Suppose that* $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *has closed range. Let* $\mathcal{X}_1$*,* $\mathcal{X}_2$ *be closed submodules of* $\mathcal{X}$ *and* $\mathcal{Y}_1$*,* $\mathcal{Y}_2$ *be closed submodules of* $\mathcal{Y}$ *such that* $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ *and* $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$*. Then the operator* $T$ *has the following matrix representations with respect to the orthogonal sums of submodules* $\mathcal{X} = \mathrm{ran}(T^*) \oplus \ker(T)$ *and* $\mathcal{Y} = \mathrm{ran}(T) \oplus \ker(T^*)$ :

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(T) \\ \ker(T^*) \end{bmatrix}.$$

*Then* $E = T_1 T_1^* + T_2 T_2^* \in \mathcal{L}(\mathrm{ran}(T))$ *is positive and invertible. Moreover,*

$$T^\dagger = \begin{bmatrix} T_1^* D^{-1} & 0 \\ T_2^* D^{-1} & 0 \end{bmatrix}.$$

$$T = \begin{bmatrix} T_1 & 0 \\ T_3 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(T^*) \\ \ker(T) \end{bmatrix} \to \begin{bmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_2 \end{bmatrix},$$

*where* $F = T_1^*T_1 + T_3^*T_3 \in \mathcal{L}(\mathrm{ran(T^*)})$ *is positive and invertible. Moreover,*

$$T^\dagger \;=\; \begin{bmatrix} F^{-1}T_1^* & F^{-1}T_3^* \\ 0 & 0 \end{bmatrix}.$$

## 2. Main Results

In this section, by using some block operator matrix techniques, we provide conditions that the product of two projections is an idempotent, and we show reverse order laws for such products of course with star replace Moore-Penrose inverses.

THEOREM 2.1. *Let* $T, S \in \mathcal{L}(\mathcal{X})$. *Then the following conditions are equivalent:*

1) $TSS^\dagger T^\dagger TS = TS$,
2) $S^\dagger T^\dagger TSS^\dagger T^\dagger = S^\dagger T^\dagger$,
3) $T^\dagger TSS^\dagger = SS^\dagger T^\dagger T$,
4) $T^\dagger TSS^\dagger$ *is an idempotent,*
5) $SS^\dagger T^\dagger T$ *is an idempotent.*

PROPOSITION 2.2. *Let* $T \in \mathcal{L}(\mathcal{X})$ *has closed range. Then, the following statements are equivalent:*

i) $S \in \mathcal{L}(\mathcal{X})$ *is the Moore-Penrose inverse of* $T$,
ii) $T = TT^*S^*$ *and* $S^* = TSS^*$.

PROPOSITION 2.3. *Let* $T, S \in \mathcal{L}(\mathcal{X})$ *such that* $S$ *has closed range. Necessary and sufficient condition for* $T$ *to commute with* $S$ *and* $S^*$ *is that* $T$ *commutes with* $S^\dagger$ *and* $S^{\dagger *}$.

PROOF. Since $S^*S$ has closed range, $(S^*S)^\dagger$ exists. Moreover,

$$S^\dagger = (S^*S)^\dagger S^*.$$

Since $T$ commutes with $S$ and $S^*$, then

$$\begin{aligned} TS^*S &= S^*TS \\ &= S^*ST, \end{aligned}$$

so $T$ commutes with $S^*S$. In addition, since $S^*S$ is Moore-Penrose invertible, $T$ commutes with $(S^*S)^\dagger$. Then

$$\begin{aligned} TS^\dagger &= TS^*(SS^*)^\dagger \\ &= S^*T(SS^*)^\dagger \\ &= S^*(SS^*)^\dagger T \\ &= S^*(S^*)^\dagger S^\dagger T \\ &= S^*(S^*)^\dagger S^\dagger T \\ &= S^\dagger T, \end{aligned}$$

so $T$ commutes with $S^\dagger = (S^*S)^\dagger S^*$.

In addition, since $S^* \in \mathcal{L}(\mathcal{X})$ has closed range and $(S^*)^* = S$, according to what has been proved, $T$ commutes with $(S^*)^\dagger = S^{\dagger *}$.

On the other hand, if $T$ commutes with $S^\dagger$ and $S^{\dagger *}$, then since $(S^\dagger)^\dagger = S$ and $(S^\dagger)^{\dagger *} = S^*$, $T$ commutes with $S$ and $S^*$. □

Theorem 2.4. *Suppose that* $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *with closed ranges,* $T = TS^\dagger T$ *and* $S = ST^\dagger S$, *then* $ST^\dagger = (TS^\dagger)^*$ *and* $(S^\dagger T)^* = T^\dagger S$.

Theorem 2.5. *Suppose that* $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *with closed ranges,* $TT^*T = TS^*T$ *and* $SS^*S = ST^*S$, *then* $ST^\dagger = (TS^\dagger)^*$ *and* $(S^\dagger T)^* = T^\dagger S$.

It should be remarked here, in general, do not yield that $TT^*T = TS^*T$ and $T = TS^\dagger T$ are equivalent, the following corollary provides conditions that these equalities coincide.

Corollary 2.6. *Suppose that* $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *with closed ranges such that* $S^\dagger ST^\dagger S = S^\dagger ST^*(S^\dagger)^*$ *then* $TT^*T = TS^*T$ *iff* $T = TS^\dagger T$.

Block matrix forms of operators conclude that we can provide a condition that adjoint plays a role reverse order law for Moore-Penrose inverse of the operator. Also, we give an explicit formula for the Moore-Penrose product of $S^\dagger$ and $T$, in the case it is idempotent.

In the following theorem we state conditions for which $(ST^\dagger)^* = TS^\dagger$ holds.

Theorem 2.7. *Let* $\mathcal{X}, \mathcal{Y}$ *be Hilbert* $\mathcal{A}$*-modules and* $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *have closed ranges such that* $T^*T = T^*S$ *and* $ST^\dagger = TT^\dagger$, *then* $(ST^\dagger)^* = TS^\dagger$.

Now, we give an explicit formula for Moore-Penrose product of $S^\dagger$ and $T$, in the case it is idempotent.

Theorem 2.8. *Suppose that* $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *have closed ranges, then*

i) *If* $TT^\dagger = ST^\dagger$ *then* $(S^\dagger T)^\dagger$ *is idempotent and*

$$(S^\dagger T)^\dagger = (S^\dagger T)^* - P_{\mathrm{ran(S^*)}}[(1 - P_{\mathrm{ran(T^*)}})(1 - P_{\mathrm{ran(S^*)}})]^\dagger P_{\mathrm{ran(S^*)}}.$$

ii) *If* $T^*T = T^*S$ *then* $(TS^\dagger)^\dagger$ *is idempotent and*

$$(TS^\dagger)^\dagger = (TS^\dagger)^* - P_{\mathrm{ran(S)}}[(1 - P_{\mathrm{ran(S)}})(1 - P_{\mathrm{ran(T)}})]^\dagger P_{\mathrm{ran(T)}}.$$

Suppose that $\mathcal{M}$ is a closed orthogonal complemented submodule of $\mathcal{X}$ and $P_{\mathcal{M}}$ denotes the unique projection onto $\mathcal{M}$. For every $T \in \mathcal{L}(\mathcal{X})$, we denote by

$$P_T = P_{\overline{\mathrm{ran(T)}}}.$$

Theorem 2.9. *Let* $T, S \in \mathcal{L}(\mathcal{X})$ *such that* $P_T = P_{T^*}$, $T^*T = T^*S$ *and* $TT^* = ST^*$ *and* $f$ *be complex analytic function that defined in a neighborhood of* $\{0\} \cup \sigma(A) \cup \sigma(B)$ *such that* $f(0) = 0$. *Then*

$$f(T^*)f(T) = f(T^*)f(S) \quad \text{and} \quad f(T)f(T^*) = f(S)f(T^*).$$

*Moreover, if* $f$ *is injective, then* $T^*T = T^*S$ *and* $TT^* = ST^*$ *if and only if* $f(T^*)f(T) = f(T^*)f(S)$ *and* $f(T)f(T^*) = f(S)f(T^*)$.

Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $S \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ with closed ranges, when $(TS)^* = S^\dagger T^\dagger$ holds?

**Acknowledgement**

I sincerely thank the organizers of the 51st Annual Iranian Mathematics Conference.

## References

1. G. Dolinar and J. Marovt, *Star partial order on $B(H)$*, Linear Algebra Appl. **434** (1) (2011) 319–326.
2. M. P. Drazin, *Natural structures on semigroups with involution*, Bull. Amer. Math. Soc. **84** (1) (1978) 139–141.
3. R. E. Hartwig, *How to partially order regular elements?*, Math. Japon. **25** (1) (1980) 1–13.
4. E. C. Lance, *Hilbert $C^*$-Modules*, LMS Lecture Note Series, Vol. 210, Cambridge University Press, England, 1995.
5. M. Jalaeian, M. Mohammadzadeh Karizaki and M. Hassani, *Conditions that the product of operators is an EP operator in Hilbert $C^*$-module*, Linear Multilinear Algebra **68** (10) (2020) 1990–2004.
6. Q. Xu and L. Sheng, *Positive semi-definite matrices of adjointable operators on Hilbert $C^*$-modules*, Linear Algebra Appl. **428** (4) (2008) 992–1000.

E-mail: m.mohammadzadeh@torbath.ac.ir
E-mail: j.farokhi@birjandut.ac.ir
E-mail: mahdialiakbari@torbath.ac.ir

# Subspace-Mixing Operators and Subspace-Hypercyclicity Criterion

Mansooreh Moosapoor*
Department of Mathematics, Farhangian University, Tehran, Iran

Abstract. In this paper, we investigate subspace-mixing operators. We prove that if an operator is invariant under a subspace and it satisfies the conditions of subspace-hypercyclicity criterion with respect to a syndetic sequence, then it is subspace-mixing.

**Keywords:** Subspace-mixing operators, Subspace-hypercyclicity criterion, Mixing operators.
**AMS Mathematical Subject Classification [2010]:** 47A16, 47B37, 37B99.

## 1. Introduction

Let $X$ be an infinite-dimensional Banach space and let $B(X)$ be the set of linear and continuous operators on $X$. We say an operator $T \in B(X)$ is mixing if for any two non-empty open sets $U$ and $V$, there exists a natural number $N$ such that $T^n(U) \cap V \neq \phi$ for any $n \geq N$.

One can see [2] for more information about mixing operators. Mathematicians are constructed various examples of mixing operators as follows.

Theorem 1.1. [3] *Let $\{w_n\}$ be any sequence of positive numbers. Let $B$ be the backward shift on $l^p$, $1 \leq p < \infty$ with weights $\{w_n\}$. Then $I + B$ is mixing.*

Moreover, Grivaux proved in [3] that any separable Banach space $X$, supports a mixing operator.

Talebi and Moosapoor defined subspace-mixing operators as follows.

Definition 1.2. [7] Let $T \in B(X)$ and let $M$ be a non-zero and closed subspace of $X$. We say $T$ is $M$-mixing if for any two non-empty and relatively open sets $U \subseteq M$ and $V \subseteq M$, there exists a natural number $N$ such that $T^n(U) \cap V \neq \phi$ for any $n \geq N$.

In the next theorem, we see a sufficient condition for an operator to be subspace-mixing.

Theorem 1.3. [6] *Let $T \in B(X)$ and let $M$ be a closed subspace of $X$. Suppose that, there are subsets $X_0 \subseteq M$ and $Y_0 \subseteq M$ such that $X_0$ and $Y_0$ are dense in $M$ and there is a map $S : Y_0 \to Y_0$ such that:*

(i) $T^n x \to 0$ *for any* $x \in X_0$,
(ii) $S^n y \to 0$ *for any* $y \in Y_0$,
(iii) $TSy = y$ *for any* $y \in Y_0$.

*Speaker

*Then, $T$ is $M$-mixing.*

One can see [6] for other interesting facts about subspace-mixing operators.

## 2. Main Results

Madore and Martinez Avendano stated a subspace-hypercyclic criterion. If an operator satisfies in their criterion, then it is subspace-transitive and subspace-hypercyclic. We say an operator $T$ is subspace-transitive with respect to a closed and non-zero subspace $M$ of $X$ if $T^{-n}(U) \cap V$ contains a non-empty and relatively open subset of $M$, where $U$ and $V$ are non-empty and relatively open subsets of $M$.

An operator $T$ is called $M$-hypercyclic if there exists an element $x \in X$ such that $orb(T,x) \cap M$ is dense in $M$. It is proved in [4] that subspace-transitive operators are subspace-hypercyclic. One can see [1] and [5] for interesting theorems about this matter.

Authors in [4] stated a sufficient condition for subspace-hypercyclicity that is named subspace-hypercyclicity criterion. In the next theorem, we recall it.

Theorem 2.1. [4] *Let $T \in B(X)$ and let $M$ be a non-zero subspace of $X$. Assume there exist $Y$ and $Z$, dense subsets of $M$ and an increasing sequence of positive integers $\{n_k\}$ such that:*

(i) *$T^{n_k} y \to 0$ for all $y \in Y$,*

(ii) *for each $z \in Z$ , there exists a sequence $\{x_k\}$ in $M$ such that*

$$x_k \to 0 \quad and \quad T^{n_k} x_k \to z,$$

(iii) *$M$ is an invariant subspace for $T^{n_k}$ for all $k \in \mathbb{N}$.*

*Then, $T$ is subspace-transitive with respect to $M$ and hence $T$ is subspace-hypercyclic with respect to $M$.*

We prove that if $T$ satisfies conditions (i) and (ii) of Theorem 2.1 with respect to a syndetic sequence $\{n_k\}$ and if $T(M) \subseteq M$, then $T$ is subspace-mixing. Recall that an increasing sequence of positive integers $\{n_k\}$ is called syndetic if

$$sup_k\{n_{k+1} - n_k\} < \infty.$$

Theorem 2.2. *Let $T \in B(X)$ and let $M$ be a non-zero and closed subspace of $X$ such that $T(M) \subseteq M$. Assume that there exist $Y$ and $Z$, dense subsets of $M$ and a syndetic sequence of positive integers $\{n_k\}$ such that:*

(i) *$T^{n_k} y \to 0$ for all $y \in Y$,*

(ii) *for each $z \in Z$ , there exists a sequence $\{x_k\}$ in $M$ such that*

$$x_k \to 0 \quad and \quad T^{n_k} x_k \to z,$$

*then, $T$ is subspace-mixing with respect to $M$.*

Proof. Let $U$ and $V$ be two relatively open sets in $M$. $Y$ is dense in $M$. So, there exists $y \in U \cap Y$. Hence, $T^{n_k} y \to 0$.

On the other hand, $\{n_k\}$ is syndetic. So, $m := sup_k\{n_{k+1} - n_k\}$ is a positive integer. By hypothesis, $T(M) \subseteq M$. So $T^n(M) \subseteq M$ for any natural number $n$. Hence, $T^n|_M$ is continuous and therefore $T^{-n}(V)$ is an open set in $M$. Then for $i = 0, 1, \ldots, m$, there are open sets $V_0, V_1, \ldots, V_m$ in $M$ such that $V_i \subseteq T^{-i}(V)$

and hence $T^i(V_i) \subseteq V$. Also, $Z$ is dense in $M$. So there exists $z_i \in V_i \cap Z$ for any $0 \leq i \leq m$. Hence, for any $0 \leq i \leq m$, there exists $\{x_k^{(i)}\}$ such that $x_k^{(i)} \to 0$ and $T^{n_k} x_k^{(i)} \to z_i$ for $0 \leq i \leq m$.

Moreover, $U$ and $V$ are relatively open subsets of $M$. So, there exists $\varepsilon > 0$ such that

$$B(y, \varepsilon) \cap M \subseteq U \quad \text{and} \quad B(z_i, \varepsilon) \cap M \subseteq V_i.$$

There exists a positive integer $k_0$ such that for any $k \geq k_0$,

$$||T^{n_k}(y)|| < \frac{\varepsilon}{2} \quad \text{and} \quad ||x_k^{(i)}|| < \frac{\varepsilon}{2} \quad \text{and} \quad ||T^{n_k} x_k^{(i)} - z_i|| < \frac{\varepsilon}{2}.$$

Consider $N := n_{k_0}$. Let $n \geq N$. So, there exists $k \geq k_0$ such that $n = n_k + r$, where $0 \leq r \leq m$. Suppose that $z_n = y + x_n^{(r)}$. Hence,

$$||z_n - y|| = ||x_n^{(r)}|| < \frac{\varepsilon}{2}.$$

Therefore, $z_n \in B(y, \varepsilon) \cap M \subseteq U$. So, $z_n \in U$. On the other hand,

$$T^n(z_n) = T^r(T^{n_k}(z_n)) = T^r(T^{n_k} y + T^{n_k}(x_k^{(r)})). \tag{1}$$

But,

$$||T^{n_k} y + T^{n_k}(x_k^{(r)}) - z_r|| \leq ||T^{n_k} y|| + ||T^{n_k}(x_k^{(r)}) - z_r|| < \varepsilon.$$

Hence, $T^{n_k} y + T^{n_k}(x_k^{(r)})$ belongs to $V_r$. But $T^r(V_r) \subseteq V$. So, by (1), $T^n(z_n)$ belongs to $V$. Therefore, $T^n(U) \cap V \neq \phi$ for any $n \geq N$. Hence, $T$ is an $M$-mixing operator. □

Corollary 2.3. *Let $T \in B(X)$. If $T$ satisfies in subspace-hypercyclicity criterion with respect to a closed and non-trivial subspace $M$ of $X$ and a syndetic sequence $\{n_k\}$ such that $T(M) \subseteq M$, then $T$ is $M$-mixing.*

Moreover, we can conclude that $T \oplus T$ is subspace-mixing as follows.

Corollary 2.4. *Let $T \in B(X)$. If $T$ satisfies in subspace-hypercyclicity criterion with respect to a closed and non-trivial subspace $M$ of $X$ and a syndetic sequence $\{n_k\}$ such that $T(M) \subseteq M$, then $T \oplus T$ is $M$-mixing.*

Proof. Let $U_1, U_2 \subseteq M$ and $V_1, V_2 \subseteq M$ be relatively open and non-empty sets. Similar to proof of Theorem 2.2, we can find a positive integer $N_1$ such that

$$T^n(U_1) \cap V_1 \neq \phi \quad \text{for any } n \geq N_1. \tag{2}$$

Also, we can find a positive integer $N_2$ such that

$$T^n(U_2) \cap V_2 \neq \phi \quad \text{for any } n \geq N_2. \tag{3}$$

If we consider $N := \max\{N_1, N_2\}$ then by (2) and (3) for any $n \geq N$ we have,

$$\begin{aligned}&(T \oplus T)^n(U_1 \oplus U_2) \cap (V_1 \oplus V_2)\\ &= T^n(U_1 \cap V_1) \oplus T^n(U_2 \cap V_2) \neq \emptyset.\end{aligned}$$

So, $T \oplus T$ is $M$-mixing. □

## References


1. N. Bamerni, V. Kadets and A. Kilicman, *Hypercyclic operators are subspace hypercyclic*, J. Math. Anal. Appl. **435** (2) (2016) 1812–1815.
2. G. Costakis and M. Sambarino, *Topologically mixing hypercyclic operators*, Proc. Amer. Math. Soc. **132** (2) (2003) 385–389.
3. S. Grivaux, *Hypercyclic operators, mixing operators, and the bounded steps problem*, J. Operator Theory **54** (1) (2005) 147–168.
4. B. F. Madore and R. A. Martinez-Avendano, *Subspace hypercyclicity*, J. Math. Anal. Appl. **373** (2) (2011) 502–511.
5. M. Moosapoor, *Hereditarily subspace-hypercyclic operators*, Int. J. Pure Appl. Math. **108** (3) (2016) 621–628.
6. M. Moosapoor, *Sufficient conditions for subspace-hypercyclicity*, Int. J. Pure Appl. Math. **99** (4) (2015) 445–454.
7. S. Talebi and M. Moosapoor, *Subspace-chaotic and subspace-weakly mixing operators*, Int. J. Pure Appl. Math. **78** (6) (2012) 879–885.


E-mail: mosapor110@gmail.com; m.mousapour@cfu.ac.ir

# $\sigma$-Derivations of Operator Algebras and an Application

Maysam Mosadeq*
Department of Mathematics, Behbahan Branch, Islamic Azad University, Behbahan, Iran

ABSTRACT. Let $\sigma$ be a bijective bounded linear operator on a Banach algebra $\mathcal{A}$. In this talk, we closely examine the concept of $\sigma$-one parameter groups of bounded linear operators as a generalization of one parameter groups and analyze their basic properties. We also, describe a $\sigma$-$C^*$-dynamical system as a uniformly continuous $\sigma$-one parameter group of $*$-linear automorphisms on a $C^*$-algebra and associate with each so-called $\sigma$-$C^*$-dynamical system a $\sigma$-derivation, named as its infinitesimal generator. Finally, as an application, we characterize each $\sigma$-$C^*$-dynamical system on the concrete $C^*$-algebra $\mathcal{A} := \mathbf{B}(H)$, where $H$ is a Hilbert space.
**Keywords:** $C^*$-Dynamical systems, (inner) $\sigma$-Derivation, One parameter group of operators, Operator algebra.
**AMS Mathematical Subject Classification [2010]:** 47D03, 46L55, 46L57.

## 1. Introduction

Let $\mathcal{A}$ be a Banach space. A one parameter group of bounded linear operators on $\mathcal{A}$ is a mapping $t \mapsto \varphi_t$ from the additive group $\mathbb{R}$ of real numbers into the set $\mathbf{B}(\mathcal{A})$ of all bounded linear operators on $\mathcal{A}$ such that $\varphi_0 = I$, where $I$ is the identity operator on $\mathcal{A}$, and $\varphi_{t+s} = \varphi_t\varphi_s$ for every $t, s \in \mathbb{R}$. The one parameter group $\{\varphi_t\}_{t\in\mathbb{R}}$ is called uniformly (resp. strongly) continuous if $\lim\limits_{t\to 0} \| \varphi_t - I \|= 0$ (resp. $\lim\limits_{t\to 0} \varphi_t(a) = I(a)$, for each $a \in \mathcal{A}$). The infinitesimal generator $d$ of the one parameter group $\{\varphi_t\}_{t\in\mathbb{R}}$ is a mapping $d : D(d) \subseteq \mathcal{A} \to \mathcal{A}$ such that $d(a) = \lim\limits_{t\to 0} \dfrac{\varphi_t(a) - a}{t}$ where

$$D(d) = \{a \in \mathcal{A} : \lim_{t\to 0} \frac{\varphi_t(a) - a}{t} exists\}.$$

The one parameter group $\{\varphi_t\}_{t\in\mathbb{R}}$ is uniformly continuous if and only if its infinitesimal generator is an everywhere defined bounded linear operator on $\mathcal{A}$. In fact, every uniformly continuous one parameter group on $\mathcal{A}$ is necessarily of the form $\{e^{td}\}_{t\in\mathbb{R}}$ for some bounded linear operator $d : \mathcal{A} \to \mathcal{A}$ (See [10, Theorems 1.1.2, 1.1.3 and Corollary 1.1.4]).

One parameter groups of bounded linear operators and their extensions are of highly considerable magnitude because of their applications in the theory of dynamical systems. The classical $C^*$-dynamical systems are expressed by means of strongly continuous one parameter groups of $*$-automorphisms on $C^*$-algebras.

*Speaker

On the other hand, the infinitesimal generator $d$ of a $C^*$-dynamical system is a closed densely defined $*$-derivation.

Recently, various generalized notions of derivations have been investigated in the context of Banach algebras. As an idea, let $\sigma$ be a linear homomorphism on an algebra $\mathcal{A}$ and $d : \mathcal{A} \to \mathcal{A}$ be a derivation. Then, the mapping $\delta : \mathcal{A} \to \mathcal{A}$ defined by $\delta(a) := d(\sigma(a))$ satisfies the equation $\delta(ab) = \delta(a)\sigma(b) + \sigma(a)\delta(b)$ for all $a, b \in \mathcal{A}$. This motivates us to consider the following definition.

Let $\mathcal{A}$ be a $*$-Banach algebra and $\sigma$ be a $*$-linear operator on $\mathcal{A}$. A $*$-linear map $\delta$ from a $*$-subalgebra $D(\delta)$ of $\mathcal{A}$ into $\mathcal{A}$ is called a *$\sigma$-derivations* if $\delta(ab) = \delta(a)\sigma(b)+\sigma(a)\delta(b)$ for all $a, b \in D(\delta)$. For instance, let $\sigma$ be a linear $*$-endomorphism and $h$ be an arbitrary self-adjoint element of $\mathcal{A}$. Then, the mapping $\delta : \mathcal{A} \to \mathcal{A}$ defined by $\delta(a) = i[h, \sigma(a)]$, where $[h, \sigma(a)]$ is the commutator $h\sigma(a) - \sigma(a)h$, is a $\sigma$-derivation which is called *inner*. Moreover, when $\sigma$ is an automorphism and $\delta : \mathcal{A} \to \mathcal{A}$ be a $\sigma$-derivation, we can consider $d := \delta\sigma^{-1}$ and find out that $d$ is an ordinary derivation (See [**3, 7, 9**] and references therein).

In each case of generalization of derivation, a noted point which draws the attention of analysts is trying to represent a suitable dynamical system whose infinitesimal generator is exactly the desired extended derivation as well as being an extension of a $C^*$-dynamical system. Such dynamical system is usually provided by adjoining a suitable property to (an extension of) a uniformly (strongly) continuous one parameter group of bounded linear operators. Some approaches to preparing new dynamical systems and their applications have been explained in [**4, 5, 6, 8**] and references therein.

In this talk, we closely examine the concept of $\sigma$-one parameter groups of bounded linear operators as a generalization of one parameter groups and analyze their basic properties. We also, describe a $\sigma$-$C^*$-dynamical system as a uniformly continuous $\sigma$-one parameter group of $*$-linear automorphisms on a $C^*$-algebra and associate with each so-called $\sigma$-$C^*$-dynamical system a $\sigma$-derivation, named as its infinitesimal generator. Finally, as an application, we characterize each $\sigma$-$C^*$-dynamical system on the concrete $C^*$-algebra $\mathcal{A} := \mathbf{B}(H)$, where $H$ is a Hilbert space.

## 2. Main Results

Definition 2.1. Let $\mathcal{A}$ be a Banach space and $\sigma : \mathcal{A} \to \mathcal{A}$ be a bijective bounded linear operator. A *$\sigma$-one parameter group* is a one parameter family $\{\alpha_t\}_{t\in\mathbb{R}}$ of bounded linear operators on $\mathcal{A}$ such that

(i) $\alpha_0 = \sigma$;
(ii) $\sigma\alpha_{t+s} = \alpha_t\alpha_s$ for every $t, s \in \mathbb{R}$.

The $\sigma$-one parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ is said to be

(i) *uniformly continuous* if $\lim_{t\to 0} \|\alpha_t - \sigma\| = 0$,
(ii) *strongly continuous* or $C_0$-$\sigma$-one parameter group if $\lim_{t\to 0} \alpha_t(a) = \sigma(a)$ for each $a \in \mathcal{A}$.

We define the *infinitesimal generator* $\delta$ of the $\sigma$-one parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ as a mapping $\delta : D(\delta) \subseteq \mathcal{A} \to \mathcal{A}$ such that $\delta(a) = \lim\limits_{t\to 0} \dfrac{\alpha_t(a) - \sigma(a)}{t}$ where

$$D(\delta) = \{a \in \mathcal{A} \textit{ such that } \lim_{t\to 0} \frac{\alpha_t(a) - \sigma(a)}{t} \textit{exists}\}.$$

If $\{\alpha_t\}_{t\in\mathbb{R}}$ is a $\sigma$-one parameter group with the generator $\delta$, then one can easily see that

(i) $\sigma\alpha_t = \alpha_t\sigma$ and $\sigma^{-1}\alpha_t = \alpha_t\sigma^{-1}$ for each $t \in \mathbb{R}$.

(ii) $\alpha_t(\mathcal{A}) = \sigma(\mathcal{A})$ and $ker(\alpha_t) = ker(\sigma)$ for each $t \in \mathbb{R}$.

(iii) $\sigma(\delta(a)) = \delta(\sigma(a))$ and $\sigma^{-1}(\delta(a)) = \delta\left(\sigma^{-1}(a)\right)$ for each $a \in D(\delta)$.

It is necessary to mention that the title of $\sigma$-one parameter group was first applied by Janfada in 2008 [2]. However, one of the faults of his definition is that $\sigma$ can not be a injective operator. More precisely, applying his definition of $\sigma$-one parameter group, it follows easily that $\sigma^2 = \sigma$. So, if $\sigma$ is an injective operator, then $\sigma$ equals immediately to the identity operator on $\mathcal{A}$ and therefore, each $\sigma$-one parameter group is nothing more than a one parameter group in the usual sense. The above restriction motivate us to demonstrate the aforementioned definition for a $\sigma$-one parameter group.

Example 2.2. Let $\mathcal{B}$ be a Banach space and take $\mathcal{A} := \mathcal{B} \times \mathcal{B}$. Suppose that $\{\phi_t\}_{t\in\mathbb{R}}$ is a one parameter group on $\mathcal{B}$ and consider the associated one parameter group $\{\phi_t \oplus \phi_t\}_{t\in\mathbb{R}}$ on $\mathcal{A}$. Define $\sigma : \mathcal{A} \to \mathcal{A}$ by $\sigma(a, b) := (b, a)$. Then, $\sigma$ is a bijective bounded linear operator on $\mathcal{A}$ and the one parameter family $\{\alpha_t\}_{t\in\mathbb{R}}$ defined by $\alpha_t := (\phi_t \oplus \phi_t)\sigma$ is a $\sigma$-one parameter group on $\mathcal{A}$ with the same continuity of $\{\phi_t \oplus \phi_t\}_{t\in\mathbb{R}}$.

The following lemma shows that with each $\sigma$-one parameter group one can associate a one parameter group.

Lemma 2.3. *Let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a uniformly (resp. strongly) continuous $\sigma$-one parameter group on $\mathcal{A}$ with the generator $\delta$. Then, the one parameter family $\{\varphi_t\}_{t\in\mathbb{R}}$ of bounded linear operators on $\mathcal{A}$ defined by $\varphi_t(a) := \alpha_t\left(\sigma^{-1}(a)\right)$ is a uniformly (resp. strongly) continuous one parameter group on $\mathcal{A}$ and the mapping $d : \sigma\big(D(\delta)\big) \subseteq \mathcal{A} \to \mathcal{A}$ defined by $d(\sigma(a)) = \delta(a)$ is its generator.*

Applying the previous lemma we have the following theorem.

Theorem 2.4. *Let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a $C_0$-$\sigma$-one parameter group on $\mathcal{A}$ with the generator $\delta$. Then,*

(i) $\displaystyle\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \alpha_s(a)ds = \alpha_t(a)$.

(ii) *For each $a \in \mathcal{A}$, $\displaystyle\int_0^t \alpha_s\left(\sigma^{-1}(a)\right) ds \in D(\delta)$ and $\displaystyle\delta\left(\int_0^t \alpha_s\left(\sigma^{-1}(a)\right) ds\right) = \alpha_t(a) - \sigma(a)$.*

(iii) *For each $a \in D(\delta)$, $\alpha_t\left(\sigma^{-1}(a)\right)$ and $\alpha_t\Big(\delta\left(\sigma^{-1}(a)\right)\Big) = \delta\Big(\alpha_t\left(\sigma^{-1}(a)\right)\Big)$.*

(iv) *For each $a \in D(\delta)$, $\displaystyle\alpha_t(a) - \alpha_s(a) = \int_s^t \alpha_\tau\Big(\delta\left(\sigma^{-1}(a)\right)\Big)d\tau$.*

The next result manifests a uniqueness theorem in the setting of $\sigma$-one parameter groups.

Theorem 2.5. *Let $\{\alpha_t\}_{t\in\mathbb{R}}$ and $\{\beta_t\}_{t\in\mathbb{R}}$ be two uniformly (resp. strongly) continuous $\sigma$-one parameter groups with the same generator $\delta$. Then, $\alpha_t = \beta_t$ ($t \in \mathbb{R}$).*

From now on, $\mathcal{A}$ is a $C^*$-algebra and $\sigma$ is a $*$-linear automorphism on $\mathcal{A}$.

Definition 2.6. A *$\sigma$-$C^*$-dynamical system* is a uniformly continuous $\sigma$-one parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ of $*$-linear automorphisms on the $C^*$-algebra $\mathcal{A}$.

According to the notations which mentioned in Lemma 2.3, for each $\sigma$-$C^*$-dynamical system $\{\alpha_t\}_{t\in\mathbb{R}}$, there exists a $C^*$-dynamical system $\{\varphi_t\}_{t\in\mathbb{R}}$ on $\mathcal{A}$ defined by $\varphi_t(a) := \alpha_t\left(\sigma^{-1}(a)\right)$.

On the other hand, the following lemma provides a method to construct a $\sigma$-$C^*$-dynamical system from a classical $C^*$-dynamical system.

Lemma 2.7. *Let $\sigma : \mathcal{A} \to \mathcal{A}$ be a $*$-linear automorphism and $\{\varphi_t\}_{t\in\mathbb{R}}$ be a $C^*$-dynamical system on $\mathcal{A}$ such that $\varphi_t\sigma = \sigma\varphi_t$. Then, $\{\varphi_t\}_{t\in\mathbb{R}}$ induces the $\sigma$-$C^*$-dynamical system $\{\alpha_t\}_{t\in\mathbb{R}}$ on $\mathcal{A}$ defined by $\alpha_t(a) := \varphi_t(\sigma(a))$.*

Theorem 2.8. *Let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a $\sigma$-$C^*$-dynamical system on $\mathcal{A}$ with the infinitesimal generator $\delta$. Then, $\delta$ is an everywhere defined bounded $*$-$\sigma$-derivation.*

Definition 2.9. A $\sigma$-inner automorphism implemented by a unitary element $u$ of $\mathcal{A}$ is a $*$-linear automorphism $\alpha : \mathcal{A} \to \mathcal{A}$ such that $\alpha(a) = u\sigma(a)u^*$ for every $a \in \mathcal{A}$.

Let $H$ be a Hilbert space. It is known that the algebra $\mathbf{B}(H)$ with respect to the operator norm and the natural involution given by the Hilbert adjoint operation is a unital $C^*$-algebra. On the other hand, Due to the Gelgand-Naimark-Segal representation, each non-commutative $C^*$-algebra can be regarded as a $C^*$-subalgebra of $\mathbf{B}(H)$, for some Hilbert space $H$. So, the study of $C^*$-dynamical systems on $\mathbf{B}(H)$ has an important role to survey of $C^*$-dynamical systems in general. Moreover, it is one of the key ideas of quantum mechanics to use uniformly continuous one parameter groups of unitary operators on a Hilbert space $H$ to implement new dynamical systems on the operator algebra $\mathbf{B}(H)$.

In the rest of the paper, we investigate this construction for a $\sigma$-$C^*$-dynamical system on the concrete $C^*$-algebra $\mathcal{A} := \mathbf{B}(H)$.

Theorem 2.10. *Let $\{U_t\}_{t\in\mathbb{R}}$ be a uniformly continuous one parameter group of unitary operators on $\mathbf{B}(H)$, and $\{\alpha_t\}_{t\in\mathbb{R}}$ be the $\sigma$-$C^*$-dynamical system implemented by the unitary operators group $\{U_t\}_{t\in\mathbb{R}}$ of $\sigma$-inner automorphisms with the generator $\delta$. Then, $\delta$ is an inner $\sigma$-derivation.*

It is now a pleasant surprise that each $\sigma$-$C^*$-dynamical system on $\mathbf{B}(H)$ is of this form, i.e., it is implemented by a unitary operators group on $H$. To achieve this nontrivial result, first note that each bounded derivation on $\mathbf{B}(H)$ is inner see [**1**, Lemma 1.3.16.2]. So, we can characterize bounded $\sigma$-derivations on the $C^*$-algebra $\mathbf{B}(H)$ as follows.

Theorem 2.11. *Let $\delta$ be bounded $*$-$\sigma$-derivations on $\mathbf{B}(H)$. Then, there is a self-adjoint operator $A \in \mathbf{B}(H)$ such that $\delta(T) = i[A, \sigma(T)]$.*

Applying the previous theorem, one can obtain the following main result.

THEOREM 2.12. *Let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a $\sigma$-one parameter group on $\mathbf{B}(H)$. Then, the following properties are equivalent.*

i) *$\{\alpha_t\}_{t\in\mathbb{R}}$ is a $\sigma$-$C^*$-dynamical system on $\mathbf{B}(H)$.*

ii) *There is a self-adjoint operator $A \in \mathbf{B}(H)$ such that*

$$\alpha_t(T) = e^{itA}\sigma(T)e^{-itA}.$$

## References

1. K. J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, New York, 2000.
2. M. Janfada, *$\sigma$-One parameter semigroup of bounded operators on Banach spaces*, J. Anal. Appl. **6** (2008) 129–137.
3. M. Mirzavaziri and M. S. Moslehian, *$\sigma$-derivations in Banach algebras*, Bull. Iranian Math. Soc. **32** (2006) 65–78.
4. M. Mosadeq, *Two parameter $\sigma$-$C^*$-Dynamical systems and application*, J. Math. Ext. **10** (2016) 63–76.
5. M. Mosadeq, *Some types of derivations and their appearance as the infinitesimal generators*, In 9th Conf. Math. PNU, Kerman, Iran (2017) pp.119–121.
6. M. Mosadeq, M. Hassani and A. Niknam, *$(\sigma,\gamma)$-Generalized dynamics on modules*, J. Dyn. Syst. Geom. Theor. **9** (2011) 171–184.
7. M. Mosadeq, M. Hassani and A. Niknam, *Approximately inner $\sigma$-dynamics on $C^*$-algebras*, J. Mahani. Math. Res. Cent. **1** (2012) 55–63.
8. M. Mosadeq, M. Hassani and A. Niknam, *Approximately quasi inner generalized dynamics on modules*, J. Sci. Islam. Repub. Iran. **24** (2012) 245–250.
9. M. Mosadeq, M. Hassani and A. Niknam, *Closability of module $\sigma$-derivations*, J. Math. Ext. **6** (2012) 81–93.
10. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

E-mail: mosadeq@mshdiau.ac.ir; maysammosaddegh@yahoo.com

# Some New Fixed Point Theorems in Midconvex Subgroups of a Banach Group

Alireza Pourmoslemi
Department of Mathematics, Payame Noor University, Tehran, Iran
and Tahere Nazari*
Department of Mathematics, Payame Noor University, Tehran, Iran

Abstract. In this paper, we introduce and prove some new fixed point theorems in normed and Banach groups. We present fixed points in midconvex and closed subsets of a Banach group.
**Keywords:** Banach group, Fixed point, Normed group, Midconvex subset.
**AMS Mathematical Subject Classification [2010]:** 47H10, 22A10.

## 1. Introduction and Preliminaries

Let $(\mathcal{L}, ., e, ()^{-1})$ be a group (written multiplicatively, with identity element e) and $\vartheta$ a self mapping on $\mathcal{L}$. If there exists an element $w \in \mathcal{L}$ such that $\vartheta(w) = w$, then element $w$ is said to be a fixed point of $\vartheta$ and define the $n^{th}$ iterate of $\vartheta$ as $\vartheta^0 = I$ (the identity map) and $\vartheta^n = \vartheta^{n-1} o \vartheta$, for $n \geq 1$.

Fixed point theory for non-expansive and related mappings plays a significant role in the development of the functional analysis and its applications. One well known type of this theorems is Banach fixed point theorems [1]. On the other hand, group-norms have also played a role in the theory of topological groups [2, 4]. The Birkhoff-Kakutanis metrization theorem for groups states that each first-countable Hausdorf group has a right invariant metric [3]. The term group-norm probably first appeared in Pettiss paper in 1950 [5]. Some results on the existence and uniqueness of fixed points on normed groups and Banach group are proved in this paper. We begin with some basic notions which will be needed in this paper.

Definition 1.1. [2] Let $\mathcal{L}$ be a group. A norm on a group $\mathcal{L}$ is a function $\|\cdot\| : \mathcal{L} \to \mathbb{R}$ with the following properties:

(1) $\|w\| \geq 0,\ for\ all\ w \in \mathcal{L}$,
(2) $\|w\| = \|w^{-1}\|,\ for\ all\ w \in \mathcal{L}$,
(3) $\|wk\| \leq \|w\| + \|k\|,\ for\ all\ w, k \in \mathcal{L}$,
(4) $\|w\| = 0\ implies\ that\ w = e$.

A normed group $(\mathcal{L}, \|\cdot\|)$ is a group $\mathcal{L}$ equipped with a norm $\|\cdot\|$. By setting $d(w, k) := \|w^{-1}k\|$, it is easy to see that norms are in bijection with left-invariant metrics on $\mathcal{L}$.

*Speaker

Note that the group-norm generates a right and a left norm topology via the right-invariant and left-invariant metrics $d_r(w,k) := \|wk^{-1}\|$ and $d_l(w,k) := \|w^{-1}k\| = d_r(w^{-1}, k^{-1})$. A group-norm is $\mathbb{N}$-homogeneous if for each $n \in \mathbb{N}$,

$$\|w^n\| = n\|w\| \ (\forall w \in \mathcal{L}).$$

Now, let $(\mathcal{L}, \|\cdot\|)$ be a normed group and $w \in \mathcal{L}$. The set

$$B_o(w,r) := \{k \in \mathcal{L} : \|kw^{-1}\| < r\},$$

is called open ball with center at $w$ and the set

$$B_c(w,r) := \{k \in \mathcal{L} : \|kw^{-1}\| \leq r\},$$

is called closed ball with center at $w$ [2].

For normed group $(\mathcal{L}, \|\cdot\|)$, element $w \in \mathcal{L}$ is called limit of a sequence $w_n$

$$w = \lim_{n\to\infty} w_n,$$

if for every $\epsilon \in \mathbb{R}$, $\epsilon > 0$, there exists positive integer $n_0$ depending on $\epsilon$ such that $\|w_n w^{-1}\| < \epsilon$ for every $n > n_0$. Also, the sequence $w_n$ in $\mathcal{L}$ is called Cauchy sequence, if for every $\epsilon \in \mathbb{R}$, $\epsilon > 0$, there exists positive integer $n_0$ depending on $\epsilon$ such that $\|w_a w_b^{-1}\| < \epsilon$ for every $a, b > n_0$. So, a normed group $\mathcal{L}$ is called complete if any Cauchy sequence of elements of $\mathcal{L}$ converges in group $\mathcal{L}$, i.e. it has a limit in the group.

Definition 1.2. A Banach group is a normed group $(\mathcal{L}, \|\cdot\|)$, which is complete with respect to the metric

$$d(w,k) = \|wk^{-1}\|, \quad (w, k \in \mathcal{L}).$$

A map $\gamma : \mathcal{L} \to \mathcal{K}$, of a normed group $(\mathcal{L}, \|\cdot\|_{\mathcal{L}})$ into a normed group $(\mathcal{K}, \|\cdot\|_{\mathcal{K}})$ is called continuous, if for every as small as we please $\epsilon > 0$ there exist such $\delta > 0$, that $\|wk^{-1}\|_{\mathcal{L}} < \delta$ implies

$$\|\gamma(w)\gamma(k)^{-1}\|_{\mathcal{K}} < \epsilon.$$

## 2. Main Results

The notion of convexity in normed spaces is used to prove fixed point theorems. In this section, we prove fixed point theorems in midconvex and closed subsets of a Banach group. We start with the definition of a $\frac{1}{2}$-convex (or midconvex) subset of a group.

Definition 2.1. [2] Let $\mathcal{L}$ be a group. A subset $S$ of $\mathcal{L}$ is called $\frac{1}{2}$-convex (or midconvex), if for every $s, t \in S$ there exists an element $c \in S$, denoted by $(st)^{\frac{1}{2}}$, such that $c^2 = st$.

Lemma 2.2. *Let $(\mathcal{L}, \|\cdot\|)$ be a Banach group and $A$ be a nonempty closed subset of $\mathcal{L}$ and let $\psi : A \to A$ be a mapping such that satisfying*

$$\|\psi(w)\psi(k)^{-1}\| \leq \eta \left[\|w\psi(w)^{-1}\| + \|k\psi(k)^{-1}\|\right],$$

*for all $w, k \in \mathcal{L}$ and $0 \leq \eta < 1$. If for arbitrary point $a \in A$ there exists $b \in A$ such that $\|\psi(b)b^{-1}\| \leq r_1\|\psi(a)a^{-1}\|$ and $\|ba^{-1}\| \leq r_2\|\psi(a)a^{-1}\|$, when there exist constants $r_1, r_2 \in \mathbb{R}$ such that $0 \leq r_1 < 1$ and $r_2 > 0$, then $\psi$ has at least one fixed point.*

PROOF. For an arbitrary element $a_0 \in A$ define a sequence $(a_n)_{n=1}^{\infty} \subset A$ such that
$$\|\psi(a_{n+1})a_{n+1}^{-1}\| \leq r_1\|\psi(a_n)a_n^{-1}\|,$$
and
$$\|a_{n+1}a_n^{-1}\| \leq r_2\|\psi(a_n)a_n^{-1}\|,$$
for $n = 1, 2, ....$ It is easy to see that $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence, since
$$\|a_{n+1}a_n^{-1}\| \leq r_2\|\psi(a_n)a_n^{-1}\| \leq r_2 r_1^n\|\psi(a_0)a_0^{-1}\|.$$
Because $A$ is complete, there exists $c \in A$ such that $\lim\limits_{n\to\infty} a_n = c$. Then
$$\begin{aligned}\|\psi(c)c^{-1}\| &\leq \|\psi(c)\psi(a_n)^{-1}\| + \|\psi(a_n)a_n^{-1}\| + \|a_n c^{-1}\| \\ &\leq \eta\left[\|c\psi(c)^{-1}\| + \|a_n\psi(a_n)^{-1}\|\right] + \|\psi(a_n)a_n^{-1}\| + \|a_n c^{-1}\|,\end{aligned}$$
and
$$\begin{aligned}\|\psi(c)c^{-1}\| &\leq \frac{\eta+1}{1-\eta}\|\psi(a_n)a_n^{-1}\| + \frac{1}{1-\eta}\|a_n c^{-1}\| \\ &\leq \frac{\eta+1}{1-\eta}r_1^n\|\psi(a_0)a_0^{-1}\| + \frac{1}{1-\eta}\|a_n c^{-1}\| \to 0,\end{aligned}$$
as $n \to \infty$. So, $\psi(c) = c$. □

THEOREM 2.3. *Let $S$ be a nonempty, closed and $\frac{1}{2}$-convex subset of Banach group $(\mathcal{L}, \|\cdot\|)$ and let $\psi : S \to S$ be a mapping such that*
$$\|\psi(s)\psi(t)^{-1}\| \leq \eta\left[\|s\psi(s)^{-1}\| + \|t\psi(t)^{-1}\|\right],$$
*for all $s, t \in S$ and $\eta < 1$. If the norm is $\mathbb{N}$-homogeneous and for $s \in S$, the equation $c^2\psi(c)^{-1} = s$ has a solution in $S$, then $\psi$ has a unique fixed point in $S$.*

PROOF. For $s \in S$, let $c = (\psi(s)\psi(c))^{\frac{1}{2}}$. Then
$$\begin{aligned}\|c\psi(c)^{-1}\| &= \|(\psi(s)\psi(c))^{\frac{1}{2}}\psi(c)^{-1}\| \\ &= \|(\psi(s)\psi(c)^{-1})^{\frac{1}{2}}\| \\ &= \frac{1}{2}\|\psi(s)\psi(c)^{-1}\| \\ &\leq \frac{\eta}{2}(\|s\psi(s)^{-1}\| + \|c\psi(c)^{-1}\|).\end{aligned}$$
Hence
$$\|c\psi(c)^{-1}\| \leq \frac{\frac{\eta}{2}}{1-\frac{\eta}{2}}\|s\psi(s)^{-1}\|.$$
Using the triangle inequality we obtain
$$\|cs^{-1}\| \leq \frac{1}{2}\|\psi(c)s^{-1}\| \leq \frac{1}{2}(\|\psi(c)c^{-1}\| + \|cs^{-1}\|).$$
So,
$$\|cs^{-1}\| \leq \|c\psi(c)^{-1}\| \leq \kappa\|s\psi(s)^{-1}\|,$$
where $\kappa = \frac{\frac{\eta}{2}}{1-\frac{\eta}{2}} < 1$.
For arbitrary $s_0 \in S$, we define a sequence $(s_n)_{n=1}^{\infty} \subset S$ in the following manner:
$$s_{n+1} = (s_n\psi(s_{n+1}))^{\frac{1}{2}}.$$

By Lemma(2.2), this sequence is converges to $z$ and $\psi(z) = z$. It is obvious that $z$ is unique. □

THEOREM 2.4. *Let $S$ be a closed and $\frac{1}{2}$-convex subset of a Banach group. If the group is abelian and the norm is $\mathbb{N}$-homogeneous and $\alpha : S \to S$ be a mapping which satisfies the condition*

$$\|s\alpha(s)^{-1}\| + \|t\alpha(t)^{-1}\| \leq \kappa\|st^{-1}\|,$$

*for all $s, t \in S$, where $2 \leq \kappa < 4$, then $\alpha$ has at least one fixed point.*

PROOF. Let for arbitrary element $s_0 \in S$, a sequence $(s_n)_{n=1}^{\infty}$ be defined by

$$s_{n+1} = (s_n\alpha(s_n))^{\frac{1}{2}} \quad (n = 0, 1, 2...).$$

Then we have

$$s_n\alpha(s_n)^{-1} = s_n^2 s_n^{-1}\alpha(s_n)^{-1} = (s_n s_{n+1}^{-1})^2,$$

and since the norm is $n$-homogeneous, $\|s_n\alpha(s_n)^{-1}\| = \|(s_n s_{n+1}^{-1})^2\| = 2\|s_n s_{n+1}^{-1}\|$. So, for $s = s_{n-1}$ and $t = s_n$, we have

$$2\|s_{n-1}s_n^{-1}\| + 2\|s_n s_{n+1}^{-1}\| \leq \kappa\|s_{n-1}s_n\|.$$

Hence $\|s_n s_{n+1}^{-1}\| \leq m\|s_{n-1}s_n^{-1}\|$, where $0 \leq m = \frac{\kappa-2}{2} < 1$, as $2 \leq \kappa < 4$. Then $(s_n)_{n=1}^{\infty}$ is a Cauchy sequence in $S$ and hence converges to some $z \in S$. Since

$$\|z\alpha(s_n)^{-1}\| \leq \|zs_n^{-1}\| + \|s_n\alpha(s_n)^{-1}\| = \|zs_n^{-1}\| + 2\|s_n s_{n+1}^{-1}\|,$$

then

$$\lim_{n\to\infty} \alpha(s_n) = z.$$

Therefore, for $s = z$ and $t = s_n$, we have

$$\|z\alpha(z)^{-1}\| + 2\|s_n s_{n+1}^{-1}\| \leq \kappa\|zs_n^{-1}\|.$$

This implies $\alpha(z) = z$, when $n$ tends to infinity. □

COROLLARY 2.5. *Let $S$ be a closed and $\frac{1}{2}$-convex subset of a Banach group and $\alpha : S \to S$ be a mapping which satisfies the condition*

$$\|s\alpha(t)^{-1}\| + \|t\alpha(s)^{-1}\| \leq \iota\|st^{-1}\|,$$

*for all $s, t \in S$, where $0 \leq \iota < 2$. Then $\alpha$ has a fixed point.*

PROOF. Using the triangle inequality we have

$$\begin{aligned}\|s\alpha(s)^{-1}\| + \|t\alpha(t)^{-1}\| &= \|st^{-1}t\alpha(s)^{-1}\| + \|ts^{-1}s\alpha(t)^{-1}\| \\ &\leq \|st^{-1}\| + \|t\alpha(s)^{-1}\| + \|ts^{-1}\| + \|s\alpha(t)^{-1}\|.\end{aligned}$$

Thus,

$$\|s\alpha(s)^{-1}\| + \|t\alpha(t)^{-1}\| \leq \iota\|st^{-1}\| + 2\|st^{-1}\|.$$

Therefore, we conclude that $\alpha$ satisfies Theorem 2.4 with $\kappa = \iota + 2$. □

## References


1. S. Banach, *Sur les operations dans les ensembles abstraits et leurr application aux equations in tegrales*, Fund. Math. **3** (1) (1922) 133–181.
2. N. H. Bingham and A. J. Ostaszewski, *Normed versus topological groups: Dichotomy and-duality*, Dissertationes Math. **472** (2010) 1–138.
3. G. Birkhoff, *A note on topological groups*, Compositio Math. **3** (1936) 427–430.
4. D. R. Farkas, *The algebra of norms and expanding maps on groups*, J. Algebra **133** (2) (1990) 386–403.
5. B. J. Pettis, *On continuity and openness of homomorphisms in topological groups*, Ann. of Math. **52** (1950) 293–308.


E-mail: a_pourmoslemy@pnu.ac.ir
E-mail: taheree.nazari2222@gmail.com

# Construction of Controlled $K$-g-Fusion Frames in Hilbert Spaces

Gholamreza Rahimlou*
Department of Mathematics, Faculty of Tabriz Branch, Technical and Vocational University (TVU), East Azarbaijan, Iran

Abstract. Considering the importance and application of dual of frames, especially fusion frames, which cannot be defined in the usual way, we try to investigate the concept of dual for controlled generalized $K$-fusion frames.
**Keywords:** $K$-g-fusion frame, Controlled g-fusion frame, Controlled $K$-g-fusion frame, $Q$-duality.
**AMS Mathematical Subject Classification [2010]:** 42C15, 94A12, 46C05.

## 1. Introduction

In this note, we first introduce the concept of controlled $K$-g-fusion frames which are generalizations of controlled g-fusion frames in Hilbert spaces. After characterizing and constructing these frames by a bounded operator, we present the $Q$-dual of controlled $K$-g-fusion frames and we describe how to create the $Q$-dual of these frames. Throughout this paper, $H$ is a separable Hilbert spaces, $\mathcal{B}(H)$ is the collection of all bounded linear operators on $H$, $\mathcal{GL}(H)$ is the set of all bounded linear operators on $H$ which have bounded inverses, $\mathcal{GL}^+(H)$ is the set of all positive operators in $\mathcal{GL}(H)$ and $K \in \mathcal{B}(H)$. Also, $\pi_V$ is the orthogonal projection from $H$ onto a closed subspace $V \subset H$ and $\{H_i\}_{i\in\mathbb{I}}$ is a sequence of Hilbert spaces, where $\mathbb{I}$ is a subset of $\mathbb{Z}$.

Lemma 1.1. [3] *Let $V \subseteq H$ be a closed subspace, and $T$ be a linear bounded operator on $H$. Then*
$$\pi_V T^* = \pi_V T^* \pi_{\overline{TV}}.$$
*If $T$ is unitary (i.e. $T^*T = Id_H$), then*
$$\pi_{\overline{TV}} T = T\pi_V.$$

Lemma 1.2. [1] *Let $U \in \mathcal{B}(H_1, H_2)$ be a bounded operator with closed range $\mathcal{R}_U$. Then there exists a bounded operator $U^\dagger \in \mathcal{B}(H_2, H_1)$ such that*
$$UU^\dagger x = x, \quad x \in \mathcal{R}_U.$$

Lemma 1.3. [2] *Let $L_1 \in \mathcal{B}(H_1, H)$ and $L_2 \in \mathcal{B}(H_2, H)$ be operators on given Hilbert spaces. Then the following assertions are equivalent:*
1) *$\mathcal{R}(L_1) \subseteq \mathcal{R}(L_2)$,*
2) *$L_1L_1^* \leq \lambda^2 L_2L_2^*$ for some $\lambda > 0$,*
3) *there exists a mapping $U \in \mathcal{B}(H_1, H_2)$ such that $L_1 = L_2U$.*

*Speaker

Definition 1.4. [5] [$K$-g-fusion frame] Let $W = \{W_i\}_{i\in\mathbb{I}}$ be a collection of closed subspaces of $H$, $\{v_i\}_{i\in\mathbb{I}}$ be a family of weights, i.e. $v_i > 0$, $\Lambda_i \in \mathcal{B}(H, H_i)$ for each $i \in \mathbb{I}$ and $K \in \mathcal{B}(H)$. We say that $\Lambda := (W_i, \Lambda_i, v_i)$ is a $K$-g- fusion frame for $H$ if there exists $0 < A \le B < \infty$ such that for each $f \in H$

$$A\|K^*f\|^2 \le \sum_{i\in\mathbb{I}} v_i^2 \|\Lambda_i \pi_{W_i} f\|^2 \le B\|f\|^2.$$

Corresponding to this frame, the representation space is defined by

$$\mathscr{H}_2 := \big\{\{f_i\}_{i\in\mathbb{I}} \ : \ f_i \in H_i, \ \sum_{i\in\mathbb{I}} \|f_i\|^2 < \infty\big\},$$

with the inner product defined by

$$\langle \{f_i\}, \{g_i\}\rangle = \sum_{i\in\mathbb{I}} \langle f_i, g_i\rangle.$$

Definition 1.5. [4] [$(C, C')$-controlled g-fusion frame] Let $W := \{W_i\}_{i\in\mathbb{I}}$ be a family of closed subspaces of $H$ and $\{v_i\}_{i\in\mathbb{I}}$ be a family of weights i.e. $v_i > 0$ for all $i \in \mathbb{I}$. Let $\{H_i\}_{i\in\mathbb{I}}$ be a sequence of Hilbert spaces, $C, C' \in \mathcal{GL}(H)$ and $\Lambda_i \in \mathcal{B}(H, H_i)$. $\Lambda_{CC'} := (W_i, \Lambda_i, v_i)$ is a $(C, C')$-controlled g-fusion frame for $H$ if there exist constants $0 < A \le B < \infty$ such that for all $f \in H$

$$A\|f\|^2 \le \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C f\rangle \le B\|f\|^2.$$

## 2. Main Results

We introduce the concept of $(C, C')$-controlled $K$-g-fusion frame on Hilbert spaces and present the corresponding operators and we shall define duality of $(C, C')$-$K$GF and present some properties of them. Throughout this paper, $C$ and $C'$ are invertible operators in $\mathcal{GL}(H)$.

Definition 2.1. Let $W := \{W_i\}_{i\in\mathbb{I}}$ be a family of closed subspaces of $H$ and $\{v_i\}_{i\in\mathbb{I}}$ be a family of weights. Suppose that $\{H_i\}_{i\in\mathbb{I}}$ is a sequence of Hilbert spaces and $\Lambda_i \in \mathcal{B}(H, H_i)$. We call $\Lambda_{CC'K} := (W_i, \Lambda_i, v_i)$ a $(C, C')$-controlled $K$-g-fusion frame (briefly $CC'$-$K$GF) for $H$ if there exist constants $0 < A_{CC'} \le B_{CC'} < \infty$ such that for each $f \in H$

$$A_{CC'}\|K^*f\|^2 \le \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C f\rangle \le B_{CC'}\|f\|^2. \tag{1}$$

Throughout this paper, $\Lambda_{CC'K}$ will be a triple $(W_i, \Lambda_i, v_i)$ with $i \in \mathbb{I}$ unless otherwise stated. We call $\Lambda_{CC'K}$ a Parseval $CC'$-$K$GF if $A_{CC'} = B_{CC'} = 1$ or, equivalently,

$$\sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C f\rangle = \|K^*f\|^2.$$

When $K = Id_H$, we get a $C, C'$-controlled g-fusion frame for $H$. If only the second inequality (1) is required, $\Lambda_{CC'K}$ is called a $(C, C')$-controlled g-fusion Bessel sequence (briefly $CC'$-GBS) with bound $B_{CC'}$.

The synthesis and analysis operators are similar to those corresponding to controlled g-fusion frame [4]. So, if $\Lambda_{CC'K}$ is a $CC'$-GBS, then

$$T_{CC'} : \mathscr{K}^2_{\Lambda_i} \to H,$$
$$T_{CC'}\Big(v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\Big) = \sum_{i\in\mathbb{I}} v_i^2 C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'f,$$

and

$$T^*_{CC'} : H \to \mathscr{K}^2_{\Lambda_i},$$
$$T^*_{CC'}f = \{v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\}_{i\in\mathbb{I}},$$

where

$$\mathscr{K}^2_{\Lambda_i} := \{v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f \ : \ f\in H\}_{i\in\mathbb{I}} \subset (\bigoplus_{i\in\mathbb{I}} H)_{l^2}.$$

Therefore, the frame operator is given by

$$S_{CC'}f := T_{CC'}T^*_{CC'}f = \sum_{i\in\mathbb{I}} v_i^2 C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'f,$$

and for each $f\in H$,

$$\begin{aligned}\langle S_{CC'}f, f\rangle &= \sum_{i\in\mathbb{I}} v_i^2\langle C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'f, f\rangle\\ &= \sum_{i\in\mathbb{I}} v_i^2\langle \Lambda_i\pi_{W_i}C'f, \Lambda_i\pi_{W_i}Cf\rangle.\end{aligned}$$

Hence

$$A_{CC'}KK^* \leq S_{CC'} \leq B_{CC'}Id_H.$$

Now, we conclude that that the following result holds.

Proposition 2.2. *Let $\Lambda_{CC'K}$ be a $CC'$-GBS for $H$. Then $\Lambda_{CC'K}$ is a $CC'$-$K$GF if and only if there exists $A_{CC'}>0$ such that $S_{CC'}\geq A_{CC'}KK^*$.*

For $CC'$-$K$GF, like for $K$-frames, the operator $S_{CC'}$ is not invertible and when $K$ has closed range, $S_{CC'}$ is an invertible operator (for more details, we refer to [5]). Assume that $K$ has closed range. Since $\mathcal{B}(H)$ is a $C^*$-algebra, then $S^{-1}_{CC'}$ is positive and self-adjoint. Now, for any $f\in S_{CC'}(\mathcal{R}(K))$ we have

$$\begin{aligned}\langle Kf, f\rangle &= \langle Kf, S_{CC'}S^{-1}_{CC'}f\rangle\\ &= \langle S_{CC'}(Kf), S^{-1}_{CC'}f\rangle\\ &= \langle \sum_{i\in\mathbb{I}} v_i^2 C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'Kf, S^{-1}_{CC'}f\rangle\\ &= \sum_{i\in\mathbb{I}} v_i^2\langle S^{-1}_{CC'}C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'Kf, f\rangle.\end{aligned}$$

In the next results, we construct $K$-g-fusion frames by using a bounded linear operator.

Theorem 2.3. *Let $U\in\mathcal{B}(H)$ be an invertible operator on $H$ such that $U^*$ commutes with $C, C'$ and let $\Lambda_{CC'K}$ be a $CC'$-$K$GF for $H$ with bounds $A_{CC'}$ and $B_{CC'}$. Then, $\Gamma := (UW_i, \Lambda_i\pi_{W_i}U^*, v_i)$ is a $CC'$-$UK$GF for $H$.*

Proof. Since $U$ is invertible, $UW_j$ is a closed subspace of $H$ for each $i \in \mathbb{I}$. For $f \in H$, by applying Lemma 1.1 with $U$ instead of $T$, we have

$$\begin{aligned}\sum_{i\in\mathbb{I}} v_i^2 \big\langle \Lambda_i \pi_{W_i} U^* \pi_{UW_i} C' f,\ \Lambda_i \pi_{W_i} U^* \pi_{UW_i} C f\big\rangle &= \sum_{i\in\mathbb{I}} v_i^2 \big\langle \Lambda_i \pi_{W_i} U^* C' f,\ \Lambda_i \pi_{W_i} U^* C f\big\rangle \\ &\leq B_{CC'} \|U^* f\|^2 \\ &\leq B_{CC'} \|U\|^2 \|f\|^2.\end{aligned}$$

So, $\Gamma$ is a g-fusion Bessel sequence for $H$. On the other hand,

$$\begin{aligned}\sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} U^* \pi_{UW_i} C' f,\ \Lambda_i \pi_{W_i} U^* \pi_{UW_i} C f\rangle &= \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} U^* C' f,\ \Lambda_i \pi_{W_i} U^* C f\rangle \\ &\geq A_{CC'} \|K^* U^* f\|^2 \\ &= A_{CC'} \|(UK)^* f\|^2,\end{aligned}$$

and the proof is completed. □

Corollary 2.4. *Let $U \in \mathcal{B}(H)$ be an invertible operator on $H$ and $U^*$ commutes with $C, C'$ and $K^*$, furthermore, let $\Lambda_{CC'K}$ be a $CC'$-$KGF$ for $H$. Then, $\Gamma = (UW_i, \Lambda_i \pi_{W_i} U^*, v_i)$ is a $CC'$-$KGF$ for $H$.*

Theorem 2.5. *Let $U \in \mathcal{B}(H)$ be an unitary operator on $H$ which commutes with $C, C'$, and let $\Lambda_{CC'K}$ be a $CC'$-$KGF$ for $H$ with bounds $A_{CC'}$ and $B_{CC'}$. Then, $\Gamma = (UW_i, \Lambda_i U^{-1}, v_i)$ is a $CC'$-$(U^{-1})^* KGF$ for $H$.*

Proof. Via Lemma 1.1, we can write for every $f \in H$,

$$A_{CC'} \|K^* U^{-1} f\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i U^{-1} \pi_{UW_i} C' f, \Lambda_i U^{-1} \pi_{UW_i} C f\rangle \leq B_{CC'} \|U^{-1}\|^2 \|f\|^2.$$

□

Corollary 2.6. *Let $U \in \mathcal{B}(H)$ be an unitary operator on $H$ which commutes with $C, C'$ and $K^*$, furthermore $\Lambda_{CC'K}$ be a $CC'$-$KGF$ for $H$ with bounds $A_{CC'}$ and $B_{CC'}$. Then, $\Gamma = (UW_i, \Lambda_i U^{-1}, v_i)$ is a $CC'$-$KGF$ for $H$.*

Theorem 2.7. *If $U \in \mathcal{B}(H)$, $\mathcal{R}(U) \subseteq \mathcal{R}(K)$ and $\Lambda_{CC'K}$ is a $CC'$-$KGF$ for $H$ with bounds $A_{CC'}$ and $B_{CC'}$, then $\Lambda_{CC'K}$ is a $CC'$-$UGF$ for $H$.*

Proof. By Lemma 1.3, there exists $\lambda > 0$ such that $UU^* \leq \lambda^2 KK^*$. Thus, for each $f \in H$ we have

$$\|U^* f\|^2 = \langle UU^* f, f\rangle \leq \lambda^2 \langle KK^* f, f\rangle = \lambda^2 \|K^* f\|^2.$$

It follows that

$$\frac{A_{CC'}}{\lambda^2} \|U^* f\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C f\rangle \leq B_{CC'} \|f\|^2.$$

□

Definition 2.8. Let $\Lambda_{CC'} = (W_i, \Lambda_i, v_i)$ be a $(C, C')$-$K$GF for $H$ with the synthesis operator $T_\Lambda$. A $(C, C')$-controlled g-fusion Bessel sequence $\Theta_{CC'} :=$

$(V_i, \Theta_i, w_i)$ is called $Q$-controlled dual $K$-g-fusion frame (or brevity $Q$-dual $(C, C')$-$K$GF) for $\Lambda_{CC'}$ if there exists a bounded linear operator $Q : \mathscr{K}^2_{\Lambda_j} \longrightarrow \mathscr{K}^2_{\Theta_j}$ such that

$$T_\Lambda Q^* T^*_\Theta = KCC'.$$

The following results present equivalent conditions of the duality with straightforward proofs.

PROPOSITION 2.9. *Let $\Theta_{CC'}$ be a $Q$-dual $(C, C')$-$K$GF for $\Lambda_{CC'}$. The following conditions are equivalent:*

1) $T_\Lambda Q^* T^*_\Theta = KCC'$,
2) $T_\Theta Q T^*_\Lambda = C'^* C^* K^*$,
3) *for each $f, f' \in H$, we have*

$$\langle KCf, C'^* f'\rangle = \langle T^*_\Theta f, QT^*_\Lambda f'\rangle = \langle Q^* T^*_\Theta f, T^*_\Lambda f'\rangle.$$

COROLLARY 2.10. *Assume $C_{op}$ and $D_{op}$ are the optimal bounds of $\Theta_{CC'}$. Then*

$$C_{op} \geq B^{-1}_{op} \|Q\|^{-2} \|C'^{-1}\|^{-2} \|C^{-1}\|^{-2} \quad \textit{and} \quad D_{op} \geq A^{-1}_{op} \|Q\|^{-2} \|C'^{-1}\|^{-2} \|C^{-1}\|^{-2},$$

*where $A_{op}$ and $B_{op}$ are the optimal bounds of $\Lambda_{CC'}$.*

## References

1. O. Christensen, *An Introduction to Frames and Riesz Bases*, 2nd ed., Applied and Numerical Harmonic Analysis, Birkhäuser Basel, Boston, 2016.
2. R. G. Douglas, *On majorization, factorization and range inclusion of operators on Hilbert spaces*, Proc. Amer. Math. Soc. **17** (2) (1966) pp. 413–415.
3. P. Găvruţa, *On the duality of fusion frames*, J. Math. Anal. Appl. **333** (2) (2007) 871–879.
4. M. Nouri, A. Rahimi and Sh. Najafizadeh, *Controlled K-frames in Hilbert spaces*, Int. J. Anal. Appl. **4** (2) (2015) 39–50.
5. V. Sadri, Gh. Rahimlou, R. Ahmadi and R. Zarghami Farfar, *Construction of g-fusion frames in Hilbert spaces*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **23** (2) (2020) 2050015. DOI:10.1142/S0219025720500150

E-mail: grahimlou@gmail.com

# On Nonsmooth Optimality Conditions and Duality in Robust Multiobjective Optimization

Maryam Saadati*
Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran
and Morteza Oveisiha
Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran

Abstract. In this paper, we introduce a new concept of generalized convexity, and establish necessary/sufficient optimality conditions for (weakly) robust efficient solutions of the considered problem. These optimality conditions are presented in terms of limiting subdifferentials of the related functions. In addition, we address Mond-Weir-type robust dual problem to the primal one, and explore weak/strong duality relations between them under assumptions of pseudo convexity.
**Keywords:** Robust nonsmooth multiobjective optimization, Optimality conditions, Duality, Limiting subdifferential.
**AMS Mathematical Subject Classification [2010]:** 65K10, 90C29, 90C46.

## 1. Introduction

*Robust optimization* has emerged as a remarkable deterministic framework for studying multiobjective optimization problems under data uncertainty [**1**, **2**]. An uncertain multiobjective optimization problem can be studied through its *robust counterpart*. The concepts of robustness for uncertain multiobjective optimization problems have been established in [**3**, **4**, **5**]. Recently, Chuong [**6**] considered nonsmooth/nonconvex uncertain multiobjective optimization problems, and introduced the concept of (strictly) generalized convexity to established optimality and duality theories with respect to limiting subdifferential for robust (weakly) Pareto solutions. Chen [**7**] studied necessary/sufficient conditions in terms of Clarke subdifferential for weakly robust efficient solutions of nonsmooth uncertain multiobjective optimization problems, and explored duality results under the generalized convexity assumptions. To the best of our knowledge, the most powerful results in this direction were established for finite-dimensional problems by exploiting various kinds of generalized convex functions. Our main purpose in this paper is to investigate a nonsmooth/nonconvex multiobjective optimization problem in arbitrary Asplund spaces under pseudo convexity assumptions.

Throughout this paper, we assume all the spaces under consideration are *Asplund* with the norm $\|\cdot\|$, and the dual pair between the space in question and its *dual* $X^*$ is denoted by $\langle\cdot,\cdot\rangle$. For a given nonempty set $\Omega \subset X$, the symbols co$\Omega$,

*Speaker

cl $\Omega$, and int$\Omega$ indicate the *convex hull*, *topological closure*, and *topological interior* of $\Omega$, respectively. The *dual cone* of $\Omega$ is the set

$$\Omega^+ := \{x^* \in X^* \mid \langle x^*, x\rangle \geq 0, \ \forall x \in \Omega\}.$$

Besides, $\mathbb{R}^n_+$ signifies the nonnegative orthant of $\mathbb{R}^n$ for $n \in \mathbb{N} := \{1, 2, \dots\}$.

Let $I := \{1, 2, \dots, n\}$ be index set. Suppose that $f : X \to Y$ be a locally *Lipschitzian* vector-valued function, and that $K \subset Y$ be a pointed (i.e., $K \bigcap (-K) = \{0\}$) closed and convex cone. We consider the following *multiobjective* optimization problem:

$$\text{(P)} \qquad \min\nolimits_K \ f(x)$$
$$\text{s.t.} \ g_i(x) \leq 0, \quad i \in I,$$

where the functions $g_i : X \to \mathbb{R}$, $i \in I$, define the constraints. This problem in the face of data *uncertainty* in the constraints can be captured by the following *uncertain multiobjective* optimization problem:

$$\text{(UP)} \qquad \min\nolimits_K \ f(x)$$
$$\text{s.t.} \ g_i(x, v) \leq 0, \quad i \in I,$$

where $x \in X$ is the vector of *decision* variable, $v$ is the vector of *uncertain* parameter and $v \in \mathcal{V}$ for some *sequentially compact* topological space $\mathcal{V}$, and $g_i : X \times \mathcal{V} \to \mathbb{R}$, $i \in I$, are given functions.

For investigating problem (UP), one usually associates the so-called *robust* counterpart:

$$\text{(RP)} \qquad \min\nolimits_K \ f(x)$$
$$\text{s.t.} \ g_i(x, v) \leq 0, \quad \forall v \in \mathcal{V}, \ i \in I.$$

A vector $x \in X$ is called a *robust feasible solution* of problem (UP) if it is a *feasible solution* of problem (RP). The robust feasible set $F$ of problem (UP) is defined by

$$F := \{x \in X \mid g_i(x, v) \leq 0, \, i \in I, \ \forall v \in \mathcal{V}\}.$$

We now recall some definitions and basic results in the literature.

Definition 1.1. [8]

i) We say that a vector $\bar{x} \in F$ is a *robust efficient solution* of problem (UP) and write $\bar{x} \in \mathcal{S}(RP)$,
$$f(x) - f(\bar{x}) \notin -K \setminus \{0\}, \quad \forall x \in F.$$

ii) A vector $\bar{x} \in F$ is called a *weakly robust efficient solution* of problem (UP) and write $\bar{x} \in \mathcal{S}^w(RP)$ if and only if
$$f(x) - f(\bar{x}) \notin -\text{int}\, K, \quad \forall x \in F.$$

Motivated by the concept of generalized convexity in [9], we introduce a similar concept of robust generalized convexity type for $f$ and $g$.

Definition 1.2.

i) $f$ is *pseudo convex* at $\bar{x} \in X$ if for any $x \in X$ and $y^* \in K^+$ the following holds:
$$\langle y^*, f\rangle(x) < \langle y^*, f\rangle(\bar{x}) \Longrightarrow \big(\langle z^*, x - \bar{x}\rangle < 0, \quad \forall z^* \in \partial\langle y^*, f\rangle(\bar{x})\big).$$

ii) $f$ is *strictly pseudo convex* at $\bar{x} \in X$ if for any $x \in X \setminus \{\bar{x}\}$ and $y^* \in K^+ \setminus \{0\}$ the following holds:

$$\langle y^*, f\rangle(x) \leq \langle y^*, f\rangle(\bar{x}) \Longrightarrow \big(\langle z^*, x-\bar{x}\rangle < 0, \quad \forall z^* \in \partial\langle y^*, f\rangle(\bar{x})\big).$$

iii) $g$ is *generalized quasi convex* at $\bar{x} \in X$ if for any $x \in X$ and $v \in \mathcal{V}$ the following holds:

$$g_i(x, v) \leq g_i(\bar{x}, v) \Longrightarrow \big(\langle v_i^*, x-\bar{x}\rangle \leq 0, \quad \forall v_i^* \in \partial_x g_i(\bar{x}, v)\big),\ i \in I.$$

DEFINITION 1.3.

i) We say that $(f, g)$ is *type I pseudo convex* at $\bar{x} \in X$ if $f$ and $g$ are pseudo convex and generalized quasi convex at $\bar{x} \in X$, respectively.
ii) We say that $(f, g)$ is *type II pseudo convex* at $\bar{x} \in X$ if $f$, $g$ are strictly pseudo convex and generalized quasi convex at $\bar{x} \in X$, respectively.

REMARK 1.4. It follows from Definitions 1.2 and 1.3 that if $(f, g)$ is type II pseudo convex at $\bar{x} \in X$, then $(f, g)$ is type I pseudo convex at $\bar{x} \in X$, but converse is not true.

Let $\Omega \subset X$ be *locally closed* around $\bar{x} \in \Omega$, i.e., there is a neighborhood $U$ of $\bar{x}$ for which $\Omega \bigcap \mathrm{cl}\,U$ is closed. The *Fréchet normal cone* $\widehat{N}(\bar{x};\Omega)$ and the *limiting/Mordukhovich normal cone* $N(\bar{x};\Omega)$ to $\Omega$ at $\bar{x} \in \Omega$ are defined by

$$\widehat{N}(\bar{x};\Omega) := \Big\{x^* \in X^* \mid \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \leq 0\Big\}, \tag{1}$$

$$N(\bar{x};\Omega) := \operatorname*{Lim\,sup}_{x \xrightarrow{\Omega} \bar{x}} \widehat{N}(x;\Omega), \tag{2}$$

where $x \xrightarrow{\Omega} \bar{x}$ stands for $x \to \bar{x}$ with $x \in \Omega$. If $\bar{x} \notin \Omega$, we put $\widehat{N}(\bar{x};\Omega) = N(\bar{x};\Omega) := \emptyset$.

For $\varphi : X \to \overline{\mathbb{R}}$, the *limiting/Mordukhovich subdifferential* of $\varphi$ at $\bar{x} \in \operatorname{dom}\varphi$ are given by

$$\partial\varphi(\bar{x}) := \big\{x^* \in X^* \mid (x^*, -1) \in N((\bar{x}, \varphi(x)); \operatorname{epi}\varphi)\big\}.$$

If $|\varphi(x)| = \infty$, then one puts $\partial\varphi(\bar{x}) := \widehat{\partial}\varphi(\bar{x}) := \emptyset$. For a further study, we refer to [10].

Throughout this paper, we assume that the following assumptions (See [6, p.131]) hold:

(A1) For a fixed $\bar{x} \in X$, $g$ is locally Lipschitz in the first argument and uniformly on $\mathcal{V}$ in the second argument, i.e., there exist an open neighborhood $U$ of $\bar{x}$ and a positive constant $\ell$ such that $\|g(y, w) - g(z, w)\| \leq \ell\|y - z\|$ for all $y, z \in U$ and $w \in \mathcal{V}$.
(A2) For each $i \in I$, the function $w \in \mathcal{V} \mapsto g_i(x, w) \in \mathbb{R}$ is upper semicontinuous for each $x \in U$.
(A3) For each $i \in I$, we define a family of real-valued functions $\phi_i, \phi : X \to \mathbb{R}$ as follows:

$$\phi_i(x) := \max_{w\in\mathcal{V}} g_i(x, w) \qquad \text{and} \qquad \phi(x) := \max_{i\in I} \phi_i(x).$$

Since $g_i$ is upper semicontinuous and $\mathcal{V}$ is sequentially compact, $\phi_i$ is well defined.

(A4) For each $i \in I$, the multifunction $(x,w) \in U \times \mathcal{V} \rightrightarrows \partial_x g_i(x,w) \subset X^*$ is closed at $(\bar{x}, v)$ for each $v \in \mathcal{V}_i(\bar{x})$, where the notation $\partial_x$ signifies the limiting subdifferential operation with respect to $x$, and $\mathcal{V}_i(\bar{x}) = \big\{v \in \mathcal{V} \mid g_i(\bar{x}, v) = \phi_i(\bar{x})\big\}$.

It is worth to mention that inspecting the proof of [6, Theorem 3.3] reveals that this proof contains a formula for limiting subdifferential of *maximum* functions in finite-dimensional spaces. The following lemma generalizes the corresponding result in arbitrary Asplund spaces.

Lemma 1.5. *Let $\mathcal{V}$ be a sequentially compact topological space, and let $g : X \times \mathcal{V} \to \mathbb{R}$ be a function such that for each fixed $w \in \mathcal{V}$, $g(\cdot, w)$ is locally Lipschitz on $X$ and for each fixed $x \in X$, $g(x, \cdot)$ is upper semicontinuous on $\mathcal{V}$. Let $\varphi(x) := \max\limits_{w \in \mathcal{V}} g(x,w)$. If the multifunction $(x,w) \in X \times \mathcal{V} \rightrightarrows \partial_x g(x,w) \subset X^*$ is closed at $(\bar{x}, v)$ for each $v \in \mathcal{V}(\bar{x})$, then the set $\operatorname{cl}\operatorname{co}\Big(\bigcup \Big\{\partial_x g(\bar{x}, v) \mid v \in \mathcal{V}(\bar{x})\Big\}\Big)$ is nonempty and*

$$\partial \varphi(\bar{x}) \subset \operatorname{cl}\operatorname{co}\Big(\bigcup \Big\{\partial_x g(\bar{x}, v) \mid v \in \mathcal{V}(\bar{x})\Big\}\Big),$$

*where $\mathcal{V}(\bar{x}) = \big\{v \in \mathcal{V} \mid g(\bar{x}, v) = \varphi(\bar{x})\big\}$.*

In the rest of this section, we state a suitable constraint qualification in the sense of robustness, which is needed to obtain a so-called *robust Karush-Kuhn-Tucker* (KKT) *condition.*

Definition 1.6. [6] Let $\bar{x} \in F$. We say that *constraint qualification* (CQ) *condition* is satisfied at $\bar{x}$ if there do not exist $\mu_i \geq 0$, $i \in I(\bar{x})$, which at least one of the $\mu_i$'s are not zero such that

$$0 \in \sum_{i \in I(\bar{x})} \mu_i \operatorname{cl}\operatorname{co}\Big(\bigcup \Big\{\partial_x g_i(\bar{x}, v) \mid v \in \mathcal{V}_i(\bar{x})\Big\}\Big),$$

Definition 1.7. A point $\bar{x} \in F$ is said to satisfy the *robust* (KKT) *condition* if there exist $y^* \in K^+ \setminus \{0\}$, $\mu := (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n_+$, and $\bar{v}_i \in \mathcal{V}$, $i \in I$, such that

$$\begin{cases} 0 \in \partial \langle y^*, f\rangle(\bar{x}) + \displaystyle\sum_{i \in I} \mu_i \operatorname{cl}\operatorname{co}\Big(\bigcup \Big\{\partial_x g_i(\bar{x}, v) \mid v \in \mathcal{V}_i(\bar{x})\Big\}\Big), \\ \mu_i \max\limits_{w \in \mathcal{V}} g_i(\bar{x}, w) = \mu_i \, g_i(\bar{x}, \bar{v}_i) = 0, \quad i \in I. \end{cases}$$

## 2. Main Results

The first theorem in this section presents a necessary optimality condition in terms of the limiting subdifferential for weakly robust efficient solutions of problem (UP).

Theorem 2.1. *Suppose that $g_i$, $i \in I$, satisfy the conditions* (A1)-(A4). *If $\bar{x} \in \mathcal{S}^w(RP)$, then there exist $y^* \in K^+$, $\mu := (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n_+$, with $\|y^*\| + \|\mu\| = 1$, and $\bar{v}_i \in \mathcal{V}$, $i \in I$, such that*

$$\begin{cases} 0 \in \partial \langle y^*, f\rangle(\bar{x}) + \displaystyle\sum_{i \in I} \mu_i \operatorname{cl}\operatorname{co}\Big(\bigcup \Big\{\partial_x g_i(\bar{x}, v) \mid v \in \mathcal{V}_i(\bar{x})\Big\}\Big), \\ \mu_i \max\limits_{w \in \mathcal{V}} g_i(\bar{x}, w) = \mu_i \, g_i(\bar{x}, \bar{v}_i) = 0, \quad i \in I. \end{cases} \tag{3}$$

*Furthermore, if the* (CQ) *is satisfied at* $\bar{x}$*, then* (3) *holds with* $y^* \neq 0$.

PROOF. Let $\bar{x} \in \mathcal{S}^w(RP)$. Exploiting the approximate extremal principle for $X \times Y$ and the weak fuzzy sum rule for the Fréchet subdifferential, we find sequences $x^{1k} \to \bar{x}$, $x^{2k} \to \bar{x}$, $y_k^* \in K^+$ with $\|y_k^*\| = 1$, $\alpha_k \in \mathbb{R}_+$, $x_{1k}^* \in \widehat{\partial}\langle y_k^*, f\rangle(x^{1k})$, and $x_{2k}^* \in \alpha_k \, \widehat{\partial}\phi(x^{2k})$ satisfying

$$
\begin{aligned}
&0 \in x_{1k}^* + x_{2k}^* + \frac{1}{k} B_{X^*}, \\
&\alpha_k \, \phi(x^{2k}) \to 0 \text{ as } k \to \infty.
\end{aligned} \tag{4}
$$

To proceed, we consider the following two possibilities for the sequence $\{\alpha_k\}$:

**Case 1:** If $\{\alpha_k\}$ is bounded, there is no loss of generality in assuming that $\alpha_k \to \alpha \in \mathbb{R}_+$ as $k \to \infty$. Moreover, since the sequence $\{y_k^*\} \subset K^+$ is bounded, by using the weak* sequential compactness of bounded sets in duals to Asplund spaces we may assume without loss of generality that $y_k^* \overset{w^*}{\to} \bar{y}^* \in K^+$ with $\|\bar{y}^*\| = 1$ as $k \to \infty$. The sequence $\{x_{1k}^*\}$ is bounded due to the boundedness of $\{y_k^*\}$ and the Lipschitz continuity of $f$ around $\bar{x}$. In this way, we can find $x_1^* \in X^*$ such that $x_{1k}^* \overset{w^*}{\to} x_1^* \in X^*$ as $k \to \infty$ and thus it stems from (4) that $x_{2k}^* \overset{w^*}{\to} x_2^* := -x_1^*$ as $k \to \infty$. For each $k \in \mathbb{N}$, the inclusion $x_{1k}^* \in \widehat{\partial}\langle y_k^*, f\rangle(x^{1k})$ means that $(x_{1k}^*, -y_k^*) \in \widehat{N}((x^{1k}, f(x^{1k})); \text{gph } f)$, see e.g., [10, Proposition 3.5]. Passing there to the limit as $k \to \infty$ and taking the definitions of normal cones (1) and (2), we get $(x_1^*, -\bar{y}^*) \in N((\bar{x}, f(\bar{x})); \text{gph } f)$, which is equivalent to

$$
x_1^* \in \partial\langle \bar{y}^*, f\rangle(\bar{x}), \tag{5}
$$

see e.g., [10, Theorem 1.90]. Similarly, we obtain $x_2^* \in \alpha \, \partial\phi(\bar{x})$. The latter inclusion with (5) imply that $0 \in \partial\langle \bar{y}^*, f\rangle(\bar{x}) + \alpha \, \partial\phi(\bar{x})$, by taking into account that $x_2^* = -x_1^*$. Invoking now the formula for the limiting subdifferential of maximum functions (See e.g., [10, Theorem 3.46]) and sum rule for the limiting subdifferential, and employing further Lemma 1.5, one has (3).

**Case 2:** Assuming next that $\{\alpha_k\}$ is unbounded. Similar to the Case 1, we get from the inclusion $x_{2k}^* \in \alpha_k \, \widehat{\partial}\phi(x^{2k})$ that $(x_{2k}^*, -\alpha_k) \in \widehat{N}((x^{2k}, \phi(x^{2k})); \text{gph } \phi)$ for each $k \in \mathbb{N}$. So

$$
\left(\frac{x_{2k}^*}{\alpha_k}, -1\right) \in \widehat{N}((x^{2k}, \phi(x^{2k})); \text{gph } \phi), \quad k \in \mathbb{N}.
$$

Letting $k \to \infty$ and noticing (1) and (2) again, we obtain that

$$
(0, -1) \in N((\bar{x}, \phi(\bar{x})); \text{gph } \phi),
$$

which amounts to $0 \in \partial\phi(\bar{x})$. Proceeding as in the proof of the Case 1, we arrive at (3) by taking $y^* := 0 \in K^+$.

Finally, let $\bar{x}$ satisfy the (CQ) in the Case 1. It follows directly from (3) that $y^* \neq 0$, which justifies the last statement of the theorem and completes the proof. □

REMARK 2.2.

i) Theorem 2.1 reduces to [6, Theorem 3.3] in the case of finite-dimensional optimization under inequality constraints.

ii) Observe that the result obtained in [7, Theorem 3.1] is expressed for problems containing Q-convexlike objective functions and the convex constraint systems in terms of the Clarke subdifferentials, but the one in Theorem 2.1 is established for nonconvex problems in the framework of Asplund spaces by applying the limiting subdifferential.

We provide a (KKT) sufficient condition for (weakly) robust efficient solutions of problem (UP).

Theorem 2.3. *Assume that $\bar{x} \in F$ satisfies the robust* (KKT) *condition.*

i) *If $(f, g)$ is type I pseudo convex at $\bar{x}$, then $\bar{x} \in \mathcal{S}^w(RP)$.*

ii) *If $(f, g)$ is type II pseudo convex at $\bar{x}$, then $\bar{x} \in \mathcal{S}(RP)$.*

Proof. Let $\bar{x} \in F$ satisfy the robust (KKT) condition. Therefore, there exist $y^* \in K^+ \setminus \{0\}$, $z^* \in \partial\langle y^*, f\rangle(\bar{x})$, $\mu_i \geq 0$, and $v_i^* \in \operatorname{cl}\operatorname{co}\Big(\bigcup\Big\{\partial_x g_i(\bar{x}, v) \mid v \in \mathcal{V}_i(\bar{x})\Big\}\Big)$, $i \in I$, such that

$$0 = z^* + \sum_{i \in I} \mu_i \, v_i^*, \tag{6}$$

$$\mu_i \max_{w \in \mathcal{V}} g_i(\bar{x}, w) = 0, \quad i \in I. \tag{7}$$

Firstly, we justify (i). Argue by contradiction that $\bar{x} \notin \mathcal{S}^w(RP)$. Hence, there is $\hat{x} \in F$ such that $f(\hat{x}) - f(\bar{x}) \in -\operatorname{int} K$. The latter gives $\langle y^*, f(\hat{x}) - f(\bar{x})\rangle < 0$. Since $(f, g)$ is the type I pseudo convex at $\bar{x}$, we deduce from this inequality that $\langle z^*, \hat{x} - \bar{x}\rangle < 0$. On the other side, it follows from (6) for $\hat{x}$ above that $0 = \langle z^*, \hat{x} - \bar{x}\rangle + \sum\limits_{i \in I} \mu_i \, \langle v_i^*, \hat{x} - \bar{x}\rangle$. The latter relations entail that $\sum\limits_{i \in I} \mu_i \, \langle v_i^*, \hat{x} - \bar{x}\rangle > 0$. So, there is $i_0 \in I$ such that $\mu_{i_0} \, \langle v_{i_0}^*, \hat{x} - \bar{x}\rangle > 0$. Taking into account that $v_{i_0}^* \in \operatorname{cl}\operatorname{co}\Big(\bigcup\Big\{\partial_x g_{i_0}(\bar{x}, v) \mid v \in \mathcal{V}_{i_0}(\bar{x})\Big\}\Big)$, we get sequence $\{v_{i_0 k}^*\} \subset \operatorname{co}\Big(\bigcup\Big\{\partial_x g_{i_0}(\bar{x}, v) \mid v \in \mathcal{V}_{i_0}(\bar{x})\Big\}\Big)$ such that $v_{i_0}^* = \lim\limits_{k \to \infty} v_{i_0 k}^*$. Hence, due to $\mu_{i_0} > 0$, there is $k_0 \in \mathbb{N}$ such that

$$\langle v_{i_0 k_0}^*, \hat{x} - \bar{x}\rangle > 0. \tag{8}$$

In addition, since $v_{i_0 k_0}^* \in \operatorname{co}\Big(\bigcup\Big\{\partial_x g_{i_0}(\bar{x}, v) \mid v \in \mathcal{V}_{i_0}(\bar{x})\Big\}\Big)$, there exist $v_p^* \in \bigcup\Big\{\partial_x g_{i_0}(\bar{x}, v) \mid v \in \mathcal{V}_{i_0}(\bar{x})\Big\}$ and $\mu_p \geq 0$ with $\sum\limits_{p=1}^{s} \mu_p = 1$, $p = 1, 2, \ldots, s$, $s \in \mathbb{N}$, such that $v_{i_0 k_0}^* = \sum\limits_{p=1}^{s} \mu_p \, v_p^*$. Combining the latter together (8), we arrive at $\sum\limits_{p=1}^{s} \mu_p \, \langle v_p^*, \hat{x} - \bar{x}\rangle > 0$. Thus, we can take $p_0 \in \{1, 2, \ldots, s\}$ such that

$$\langle v_{p_0}^*, \hat{x} - \bar{x}\rangle > 0, \tag{9}$$

and choose $v_{i_0} \in \mathcal{V}_{i_0}(\bar{x})$ satisfying $v_{p_0}^* \in \partial_x g_{i_0}(\bar{x}, v_{i_0})$ due to $v_{p_0}^* \in \bigcup\Big\{\partial_x g_{i_0}(\bar{x}, v) \mid v \in \mathcal{V}_{i_0}(\bar{x})\Big\}$. Invoking now definition of type I pseudo convexity of $(f, g)$ at $\bar{x}$, we get from (9) that

$$g_{i_0}(\hat{x}, v_{i_0}) > g_{i_0}(\bar{x}, v_{i_0}). \tag{10}$$

Note that $v_{i_0} \in \mathcal{V}_{i_0}(\bar{x})$, thus we have $g_{i_0}(\bar{x}, v_{i_0}) = \max_{w \in V} g_{i_0}(\bar{x}, w)$ which together with (7) yields $\mu_{i_0} g_{i_0}(\bar{x}, v_{i_0}) = 0$. This implies by (10) that $\mu_{i_0} g_{i_0}(\hat{x}, v_{i_0}) > 0$, and hence $g_{i_0}(\hat{x}, v_{i_0}) > 0$, which contradicts with the fact that $\hat{x} \in F$. Assertion (ii) is proved similarly to the part (i). □

We get the following sufficient optimality conditions from Remark 1.4 and Theorem 2.3.

Corollary 2.4. *Let $\bar{x} \in F$ satisfy the robust* (KKT) *condition and $(f, g)$ be type I pseudo convex at $\bar{x}$, then $\bar{x} \in \mathcal{S}(RP)$.*

Remark 2.5. Theorem 2.3 improves [6, Theorem 3.11], where the involved functions are generalized convex in the setting of finite-dimensional spaces. We establish the (KKT) sufficient optimality conditions for problem (UP) in the sense of pseudo convexity concept.

We now formulate Mond-Weir-type dual robust problem ($\mathrm{RD}_{MW}$) for (RP), and investigate weak and strong duality relations between corresponding problems under pseudo convexity assumptions.

Let $z \in X$, $y^* \in K^+ \setminus \{0\}$, and $\mu \in \mathbb{R}^n_+$. In connection with the problem (RP), we introduce a *dual robust multiobjective optimization* problem in the sense of Mond-Weir as follows:

$$(\mathrm{RD}_{MW}) \qquad \max\nolimits_K \ \left\{\bar{f}(z, y^*, \mu) := f(z) \mid (z, y^*, \mu) \in F_{MW}\right\}.$$

The feasible set $F_{MW}$ is given by

$$F_{MW} := \Big\{(z, y^*, \mu) \in X \times K^+ \setminus \{0\} \times \mathbb{R}^n_+ \mid 0 \in \partial\langle y^*, f\rangle(z) + \sum_{i \in I} \mu_i v_i^*,$$
$$v_i^* \in \operatorname{cl}\operatorname{co}\Big(\bigcup\Big\{\partial_x g_i(z, v) \mid v \in \mathcal{V}_i(z)\Big\}\Big),\ \mu_i g_i(z, v) \geq 0, i \in I\Big\}.$$

In what follows, a robust efficient solution (resp., weakly robust efficient solution) of the dual problem ($\mathrm{RD}_{MW}$) is defined similarly as in Definition 1.1 by replacing $-K$ (resp., -int $K$) by $K$ (resp., int $K$). We denote the set of robust efficient solutions (resp., weakly robust efficient solutions) of problem ($\mathrm{RD}_{MW}$) by $\mathcal{S}(RD_{MW})$ (resp., $\mathcal{S}^w(RD_{MW})$). Besides, we use the following notations:

$$u \prec v \Leftrightarrow u - v \in -\operatorname{int} K, \quad u \nprec v \text{ is the negation of } u \prec v,$$
$$u \preceq v \Leftrightarrow u - v \in -K \setminus \{0\}, \quad u \npreceq v \text{ is the negation of } u \preceq v.$$

Theorem 2.6. (Weak Duality) *Let $x \in F$, and let $(z, y^*, \mu) \in F_{MW}$.*

i) *If $(f, g)$ is type I pseudo convex at $z$, then $f(x) \nprec \bar{f}(z, y^*, \mu)$.*

ii) *If $(f, g)$ is type II pseudo convex at $z$, then $f(x) \npreceq \bar{f}(z, y^*, \mu)$.*

Proof. By $(z, y^*, \mu, \gamma) \in F_{MW}$, there exist $z^* \in \partial\langle y^*, f\rangle(z)$, $\mu_i \geq 0$ and

$$v_i^* \in \operatorname{cl}\operatorname{co}\Big(\bigcup\Big\{\partial_x g_i(z, v) \mid v \in \mathcal{V}_i(z)\Big\}\Big), \quad i \in I,$$

satisfying $0 = z^* + \sum_{i\in I} \mu_i\, v_i^*$ and $\mu_i\, g_i(z,v) \geq 0$. To justify (i), assume that $f(x) \prec \bar{f}(z,y^*,\mu)$. Hence $\langle y^*, f(x) - \bar{f}(z,y^*,\mu)\rangle < 0$ due to $y^* \neq 0$. This is nothing else but $\langle y^*, f(x) - f(z)\rangle < 0$. Since $(f,g)$ is type I pseudo convex at $z$, we deduce from the last inequality that $\langle z^*, x - z\rangle < 0$. Proceeding similarly to the proof of Theorem 2.3(i), one can obtain the result. The proof of (ii) is similar, so it is omitted. □

THEOREM 2.7. (STRONG DUALITY) *Let $\bar{x} \in \mathcal{S}^w(RP)$ be such that the* (CQ) *is satisfied at this point. Then, there exist $(\bar{y}^*,\bar{\mu}) \in K^+ \setminus \{0\} \times \mathbb{R}^n_+$ such that $(\bar{x},\bar{y}^*,\bar{\mu}) \in F_{MW}$. Furthermore,*

i) *If $(f,g)$ is type I pseudo convex at any $z \in X$, then $(\bar{x},\bar{y}^*,\bar{\mu}) \in \mathcal{S}^w(RD_{MW})$.*
ii) *If $(f,g)$ is type II pseudo convex at any $z \in X$, then $(\bar{x},\bar{y}^*,\bar{\mu}) \in \mathcal{S}(RD_{MW})$.*

PROOF. Thanks to Theorem 2.1, we find $y^* \in K^+ \setminus \{0\}$, $\mu_i \geq 0$, and $v_i^* \in \operatorname{cl}\operatorname{co}\Big(\bigcup\Big\{\partial_x g_i(\bar{x},v) \mid v \in \mathcal{V}_i(\bar{x})\Big\}\Big)$, $i \in I$, satisfying $0 \in \partial\langle y^*, f\rangle(\bar{x}) + \sum_{i\in I} \mu_i\, v_i^*$ and

$$\mu_i \max_{w\in\mathcal{V}} g_i(\bar{x},w) = 0, \quad i \in I. \tag{11}$$

Putting $\bar{y}^* := y^*$ and $\bar{\mu} := (\mu_1,\mu_2,\ldots,\mu_n)$, we have $(\bar{y}^*,\bar{\mu}) \in K^+ \setminus \{0\} \times \mathbb{R}^n_+$. Moreover, the inclusion $v \in \mathcal{V}_i(\bar{x})$ means that $g_i(\bar{x},v) = \max_{w\in\mathcal{V}} g_i(\bar{x},w)$ for all $i \in I$. Thus, it stems from (11) that $\mu_i\, g_i(\bar{x},v) = 0$, $i \in I$. So $(\bar{x},\bar{y}^*,\bar{\mu}) \in F_{MW}$. (i) As $(f,g)$ be type I pseudo convex at any $z \in X$, employing (i) of Theorem 2.6 gives us $\bar{f}(\bar{x},\bar{y}^*,\bar{\mu}) = f(\bar{x}) \not\prec \bar{f}(z,y^*,\mu)$ for each $(z,y^*,\mu) \in F_{MW}$. Hence $(\bar{x},\bar{y}^*,\bar{\mu}) \in \mathcal{S}^w(RD_{MW})$. To prove (ii), we proceed similarly to the part (i). □

## References


1. A. Ben-Tal, L. E. Ghaoui and A. Nemirovski, *Robust Optimization*, Princeton University Press, Princeton, 2009.
2. A. Ben-Tal and A. Nemirovski, *A selected topics in robust convex optimization*, Math. Program. Ser. B **112** (2008) 125–158.
3. J. Ide and E. Köbis, *Concepts of efficiency for uncertain multiobjective problems based on set order relations*, Math. Meth. Oper. Res. **80** (2014) 99–127.
4. R. Bokrantz and A. Fredriksson, *Necessary and sufficient conditions for Pareto efficiency in robust multiobjective optimization*, Eur. J. Oper. Res. **262** (2017) 682–692.
5. G. M. Lee and P. T. Son, *On nonsmooth optimality theorems for robust optimization problems*, Bull. Korean Math. Soc. **51** (1) (2014) 287–301.
6. T. D. Chuong, *Optimality and duality for robust multiobjective optimization problems*, Nonlinear Anal. **134** (2016) 127–143.
7. J. Chen, E. Köbis and J. C. Yao, *Optimality conditions and duality for robust nonsmooth multiobjective optimization problems with constraints*, J. Optim. Theory Appl. **181** (2019) 411–436.
8. M. Ehrgott, J. Ide and A. Schöbel, *Minmax robustness for multi-objective optimization problems*, Eur. J. Oper. Res. **239** (2014) 17–31.
9. M. Fakhar, M. R. Mahyarinia and J. Zafarani, *On nonsmooth robust multiobjective optimization under generalized convexity with applications to portfolio optimization*, Eur. J. Oper. Res. **265** (1) (2018) 39–48.
10. B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation. I: Basic Theory*, Springer, Berlin, 2006.

E-mail: m.saadati@edu.ikiu.ac.ir
E-mail: oveisiha@sci.ikiu.ac.ir

# Tensor Products and $BSE$-Algebras

Amir Hossein Sanatpour*
Department of Mathematics, Kharazmi University, Tehran, Iran
and Zahra Sadaat Hosseini
Department of Mathematics, Bu-Ali Sina University, Hamedan, Iran

ABSTRACT. In this paper, we investigate the $BSE$ property of tensor product $A\widehat{\otimes}_\alpha B$ of commutative Banach algebras $\mathcal{A}$ and $\mathcal{B}$. We show that if $A\widehat{\otimes}_\alpha B$ is a $BSE$-algebra, then $\mathcal{A}$ and $\mathcal{B}$ are $BSE$-algebras. In the special case, we investigate Banach algebras of vector-valued continuous functions on a compact Hausdorff space $X$, and also vector-valued polynomial Lipschitz algebras on a compact plane set $X$.

**Keywords:** $BSE$-Algebra, Tensor product, Commutative Banach algebra, Lipschitz algebra.
**AMS Mathematical Subject Classification [2010]:** 46B28, 46J15, 46J10.

## 1. Introduction

Let $\mathcal{A}$ be a commutative Banach algebra with maximal ideal space $\Phi_\mathcal{A}$ and $C_0(\Phi_\mathcal{A})$ denote the space of all continuous functions on $\Phi_\mathcal{A}$ vanishing at infinity. The algebra $\mathcal{A}$ is embedded in $C_0(\Phi_\mathcal{A})$ by considering the Gelfand transform $a \mapsto \widehat{a}$, where $\widehat{a}(\varphi) = \varphi(a)$ for each $\varphi \in \Phi_\mathcal{A}$. A commutative Banach algebra $\mathcal{A}$ is called *without order* if $a \in \mathcal{A}$ and $a\mathcal{A} = \{0\}$ implies that $a = 0$. Given a without order commutative Banach algebra $\mathcal{A}$, a bounded linear operator $T : \mathcal{A} \to \mathcal{A}$ is called a *multiplier* if $a(Tb) = T(ab)$ for all $a, b \in \mathcal{A}$. The set of all multipliers on $\mathcal{A}$ is denoted by $M(\mathcal{A})$ which is a commutative unital Banach subalgebra of $\mathcal{B}(\mathcal{A})$, the space of all bounded linear operators on $\mathcal{A}$ [7]. Larsen in [7] proved that for every $T \in M(\mathcal{A})$ there exists a unique bounded continuous function $\widehat{T}$ on $\Phi_\mathcal{A}$ such that $\widehat{(Tx)} = \widehat{T}\widehat{x}$ for all $x \in \mathcal{A}$. As an another definition of the multiplier algebra of $\mathcal{A}$, a complex-valued continuous function $T : \Phi_\mathcal{A} \to \mathbb{C}$ is a multiplier if $T \cdot \widehat{\mathcal{A}} \subseteq \widehat{\mathcal{A}}$, that is

$$\mathcal{M}(\mathcal{A}) = \{T : \Phi_\mathcal{A} \to \mathbb{C} \mid T \text{ is continuous and } T \cdot \widehat{\mathcal{A}} \subseteq \widehat{\mathcal{A}}\}.$$

A bounded continuous function $\sigma$ on $\Phi_\mathcal{A}$ is called a *$BSE$-function* if there exists a positive constant $\beta > 0$ such that for any finite numbers of $\varphi_1, \varphi_2, \ldots, \varphi_n$ in $\Phi_\mathcal{A}$ and any complex numbers $c_1, c_2, \ldots, c_n$, the following inequality holds:

$$|\sum_{i=1}^{n} c_i\sigma(\varphi_i)| \leq \beta\| \sum_{i=1}^{n} c_i\varphi_i\|_{\mathcal{A}^*}.$$

The *$BSE$-norm* of $\sigma$ is defined to be the infimum of all such $\beta$ in the above inequality and $C_{BSE}(\Phi_\mathcal{A})$ denotes the set of all $BSE$-functions. Takahasi and Hatori [8, Lemma 1] proved that $C_{BSE}(\Phi_\mathcal{A})$ with the $BSE$-norm is a commutative

*Speaker

semisimple Banach subalgebra of $C^b(\Phi_{\mathcal{A}})$, the space of all bounded continuous functions on $\Phi_{\mathcal{A}}$. The next definition is given by Takahasi and Hatori in [8].

DEFINITION 1.1. A without order commutative Banach algebra $\mathcal{A}$ is called a *BSE-algebra* if $\widehat{M(\mathcal{A})} = C_{BSE}(\Phi_{\mathcal{A}})$, where $\widehat{M(\mathcal{A})} = \{\widehat{T} : T \in M(\mathcal{A})\}$.

Bochner and Schoenberg in 1934 studied these algebras on the real line and then Eberlein in 1955 gave the extension for locally compact abelian groups $G$. Takahasi, Hatori, Kaniuth, Ulger and some other mathematicians studied this topic for the commutative Banach algebras, Banach function algebras and some other well-known algebras [1, 2, 8]. In this paper we study $BSE$ property of tensor product of two Banach algebras $\mathcal{A}$ and $\mathcal{B}$. We also investigate the $BSE$ property of tensor products in some special cases.

## 2. Main Results

For the normed spaces $X$, $Y$ and $Z$, let $B(X \times Y, Z)$ denote the vector space of all bilinear mappings from $X \times Y$ into $Z$. In the special case of $Z = \mathbb{C}$, we write $B(X \times Y)$ instead of $B(X \times Y, \mathbb{C})$. For each $x \in X$ and $y \in Y$, the linear functional $x \otimes y$ on $B(X \times Y)$ is given by

$$(x \otimes y)(T) = T(x, y),$$

for each bilinear form $T$ on $X \times Y$. The *tensor product* $X \otimes Y$ is the space of all linear functionals on $B(X \times Y)$ of the standard form

$$u = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i, \qquad (n \in \mathbb{N}, \lambda_i \in \mathbb{C}).$$

The space $X \otimes Y$ can be equipped with injective and projective norms defined as follows:

DEFINITION 2.1. For the Banach spaces $X$ and $Y$ with dual spaces $X^*$ and $Y^*$, *the injective* norm on $X \otimes Y$ is defined by

$$\|u\|_{\varepsilon} = \sup\{|\sum_{i=1}^{n} \varphi(x_i)\psi(y_i)| : \varphi \in B_{X^*}, \psi \in B_{Y^*}\},$$

where $\sum_{i=1}^{n} x_i \otimes y_i$ is any representation of $u$. Also, the *projective norm* on $X \otimes Y$ is defined by

$$\|u\|_{\pi} = \inf\{\sum_{i=1}^{n} \|x_i\|\|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i\}.$$

The completion of tensor product $X \otimes Y$ with respect to the injective and projective norm is denoted by $X\widehat{\otimes}_{\varepsilon}Y$ and $X\widehat{\otimes}_{\pi}Y$, respectively. We recall that a norm $\|\cdot\|_{\alpha}$ on $X \otimes Y$ is called a *cross norm* if for all $x \in X$ and $y \in Y$, $\|x \otimes y\|_{\alpha} = \|x\|_{\alpha}\|y\|_{\alpha}$. It is known that injective and projective norms are cross norms.

To study some special cases of tensor product spaces, we next introduce some well-known vector-valued function spaces.

Let $X$ be a compact Hausdorff space and $\mathcal{A}$ be a commutative Banach algebra. Then, $C(X,\mathcal{A})$ denotes the Banach algebra of all continuous maps from $X$ into $\mathcal{A}$ equipped with the uniform norm

$$\|f\|_\infty = \sup_{x\in X} \|f(x)\|_{\mathcal{A}}, \qquad (f \in C(X,\mathcal{A})).$$

When $(X,d)$ is a compact metric space, for each $0 < \alpha \le 1$ the vector-valued Lipschitz algebra $Lip^\alpha(X,\mathcal{A})$ is defined as follows:

$$Lip^\alpha(X,\mathcal{A}) = \{f : X \to \mathcal{A} \mid p_\alpha(f) < \infty\},$$

where

$$p_\alpha(f) = \sup_{\substack{x,y\in X\\ x\neq y}} \frac{\|f(x) - f(y)\|_{\mathcal{A}}}{d(x,y)^\alpha}.$$

It is known that $Lip^\alpha(X,\mathcal{A})$ is a Banach algebra equipped with the norm

$$\|f\|_\alpha = \|f\|_\infty + p_\alpha(f), \qquad (f \in Lip^\alpha(X,\mathcal{A})).$$

For every $0 < \alpha < 1$, the *little* vector-valued Lipschitz algebra $\ell ip^\alpha(X,\mathcal{A})$ is the closed subalgebra of $Lip^\alpha(X,\mathcal{A})$ consisting of those elements $f$ for which

$$\lim_{d(x,y)\to 0} \frac{\|f(x) - f(y)\|_{\mathcal{A}}}{d(x,y)^\alpha} = 0.$$

For more information about these algebras see [3] and the references therein.

For a unital commutative semisimple Banach algebra $\mathcal{A}$, F. Abtahi, Z. Kamali and M. Toutounchi in [1] proved that $Lip^\alpha(X,\mathcal{A})$ is a $BSE$-algebra if and only if $\mathcal{A}$ is a $BSE$-algebra. The statement of this result remains valid if we replace $Lip^\alpha(X,\mathcal{A})$ by $C(X,\mathcal{A})$. Since $C(X,\mathcal{A}) \cong C(X)\widehat{\otimes}_\varepsilon\mathcal{A}$, this interesting question arises that what is the relation between $BSE$ property of $\mathcal{A}\widehat{\otimes}_\varepsilon\mathcal{B}$ and $BSE$ properties of the algebras $\mathcal{A}$ and $\mathcal{B}$? We next show that if $\mathcal{A}\widehat{\otimes}_\varepsilon\mathcal{B}$ is a $BSE$-algebra, then $\mathcal{A}$ and $\mathcal{B}$ are $BSE$-algebras.

Before giving the next theorem, we recall that a *weak approximate identity* in a Banach algebra $\mathcal{A}$ is a net $\{e_i\}$ in $\mathcal{A}$ such that for every $\varphi \in \Phi_{\mathcal{A}}$ we have

$$\lim_i \varphi(e_i a) = \varphi(a), \qquad (a \in \mathcal{A}),$$

or $\lim_i \varphi(e_i) = 1$ [4]. A net $\{e_i\}$ in $\mathcal{A}$ is called an *approximate identity* if $\lim_i \| e_i a - a \|_{\mathcal{A}} = 0$ for all $a \in \mathcal{A}$. If in addition the net $\{e_i\}$ is bounded, it is said that $\mathcal{A}$ has a bounded (weak) approximate identity.

Note that for the commutative Banach algebras $\mathcal{A}$ and $\mathcal{B}$, the map

$$\Phi_{\mathcal{A}} \times \Phi_{\mathcal{B}} \longrightarrow \Phi_{\mathcal{A}\widehat{\otimes}_\alpha\mathcal{B}}, \qquad (\varphi,\psi) \mapsto \varphi\widehat{\otimes}_\alpha\psi,$$

is a homeomorphism for every algebra cross norm on $\mathcal{A}\widehat{\otimes}_\alpha\mathcal{B}$ [5, Theorem 2.11.2].

THEOREM 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital commutative Banach algebras and $\mathcal{A}\widehat{\otimes}_\alpha\mathcal{B}$ be a $BSE$-algebra for a cross norm $\|\cdot\|_\alpha$. Then, $\mathcal{A}$ and $\mathcal{B}$ are $BSE$-algebras.*

PROOF. Since the Banach algebra $\mathcal{A}$ is unital, it has a bounded weak approximate identity. Hence, by [8, Corollary 5], we have $\widehat{M(\mathcal{A})} \subseteq C_{BSE}(\Phi_{\mathcal{A}})$. In order to prove the converse, let $\sigma \in C_{BSE}(\Phi_{\mathcal{A}})$ and $a_0 \in \mathcal{A}$. Consider the function $\varrho : \Phi_{\mathcal{A}} \times \Phi_{\mathcal{B}} \to \mathbb{C}$ given by

$$\varrho(\varphi,\psi) = \sigma(\varphi), \qquad (\varphi \in \Phi_{\mathcal{A}}, \psi \in \Phi_{\mathcal{B}}).$$

Then, for every finite numbers of $c_1, \dots, c_n \in \mathbb{C}$ and $(\varphi_1, \psi_1), \dots, (\varphi_n, \psi_n) \in \Phi_{\mathcal{A}} \times \Phi_{\mathcal{B}}$ we get

$$|\sum_{i=1}^{n} c_i \varrho(\varphi_i, \psi_i)| \le c \| \sum_{i=1}^{n} c_i (\varphi_i, \psi_i) \|_{(\mathcal{A}\widehat{\otimes}_\alpha \mathcal{B})^*},$$

for some constant $c > 0$. It follows that $\varrho \in C_{BSE}(\Phi_{\mathcal{A}\widehat{\otimes}_\alpha \mathcal{B}})$ and so there exists an element $c_0 \otimes d_0 \in \mathcal{A}\widehat{\otimes}_\alpha \mathcal{B}$ such that

$$\varrho . \widehat{a_0 \otimes e_{\mathcal{B}}} = \widehat{c_0 \otimes d_0}. \tag{1}$$

Considering the left and right side of (1) for each $(\varphi, \psi) \in \Phi_{\mathcal{A}} \times \Phi_{\mathcal{B}}$ implies that $\sigma(\varphi)\varphi(a_0) = \varphi(c_0)\psi(d_0)$. Now, let $\psi$ be a fixed element of $\Phi_{\mathcal{B}}$ and $\lambda = \psi(d_0)$, then

$$\sigma(\varphi)\widehat{a_0}(\varphi) = \lambda \widehat{c_0}(\varphi) = \widehat{\lambda c_0}(\varphi).$$

Hence, $\sigma \in \mathcal{M}(\mathcal{A})$ and therefore $\mathcal{A}$ is a $BSE$-algebra. Similarly, one can prove that $\mathcal{B}$ is a $BSE$-algebra. □

Note that the inverse of Theorem 2.2 arises the following important question that is still open to the best of our knowledge.

Question 2.3. Let $\mathcal{A}$ and $\mathcal{B}$ be $BSE$-algebras. Is $\mathcal{A}\widehat{\otimes}_\alpha \mathcal{B}$ a $BSE$-algebra for any cross-norm $\| \cdot \|_\alpha$?

By applying a different approach from the already known results, in the next corollary we show that the converse of Theorem 2.2 is also valid for the tensor product $C(X, \mathcal{A}) \cong C(X)\widehat{\otimes}_\varepsilon \mathcal{A}$.

Corollary 2.4. *Let $X$ be a compact Hausdorff space and $\mathcal{A}$ be a unital commutative Banach algebra. Then, $C(X, \mathcal{A})$ is a $BSE$-algebra if and only if $\mathcal{A}$ is a $BSE$-algebra.*

Proof. Since $C(X, \mathcal{A}) \cong C(X)\widehat{\otimes}_\varepsilon \mathcal{A}$, if $C(X, \mathcal{A})$ is a $BSE$-algebra then $\mathcal{A}$ is a $BSE$-algebra by Theorem 2.2. Conversely, assume that $\mathcal{A}$ is a $BSE$-algebra. Then, $\mathcal{A}$ and so $C(X, \mathcal{A})$ has bounded weak approximate identity [6, Page 520] and therefore

$$\mathcal{M}(C(X, \mathcal{A})) \subseteq C_{BSE}(X \times \Phi_{\mathcal{A}}),$$

by [8, Corollary 5]. Let $\sigma \in C_{BSE}(X \times \Phi_{\mathcal{A}})$. By the definition of $\mathcal{M}(\mathcal{A})$, it is sufficient to find $g \in C(X, \mathcal{A})$ such that $\sigma = \widehat{g}$. For an arbitrary and fixed $x \in X$, consider $\sigma_x : \Phi_{\mathcal{A}} \longrightarrow \mathbb{C}$ given by

$$\sigma_x(\varphi) = \sigma(x, \varphi), \qquad (x \in X, \varphi \in \Phi_{\mathcal{A}}).$$

By applying a similar approach as in the proof of Theorem 2.2 for every $\varphi_1, \dots, \varphi_n \in \Phi_{\mathcal{A}}$ and complex numbers $c_1, \dots, c_n$ we get

$$|\sum_{i=1}^{n} c_i \sigma_x(\varphi_i)| \le d \| \sum_{i=1}^{n} c_i \varphi_i \|_{\mathcal{A}^*},$$

for some constant $d > 0$. Hence, $\sigma_x \in C_{BSE}(\Phi_{\mathcal{A}})$ and therefore there exists $a_x \in \mathcal{A}$ such that $\sigma_x = \widehat{a_x}$. Define the function $g : X \longrightarrow \mathcal{A}$ by $g(x) = a_x$ for each $x \in X$. Then, one can get $g \in C(X, \mathcal{A})$ and $\sigma = \widehat{g}$ which completes the proof. □

For a compact plane set $X$ and a unital commutative Banach algebra $\mathcal{A}$, the algebra of all polynomials on $X$ with coefficients in $\mathcal{A}$ is denoted by $P_0(X, \mathcal{A})$. It is known that

$$P_0(X, \mathcal{A}) \subseteq \ell ip^\alpha(X, \mathcal{A}) \subseteq Lip^\alpha(X, \mathcal{A}).$$

Hence, one can define the vector-valued polynomial Lipschitz algebra $Lip_P^\alpha(X, \mathcal{A})$ as the closed subalgebra of $Lip^\alpha(X, \mathcal{A})$ generated by $P_0(X, \mathcal{A})$. Similarly, the vector-valued polynomial little Lipschitz algebra $\ell ip_P^\alpha(X, \mathcal{A})$ is defined, which is equal to $Lip_P^\alpha(X, \mathcal{A})$ in the case of $0 < \alpha < 1$.

As our final results, by applying Theorem 2.2 and [**3**, Theorem 3.5], we give the following necessary conditions for the algebras $Lip_P^\alpha(X, \mathcal{A})$ and $\ell ip_P^\alpha(X, \mathcal{A})$ to be $BSE$-algebras.

THEOREM 2.5. *Let $X$ be a compact plane set and $\mathcal{A}$ be a unital commutative Banach algebra.*

(i) *If $0 < \alpha \leq 1$ and $Lip_P^\alpha(X, \mathcal{A})$ is a $BSE$-algebra, then $Lip_P^\alpha(X, \mathbb{C})$ and $\mathcal{A}$ are $BSE$-algebras,*

(ii) *If $0 < \alpha < 1$ and $\ell ip_P^\alpha(X, \mathcal{A})$ is a $BSE$-algebra, then $\ell ip_P^\alpha(X, \mathbb{C})$ and $\mathcal{A}$ are $BSE$-algebras.*

## References

1. F. Abtahi, Z. Kamali and M. Toutounchi, *The Bochner-Schoenberg-Eberlein property for vector-valued Lipschitz algebras*, J. Math. Anal. Appl. **479** (1) (2019) 1172–1181.
2. H.G. Dales and A. Ulger, *Approximate identities in Banach function algebras*, Studia Math. **226** (2) (2015) 155–187.
3. T. G. Honary, A. Nikou and A. H. Sanatpour, *On the character space of vector-valued Lipschitz algebras*, Bull. Iranian Math. Soc. **40** (6) (2014) 1453–1468.
4. J. Inoue and Sin-Ei Takahasi, *Constructions of bounded weak approximate identities for Segal algebras on $R^n$*, Harmonic/Analytic Function Spaces and Linear Operators (1998) 30–38.
5. E. Kaniuth, *A Course in Commutative Banach Algebras*, Springer-Verlag, New York, 2009.
6. T. W. Palmer, *Banach Algebras and the General Theory of $*$-Algebras*, Vol. I, Cambridge University Press, Cambridge, 1994.
7. R. Larsen, *An Introduction to the Theory of Multipliers*, Springer-Verlag, New York, 1971.
8. Sin-Ei. Takahasi and O. Hatori, *Commutative Banach algebras which satisfy a Bochner-Schenberg-Eberlin-type theorem*, Proc. Amer. Math. Soc. **110** (1) (1990) 149–158.

E-mail: a_sanatpour@khu.ac.ir; a.sanatpour@gmail.com
E-mail: z.hoseini@basu.ac.ir; zahrahoseinizh3@gmail.com

# A New Subclass of Univalent Functions Associated with the Limaçon Domain

Vali Soltani Masih*

Department of Mathematics, Payame Noor University, Tehran, Iran

Abstract. Let $\mathcal{A}$ denote the family of analytic and normalized functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the unit disk $\mathbb{D} := \{z\colon |z| < 1\}$, such that $f(z) = u + iv$ lies in a domain bounded by a Limaçon

$$\left[(u-1)^2 + v^2 - s^2t^2\right]^2 = (t-s)^2 \left[(u-1-st)^2 + v^2\right],$$

where $-1 \le s < t \le 1$ and $0 < 2|st| \le t - s$. In this work, we introduce a family of analytic univalent functions in the open unit disc $\mathbb{D}$. For functions belonging to this class, we derive several properties such as bounded for real part and the order of starlikeness and convexity.

**Keywords:** Univalent functions, Subordination, Starlike and convex functions, Domain bounded by Limaçon.

**AMS Mathematical Subject Classification [2010]:** 30C45, 30C80.

## 1. Introduction

Geometric function theory is a branch of complex analysis that proceedings and studies the geometric properties of the analytic functions. The foundation of the geometric function theory is the theory of univalent functions which is considered as one of the active fields of the current research. Most of this field is concerned with the class $\mathcal{S}$ of functions analytic and univalent in the unit disc $\mathbb{D}$. One of the most famous problems in this field was Bieberbach conjecture. For many years this problem was a challenge to the mathematicians and motivated the development of many new techniques in complex analysis. In the course of investigating Bieberbach conjecture, new classes of analytic and univalent functions such as classes of convex and starlike functions were defined and some helpful properties of these classes were comprehensively studied [3, 5].

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad for \quad z \in \mathbb{D},$$

which are analytic in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$. A functions $f \in \mathcal{A}$ is univalent if $f(z_1) \ne f(z_2)$ for all $z_1, z_2 \in \mathbb{D}$ with $z_1 \ne z_2$. The subclass of $\mathcal{A}$ consisting of all univalent functions $f$ in $\mathbb{D}$, is denoted by $\mathcal{S}$. A functions $f \in \mathcal{S}$ is said to belong to the class $\mathcal{ST}(\beta)$, called starlike functions of order $0 \le \beta < 1$, if $\Re\{zf'(z)/f(z)\} > \beta$, and is said to belong to the class $\mathcal{CV}(\beta)$, called convex functions of order $0 \le \beta < 1$, if $\Re\{1 + zf''(z)/f'(z)\} > \beta$ [2].

*Speaker

Observe that $\mathcal{ST} := \mathcal{ST}(0)$ and $\mathcal{CV} := \mathcal{CV}(0)$ represent standard starlike and convex univalent functions, respectively. Let $f$ and $g$ be analytic in $\mathbb{D}$. Then the function $f$ is said to subordinate to $g$ in $\mathbb{D}$ written by $f(z) \prec g(z)$, if there exists a self-map function $\omega(z)$ which is analytic in $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{D})$, and such that $f(z) = g(\omega(z))$ $(z \in \mathbb{D})$. If $g$ is univalent in $\mathbb{D}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$ [1].

Motivated with the geometry of the image $f(\mathbb{D})$, we introduce the classes of functions $f$ where $f(z) = u + iv$ belonging to a bounded domain by a Limaçon defined by

$$\partial\mathfrak{D}(t,s) = \Big\{u + \mathrm{i}v \in \mathbb{C}\colon \left[(u-1)^2 + v^2 - t^2s^2\right]^2 = (t-s)^2\left[(u-1-st)^2 + v^2\right]\Big\}, \tag{1}$$

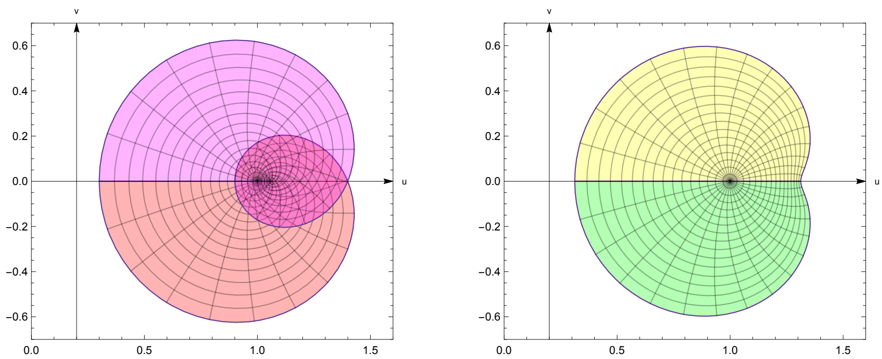
where $-1 \leq s < t \leq 1$.

## 2. Main Results

To state our aim, we introduce a family of analytic functions $\mathscr{L}_{t,s}(\cdot)$ defined by

$$\mathscr{L}_{t,s}(z) = (1 - sz)(1 + tz) \quad \textit{for} \quad z \in \mathbb{D}, \tag{2}$$

for some $-1 \leq s < t \leq 1$ and $st \neq 0$, such that maps the unit disk $\mathbb{D}$ onto a domain bounded by Limaçon $\partial\mathfrak{D}(t,s)$ given in (1) (See Figure 1). In fact, if we




(A) $s = 0.5, t = 0.8$ (B) $s = 0.25, t = 0.75$

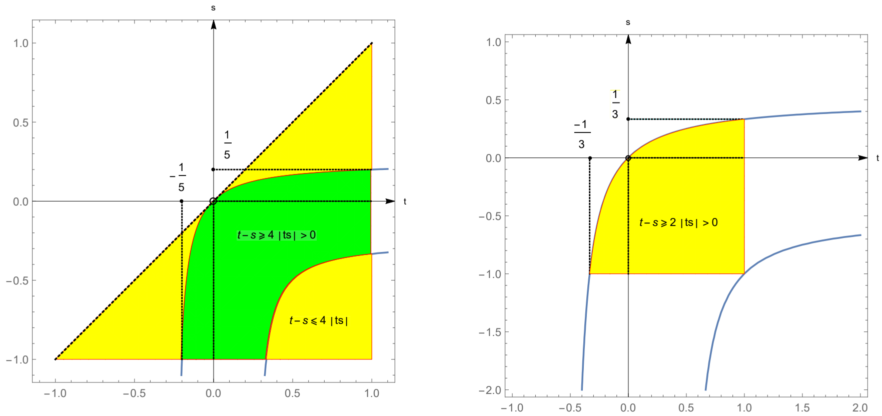
FIGURE 1. The image of $\mathbb{D}$ under $\mathscr{L}_{t,s}$.

take $z = \mathrm{e}^{\mathrm{i}\theta}$; $0 \leq \theta < 2\pi$, then

$$\begin{aligned}(1-sz)(1+tz) &= \left(1 - se^{i\theta}\right)\left(1 + te^{i\theta}\right) \\ &= (1 + (t-s)\cos\theta - ts\cos 2\theta) + \mathrm{i}((t-s)\sin\theta - ts\sin 2\theta).\end{aligned}$$

Let us denote $u(\theta) = \Re\left\{\mathscr{L}_{t,s}\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right\}$ and $v(\theta) = \mathrm{Im}\left\{\mathscr{L}_{t,s}\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right\}$ for $0 \leq \theta < 2\pi$. Then

$$u(\theta) = 1 + (t-s)\cos\theta - ts\cos 2\theta, \qquad v(\theta) = (t-s)\sin\theta - ts\sin 2\theta.$$




(A) Location of areas $t-s>4|ts|>0$, $t-s\leq 4|ts|$ and line $t=s$.

(B) Location of area $t-s\geq 2|ts|>0$.

FIGURE 2. Location of $-1\leq s<t\leq 1$.

The min or max of $u(\theta)$ are attained at the critical points of the above function, equivalently

$$u'(\theta)=(4ts\cos\theta+s-t)\sin\theta=0. \tag{3}$$

The previous expression has only critical points are $\theta=0,\theta=\pi$ and the solution of equation $\cos\theta_1=(t-s)/(4ts)$. If $t-s\leq 4|ts|$, then

$$\min_{0\leq\theta<2\pi}\Re\left\{\mathscr{L}_{t,s}\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right\}=\begin{cases}u(\theta_1)=1+ts+\dfrac{(t-s)^2}{8ts}, & \textit{for}\quad ts<0,\\ u(\pi)=(1+s)(1-t), & \textit{for}\quad ts>0,\end{cases}$$

and

$$\max_{0\leq\theta<2\pi}\Re\left\{\mathscr{L}_{t,s}\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right\}=\begin{cases}u(\theta_1), & \textit{for}\quad ts>0,\\ u(0), & \textit{for}\quad ts<0.\end{cases}$$

For $t-s>4|ts|$, the expression (3) has only critical points are $\theta=0,\theta=\pi$. Thus (See Figure 2a)

$$\min_{0\leq\theta<2\pi}\Re\left\{\mathscr{L}_{t,s}\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right\}=u(\pi),$$

and

$$\max_{0\leq\theta<2\pi}\Re\left\{\mathscr{L}_{t,s}\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right\}=u(0).$$

The above discussion can be summarized as follows.

THEOREM 2.1. *Let $\mathscr{L}_{t,s}(\cdot)$ be a function defined by* (2). *Then*

$$\min_{z\in\mathbb{D}}\Re\left\{\mathscr{L}_{t,s}(z)\right\}=m_0(t,s)=\begin{cases}1+ts+\dfrac{(t-s)^2}{8ts}, & \textit{for}\quad ts<0,\ t-s\leq 4|ts|,\\ (1+s)(1-t), & \textit{otherwise},\end{cases}$$

$$\max_{z\in\mathbb{D}} \Re\{\mathscr{L}_{t,s}(z)\} = \begin{cases} 1+ts+\dfrac{(t-s)^2}{8ts}, & \textit{for} \quad ts>0,\ t-s\le 4|ts|, \\ (1-s)(1+t), & \textit{otherwise}, \end{cases}$$

$$\mathscr{L}_{t,s}(\mathbb{D}) = \mathfrak{D}(t,s) = \left\{u+iv\colon \left[(u-1)^2+v^2-t^2s^2\right]^2 < (t-s)^2\left[(u-1-ts)^2+v^2\right]\right\}.$$

Due to the fact that the function $z+a_2z^2$ is univalent and starlike if and only if $|a_2|\le 1/2$ and convex if and only if $|a_2|\le 1/4$ [1], we conclude the following results.

THEOREM 2.2. *Let*

$$g(z) = \frac{\mathscr{L}_{t,s}(z)-1}{t-s} = z - \frac{ts}{t-s}z^2, \quad \textit{for} \quad -1\le s<t\le 1,$$

*where the function $\mathscr{L}_{t,s}(z)$ defined by (2). Then $g(z)$ is univalent if and only if $2|ts|\le t-s$ (See Figure 2b) and*

$$g(z)\in \mathcal{ST}\left(\frac{t-s-2|ts|}{t-s-|ts|}\right) \iff 0<2|ts|\le t-s,$$

*and*

$$g(z)\in \mathcal{CV}\left(\frac{t-s-4|ts|}{t-s-2|ts|}\right) \iff 0<4|ts|\le t-s.$$

By Theorem 2.2, for $s\in\left[-1,\frac{1}{3}\right]$, the functions $\mathscr{L}_{1,s}(z)=(1-sz)(1+z)$ are starlike and for $s\in\left[-1,\frac{1}{5}\right]$, the functions $\mathscr{L}_{1,s}(z)=(1-sz)(1+z)$ are convex.

Also, from Theorem 2.2, It can be seen that the smallest disk with center $(1,0)$ that contains $\mathscr{L}_{t,s}(z)$ and the largest disk with center at $(1,0)$ contained in $\mathscr{L}_{t,s}(z)$ are (See Figure 3).

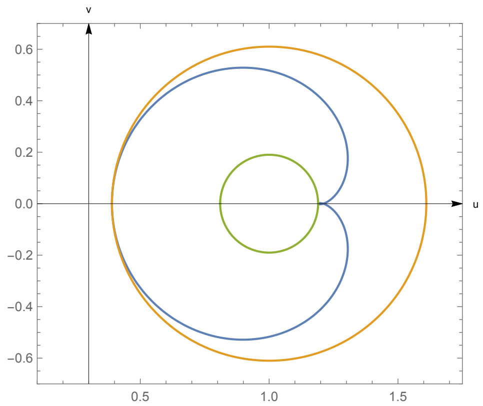



FIGURE 3. The image of $\partial\mathbb{D}$ under $\mathscr{L}_{t,s}(z)$, $[1-(1+s)(1-t)]z+1$ and $[(1-s)(1+t)-1]z+1$ for $s=0.3$, $t=0.7$.

$$\{w\in\mathbb{C}\colon |w-1|<1-(1+s)(1-t)\}\subset \mathscr{L}_{t,s}(\mathbb{D})\subset\{w\in\mathbb{C}\colon |w-1|<(1-s)(1+t)-1\}.$$

Definition 2.3. Let $\mathcal{ST}_L(t,s)$ and $\mathcal{CV}_L(t,s)$ denote the subfamily of $\mathcal{A}$ consisting of the functions $f$, satisfying the condition

$$(4)\quad \frac{zf'(z)}{f(z)} \prec \mathscr{L}_{t,s}(z), \quad 1+\frac{zf''(z)}{f'(z)} \prec \mathscr{L}_{t,s}(z) \quad \textit{for} \quad z\in\mathbb{D},\ 0<2|ts|\le t-s,$$

respectively. From Theorem 2.1, we obtain

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > m_0(t,s), \qquad \Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > m_0(t,s) \quad \textit{for} \quad z\in\mathbb{D},$$

where $f\in\mathcal{ST}_L(t,s)$ or $f\in\mathcal{CV}_L(t,s)$, respectively and $m_0(t,s)$ is taken from Theorem 2.1 and assumed to be positive. Geometrically, the conditions (4) means that the expression $zf'(z)/f(z)$ or $1+zf''(z)/f'(z)$ lies in a domain bounded by the Limaçon $\partial\mathfrak{D}(t,s)$. A special case of the function $\mathscr{L}_{t,s}(z)$ and the classes $\mathcal{ST}_L(t,s)$ and $\mathcal{CV}_L(t,s)$ where $s=-t$ considered in [4].

Let us mention some important consequences of the Theorem 2.1 and Theorem 2.2. According to the Theorem 2.2, the functions $\mathscr{L}_{t,s}(\cdot)$ has symmetric domain with respect to the real axis and starlike and convex with respect to $\mathscr{L}_{t,s}(0)=1$, $\mathscr{L}'_{t,s}(0)=t-s>0$.

**Acknowledgment**

The author would like to thank the anonymous reviewer for his/her comments.

**References**

1. P. L. Duren, *Univalent Functions*, Springer-Verlag, New York, 2001.
2. I. S. Jack, *Functions starlike and convex of order $\alpha$*, J. Lond. Math. Soc. **23** (1971) 469–474.
3. S. Kanas, V. S. Masih and A. Ebadian, *Relations of a planar domain bounded by hyperbola with family of holomorphic functions*, J. Inequal. Appl. **246** (2019) 246.
4. V. S. Masih and S. Kanas, *Subclasses of starlike and convex functions associated with the Limaçon domain*, Symmetry **12** (6) (2020) 942.
5. G. Singh, G. Singh and G. Singh, *A new subclass of univalent functions*, Ufa Math. J. **11** (1) (2019) 133–140.

E-mail: masihvali@gmail.com; v_soltani@pnu.ac.ir

# Projectivity of Some Banach Spaces Related to Locally Compact Groups

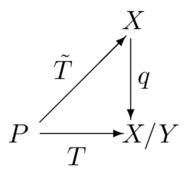
Sima Soltani Renani*
Department of Mathematical Sciences, Isfahan University of Technology, Isfahan, Iran
and Zahra Yari
Department of Mathematical Sciences, Isfahan University of Technology, Isfahan, Iran

Abstract. For a locally compact group $G$ we investigate some geometric properties of Banach spaces $L_0^\infty(G)$ and $L_0^\infty(G)^*$.
**Keywords:** Projective Banach space, Phillips property, Locally compact group.
**AMS Mathematical Subject Classification [2010]:** 22B20, 22D05, 22D15.

## 1. Introduction

Studying the geometrical properties of Banach spaces is one of the most important research fields in the theory of Banach spaces. Projectivity is one of these properties that was paid attention so far. Here is the definition of a projective Banach space. Note that for Banach spaces $V$ and $W$, we denote the space of all linear and bounded operators from $V$ to $W$ by $\mathcal{L}(V, W)$.

Definition 1.1. A Banach space $P$ is called projective if for every Banach space $X$, a closed subspace $Y$ and an $\epsilon > 0$, every contractive operator $T \in \mathcal{L}(P, X/Y)$ lifts to a bounded operator $\tilde{T} \in \mathcal{L}(P, X)$ with $\|\tilde{T}\| \leq (1+\epsilon)$ such that the following digram commutes.




where $q : X \longrightarrow X/Y$ is the canonical surjection.

Grothendieck completely characterized projective Banach spaces by showing that $P$ is projective if and only if $P$ is isometrically isomorphic to $\ell^1(\Omega)$ for some $\Omega$ [5]. There is a rich literature connected with this concept; see for example [3].

On the other hand, several mathematicians defined new geometric properties of Banach spaces such as the Schur, Phillips, Dunford-Pettis, and so on; see [5] and [2]. Moreover, projectivity of Banach spaces has various relations with another

*Speaker

geometric properties. For instance, we can see some of these relations in [3] and [5]. Freedman and Ülger in [4] introduced the Phillips and the weak Phillips properties and, then, Ülger in [9] presented further results on the weak Phillips property. The authors in [4] also studied the Schur property and gained its relation to the Phillips and Dunford-Pettis properties.

In this paper we aim to study the Phillips property and projectivity of certain Banach spaces related to a locally compact group.

## 2. Main Results

In this section, in order to present our main results, we first bring some definitions.

Definition 2.1. Let $X$ be a Banach space.

i) $X$ is said to have the Schur property if any weakly convergent sequence is norm convergent,

ii) $X$ has the Phillips property if the canonical projection $p : X^{***} \longrightarrow X^*$ is sequentially weak*-norm continuous.

We consider the following fact about the Phillips property of a Banach space [4].

Proposition 2.2. *Let $X$ be a Banach space. If $X$ has the Phillips property, then it is not complemented in any dual space.*

We also have the following Theorem about the Schur and the Phillips property of a $C^*$-algebra.

Theorem 2.3. [4, Lemma 3.1] *Let $\mathcal{A}$ be a $C^*$-algebra. Then the following statements are equivalent:*

i) *$\mathcal{A}$ has the Phillips property,*

ii) *$\mathcal{A}^*$ has the Schur property.*

Let $G$ be a locally compact group with the left Haar measure $\lambda$, $L^\infty(G)$ be the space of all measurable bounded functions with essential supremum norm, $M(G)$ be the measure algebra of all bounded regular Borel measures on $G$ and $L^1(G)$ be the group algebra of all $\lambda$-integrable functions on $G$.

The following simple Proposition is about projectivity of $M(G)$ and $L^1(G)$.

Proposition 2.4. [3, Proposition 3.2] *Let $G$ be a locally compact group. Then the following statements are equivalent.*

i) *$M(G)$ is projective,*

ii) *$L^1(G)$ is projective,*

iii) *$G$ is discrete.*

Consider a locally compact group $G$. $L_0^\infty(G)$ is the space of all $\phi \in L^\infty(G)$ that vanish at infinity: that is, for each $\epsilon > 0$, there is a compact subset $K$ of $G$ for which $\|\phi\chi_{G\setminus K}\| < \epsilon$, where $\chi_{G\setminus K}$ denotes the characteristic function of $G\setminus K$ on $G$. $L_0^\infty(G)$ is the closed subspace of $L^\infty(G)$ that was introduced and studied in [7]. Further survey on this space can be find in [1] and [8]. The dual space of $L_0^\infty(G)$ is denoted by $L_0^\infty(G)^*$. It was shown in [7] that $L_0^\infty(G)^*$ is a

Banach algebra, $L^1(G)$ is as a closed subspace of $L_0^\infty(G)^*$ and if $G$ is discrete, then $L^1(G) = L_0^\infty(G)^*$. Furthermore, $L_0^\infty(G)$ is also a $C^*$-algebra. In addition, we consider the space $C_0(G)$ of all continuous functions on $G$ vanishing at infinity. Note that if the group $G$ is discrete, then $L_0^\infty(G) = C_0(G)$.

Eventually, we give some mentioned properties of $L_0^\infty(G)$, $L_0^\infty(G)^*$ and $C_0(G)$ and relate them to locally compact group $G$. Our following theorem links Theorem 2.3 and Proposition 2.4 about the Banach space $L_0^\infty(G)$.

Theorem 2.5. *Let $G$ be a locally compact group. Then the following statements are equivalent.*

i) *$L_0^\infty(G)^*$ is a projective Banach space,*
ii) *$L^1(G)$ is a projective Banach space,*
iii) *$C_0(G)$ has the Phillips property,*
iv) *$L_0^\infty(G)$ has the Phillips property,*
v) *$G$ is discrete.*

Proof. (i) $\Rightarrow$ (ii): Suppose that $L_0^\infty(G)^*$ is projective. By [3] and [5], we conclude that every projective Banach space has the Schur property. So $L_0^\infty(G)^*$ has the Schur property. Moreover, $L^1(G)$ is a closed subspace of $L_0^\infty(G)^*$ and the Schur property is inherited by closed subspaces ([3] and [5]). Thus $L^1(G)$ has the Schur property. By [6, Theorem 5.1] we conclude that $G$ is discrete and therefore $L_0^\infty(G)^* = L^1(G)$. Eventually, projectivity of $L^1(G)$ is implied by projectivity of $L_0^\infty(G)^*$.
(ii) $\Rightarrow$ (iii): Let $L^1(G)$ be projective. Since every projective Banach space has the Schur property, it follows that $L^1(G)$ has also the Schur property. Theorem 5.1 from [6] now say that $G$ is a discrete group. Thus in this case, $c_0 = C_0(G)$. By implying [4] we deduce that $c_0$ has the Phillips property and therefore $C_0(G)$ has also the Phillips property.

(iii) $\Rightarrow$ (iv): Suppose that $C_0(G)$ has the Phillips property. Since $C_0(G)$ is a $C^*$-algebra, by Theorem 2.3 that $C_0(G)^* = M(G)$ has the Schur property. By using [6, Theorem 5.1] we deduce that $G$ is discrete and therefore $C_0(G) = L_0^\infty(G)$. Thus $L_0^\infty(G)$ has the Phillips property.

(iv) $\Rightarrow$ (v): Assume that $L_0^\infty(G)$ has the Phillips property. Since $L_0^\infty(G)$ is a $C^*$-algebra, it follows by Theorem 2.3 that $L_0^\infty(G)^*$ has the Schur property. $L^1(G)$ is a closed subspace of $L_0^\infty(G)^*$. Moreover, the Schur property is inherited by closed subspaces. Thus $L^1(G)$ has the Schur property. Finally, discreteness of $G$ is implied by [6, Theorem 5.1].

(v) $\Rightarrow$ (i): Let $G$ be a discrete locally compact group. Therefore $L_0^\infty(G)^* = L^1(G)$ and by Theorem 2.4, $L^1(G)$ is projective. So $L_0^\infty(G)^*$ is also projective. □

If $G$ is a discrete locally compact group, then previous theorem says that $L_0^\infty(G)$ has the Phillips property. Therefore by using Proposition 2.2 we conclude that it is not complemented in any dual space. In other words, we have:

Corollary 2.6. *Let $G$ be a locally compact group. If $L_0^\infty(G)$ is complemented in a dual space, then $G$ is not discrete.*

## References

1. F. Bahrami, R. Nasr-Isfahani and S. Soltani Renani, *Homological properties of certain Banach modules over group algebras*, Proc. Edinb. Math. Soc. (2) **54** (2011) 321–328.
2. J. Diestel, *Sequences and Series in Banach Spaces*, Springer, New York, 1993.
3. B. Forrest, *Projective operator spaces, almost periodicity and completely complemented ideals in the Fourier algebra*, Rocky Mountain J. Math. **41** (2011) 155–176.
4. W. Freedman and A. Ülger, *The Phillips properties*, Proc. Amer. Math. Soc. **128** (2000) 2137–2145.
5. A. Grothendieck, *Une caractérisation vectorielle-métrique des espaces* $L^1$, Canad. J. Math. **7** (1955) 552–561.
6. A. T. M. Lau and A. Ülger, *Some geometric properties on the Fourier and Fourier-Stieltjes algebras of locally compact groups, Arens regularity and related problems*, Trans. Amer. Math. Soc. **337** (1993) 321–359.
7. A. T. M. Lau and J. Pym, *Concerning the second dual of the group algebra of a locally compact group*, J. London Math. Soc. (2) **41** (1990) 445–460.
8. S. Maghsoudi and R. Nasr-Isfahani, *On the maximal and minimal left ideals of certain Banach algebras on locally compact groups*, Results Math. **62** (2012) 157–165.
9. A. Ülger, *The weak Phillips property*, Colloq. Math. **87** (2001) 147–158.

E-mail: simasoltani@iut.ac.ir
E-mail: zahra.yari@math.iut.ac.ir

# $\varphi$-Connes Module Amenability of Dual Banach Algebras and $\varphi$-Splitting

Ali Ghaffari
Faculty of Mathematical Sciences, University of Semnan, Semnan, Iran
and Ebrahim Tamimi*
Department of Mathematics, University of Semnan, Semnan, Iran

Abstract. In this talk, we define $\varphi$-Connes module amenability of a dual Banach algebra $\mathcal{A}$, where $\varphi$ is a $\omega^*$-continuous bounded module homomorphism from $\mathcal{A}$ onto itself. We obtain the relation between $\varphi$-Connes module amenability of $\mathcal{A}$ and $\varphi$-splitting of the certain short exact sequence. We show that if $S$ is a weakly cancellative inverse semigroup with subsemigroup $E_S$ of idempotents and $l^1(S)$ as a Banach module over $l^1(E_S)$ is $\chi$-Connes module amenable, then the short exact sequence is $\chi$-splitting that $\chi$ is a $\omega^*$-continuous bounded module homomorphism from $l^1(S)$ onto itself.

**Keywords:** Dual Banach algebra, Connes module amenability, Short exact sequence, Semigroup algebra, $\varphi$-Splitting.
**AMS Mathematical Subject Classification [2010]:** 22D15, 43A10.

## 1. Introduction

In [6], the Connes amenability of certain Banach algebras in terms of normal virtual diagonals is characterized by Effros. Ghaffari and Javadi in [7], investigated $\phi$-Connes amenability for dual Banach algebras, where $\phi$ is an homomorphism from a Banach algebra on $\mathbb{C}$. Also, several characterizations of $\widehat{\chi}$-Connes amenability of semigroup algebras were introduced by these two authors, where $\chi$ is a nonzero bounded continuous character on unital weakly cnacellative semigroup $S$ and the map $\widehat{\chi}$ is defined on semigroup algebra $l^1(S)$. Weak module amenability for semigroup algebras is studied by Amini and Ebrahimi bagha in [1].

Recently, in [8], Ghaffari et al. investigated $\psi$-Connes module amenability of dual Banach algebras that $\psi$ is a $\omega^*$-continuous bounded module homomorphism from a Banach algebra on itself. In [5, pro 4.4], the author proved that a Banach algebra is Connes amenable if and only if the short exact sequence splits. In [2], the concept of module amenability for Banach algebras is introduced. Also, it is proved that when S is an inverse semigroup with subsemigroup $E_S$ of idempotents, then $l^1(S)$ as a Banach module over $\mathcal{U} = l^1(E_S)$ is module amenable if and only if $S$ is amenable. For more information and details of module amenability, we may refer the reader to [2, 3].

In this talk, we study the relation between $\varphi$-splitting and $\varphi$-Connes module amenability, where $\varphi$ is a $\omega^*$-continuous bounded module homomorphism from Banach algebra $\mathcal{A}$ onto $\mathcal{A}$. In fact, we give a characterization of $\varphi$-Connes module amenability of a dual Banach algebra in terms of so-called $\varphi$-splitting of the certain

*Speaker

short exact sequences (Theorem 2.8). Also, the mentioned concepts and details are shown for semigroup algebras in Theorem 2.10. In Theorem 2.9, by letting that $\mathcal{A}$ and $\mathcal{B}$ are $\varphi$ and $\psi$-Connes module amenable Banach algebras respectively, that both of $\varphi : \mathcal{A} \to \mathcal{A}$ and $\psi : \mathcal{B} \to \mathcal{B}$, are $\omega^*$-continuous bounded module homomorphisms, we show that this property is transferred from $\mathcal{A}$ and $\mathcal{B}$ to the special tensor product of their. In finally, it is presented a corollary and an example in this direction.

A Banach $\mathcal{A}$-bimodule $E$ is dual if there is a closed submodule $E_* \subseteq E^*$ such that $E = (E_*)^*$. We say $E_*$ predual of $E$. Throughout the talk, $\Delta(\mathcal{A})$ and $\Delta_{\omega^*}(\mathcal{A})$ will denote the sets of all homomorphisms and $\omega^*$-continuous homomorphisms from the Banach algebra $\mathcal{A}$ onto $\mathbb{C}$, respectively.

## 2. Main Results

The following definitions are analogue to [8]. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra, and $\mathcal{U}$ be a Banach algebra such that $\mathcal{A}$ is a Banach $\mathcal{U}$-bimodule via,

$$\alpha.(ab) = (\alpha.a).b, \quad (\alpha\beta).a = \alpha.(\beta.a), \qquad (a, b \in \mathcal{A}, \alpha, \beta \in \mathcal{U}).$$

A discrete semigroup $S$ is called an inverse semigroup if for each $t \in S$ there is a unique element $t^* \in S$ such that $tt^*t = t$ and $t^*tt^* = t^*$. The set of idempotent elements of $S$ is denoted by $E_S = \{e \in S; \;\; e = e^* = e^2\}$.
Let $E$ be a dual Banach $\mathcal{A}$-bimodule. $E$ is called normal if for each $x \in E$, the maps

$$\mathcal{A} \to E; \qquad a \to a.x, \quad a \to x.a,$$

are $\omega^*$- continuous. If moreover $E$ is a $\mathcal{U}$-bimodule such that for $a \in \mathcal{A}, \alpha \in \mathcal{U}$ and $x \in E$

$$\alpha.(a.x) = (\alpha.a).x, \quad (a.\alpha).x = a.(\alpha.x), \quad (\alpha.x).a = \alpha.(x.a),$$

then $E$ is called a normal Banach left $\mathcal{A}$-$\mathcal{U}$-module. Similarly for the right and two sided actions. Also, $E$ is called commutative, if

$$\alpha.x = x.\alpha, \qquad (\alpha \in \mathcal{U}, x \in E).$$

A module homomorphism from $\mathcal{A}$ to $\mathcal{A}$ is a map $\varphi : \mathcal{A} \to \mathcal{A}$ with

$$\varphi(\alpha.a + b.\beta) = \alpha.\varphi(a) + \varphi(b).\beta, \quad \varphi(ab) = \varphi(a)\varphi(b) \quad (a, b \in \mathcal{A}, \alpha, \beta \in \mathcal{U}).$$

It is obvious that multiplication in $\mathcal{A}$ is $\omega^*$-continuous. Consider $\mathcal{A}$ as dual $\mathcal{A}$-module with predual $\mathcal{A}_*$. So, we shall suppose that $\mathcal{A}$ takes $\omega^*$-topology. $\mathcal{HOM}^b_{\omega^*}(\mathcal{A})$ will denotes the space of all bounded module homomorphisms from $\mathcal{A}$ to $\mathcal{A}$ that are $\omega^*$-continuous.
Now, in the following we present some definitions.

Definition 2.1. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra, $\varphi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{A})$ and $S$ is an inverse semigroup with subsemigroup $E_S$ of idempotents. let that $E$ be a dual Banach $\mathcal{A}$-bimodule. A bounded map $D_{\mathcal{U}} : \mathcal{A} \to E$ is called a module $\varphi$-derivation if

$$\begin{aligned} D_{\mathcal{U}}(\alpha.a \pm b.\beta) &= \alpha.D_{\mathcal{U}}(a) \pm D_{\mathcal{U}}(b).\beta, \\ D_{\mathcal{U}}(ab) &= D_{\mathcal{U}}(a).\varphi(b) + \varphi(a).D_{\mathcal{U}}(b), \quad (a, b \in \mathcal{A}, \alpha, \beta \in \mathcal{U}). \end{aligned}$$

When $E$ is commutative, each $x \in E$ defines a module $\varphi$-derivation

$$(D_{\mathcal{U}})_x(a) = \varphi(a).x - x.\varphi(a), \qquad (a \in \mathcal{A}).$$

Derivations of this form are called inner module $\varphi$-derivation.

Definition 2.2. Let $\mathcal{A}$ be a dual Banach algebra, $\mathcal{U}$ be a Banach algebra such that $\mathcal{A}$ is a Banach $\mathcal{U}$-module and $\varphi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{A})$. $\mathcal{A}$ is called $\varphi$-Connes module amenable if for any commutative normal Banach $\mathcal{A}$-$\mathcal{U}$-module E, each $\omega^*$-continuous module $\varphi$-derivation $D_{\mathcal{U}} : \mathcal{A} \to E$ is inner.

Recall that if $\varphi$ is identity map on $\mathcal{A}$, then $id$-Connes module amenability is called Connes module amenability. Also, by the proof of [2, Proposition 2.1], Connes amenability of $\mathcal{A}$ implies its Connes module amenability in the case where $\mathcal{U}$ has a bounded approximate identity for $\mathcal{A}$. In continuation, example 2.12 shows that the converse is false. The following definitions are from [5].

Definition 2.3. Let $\mathcal{A}$ be a Banach algebra, and let $3 \leq n \in \mathbb{N}$. A sequence

$$\mathcal{A}_1 \xrightarrow{\varphi_1} \mathcal{A}_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{n-1}} \mathcal{A}_n,$$

of $\mathcal{A}$-bimodules $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ and $\mathcal{A}$-bimodule homomorphisms $\varphi_i : \mathcal{A}_i \to \mathcal{A}_{i+1}$ for $i \in \{2, \ldots, n-1\}$ is called exact at position $i = 2, \ldots, n-1$ if $\varphi_{i-1} = ker\varphi_i$. (1) is called exact if it is exact at every position $i = 2, \ldots, n-1$.

If the mentioned above sequence has at least three non-zero terms. Then it is called a short exact sequence. For example,

$$0 \to \mathcal{A}_1 \xrightarrow{\varphi} \mathcal{A}_2 \xrightarrow{\psi} \mathcal{A}_3 \to 0,$$

is called a short exact sequence. In the following we define the admissible and the splitting short exact sequence.

Definition 2.4. Let $\mathcal{A}$ be a Banach algebra. A short exact sequence

$$\Theta : 0 \to \mathcal{A}_1 \xrightarrow{\varphi_1} \mathcal{A}_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{n-1}} \mathcal{A}_n \to 0,$$

of Banach $\mathcal{A}$-bimodules $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ and $\mathcal{A}$-bimodule homomorphisms $\varphi_i : \mathcal{A}_i \to \mathcal{A}_{i+1}$ for $i = 1, 2, \ldots, n-1$ is admissible, if there exists a bounded linear map $\rho : \mathcal{A}_{i+1} \to \mathcal{A}_i$ such that $\rho o \varphi_i$ on $\mathcal{A}_i$ for $i = 1, 2, \ldots, n-1$ is the identity map on $\mathcal{A}_{i+1}$. Further, $\Theta$ splits if we may choose $\rho$ to be an $\mathcal{A}$-bimodule homomorphism.

We recall that for Banach algebra $\mathcal{A}$ the projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule in the canonical way. Then the map $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}$ defined by $\pi(a \otimes b) = ab$ , is an $\mathcal{A}$-bimodule homomorphism.

Example 2.5. i) Let $\mathcal{A}$ be a unital Banach algebra. The short exact sequence of Banach $\mathcal{A}$-bimodules, $0 \to ker\pi \to \mathcal{A}\widehat{\otimes}\mathcal{A} \xrightarrow{\pi} \mathcal{A} \to 0$, is admissible.

ii) Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a unital dual Banach algebra. Then the short exact sequence

$$\sum_{\varphi} : 0 \longrightarrow \mathcal{A}_* \xrightarrow{\pi^*} \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*) \longrightarrow \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)/\pi^*(\mathcal{A}_*) \longrightarrow 0,$$

of $\mathcal{A}$ -bimodules is admissible.

Definition 2.6. Let $S$ be a weakly cancellative semigroup, $S$ be an inverse semigroup with idempotents $E_S$. Let $\chi \in \mathcal{HOM}^b_{\omega^*}(l^1(S))$ and $l^1(S)$ be a Banach $l^1(E_S)$-module. An element $M \in \sigma wc((l^1(S)\widehat{\otimes}l^1(S))^*)^*$ is a $\chi - \sigma wc$- virtual diagonal for $l^1(S)$ if

$$\delta_s.M = \chi(\delta_s)M, \qquad \langle \chi \otimes \chi, M\rangle = 1, \qquad (\delta_s \in l^1(S)).$$

Let $l^1(S) = (l^1(S)_*)^*$ be a unital dual Banach algebra. Then we consider the following short exact sequence of $l^1(S)$ -bimodules,

$$\sum_\chi : 0 \to l^1(S)_* \xrightarrow{\pi^*_\chi} \sigma wc((l^1(S)\widehat{\otimes}l^1(S))^*) \to \sigma wc((l^1(S)\widehat{\otimes}l^1(S))^*)/\pi^*_\chi(l^1(S)_*) \to 0.$$

Now, we present an important definition.

Definition 2.7. Let $S$ be a weakly cancellative inverse semigroup. Let $l^1(S) = (c_0(S))^*$ be a unital dual Banach algebra, and let $\chi \in \mathcal{HOM}^b_{\omega^*}(l^1(S))$. We say that $\sum_\chi$ $\chi$-splits if there exists a bounded linear map $\rho : \sigma wc((l^1(S)\widehat{\otimes}l^1(S))^*) \to l^1(S)_* = c_0(S)$ such that $\rho o \pi^*_\chi(\chi) = \chi$ and $\rho(T.\delta_s) = \chi(\delta_s)\rho(T)$, for all $\delta_s \in l^1(S)$, $T \in \sigma wc((l^1(S)\widehat{\otimes}l^1(S))^*)$ and $\pi^*_\chi : l^1(S) \otimes l^1(S) \to l^1(S)$.

Theorem 2.8. *Let $\mathcal{A}$ be a dual Arens regular Banach algebra and $\varphi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{A})$. Then $\mathcal{A}$ is $\varphi$-Connes module amenable if and only if the short exact sequences $\Sigma_\varphi$ $\varphi$-splits.*

Suppose that $\mathcal{A}, \mathcal{B}$ and $\mathcal{U}$ be dual Banach algebras such that $\mathcal{A}$ and $\mathcal{B}$ be dual Banach $\mathcal{U}$-modules and $\mathcal{A}\widehat{\otimes}\mathcal{B}$ denotes the projective tensor product of $\mathcal{A}$ and $\mathcal{B}$. Let $I$ be the closed ideal of $\mathcal{A}\widehat{\otimes}\mathcal{B}$ generated by elements of the form $\alpha.(a\otimes b)-(a\otimes b).\alpha$ for $a \in \mathcal{A}, b \in \mathcal{B}$ and $\alpha \in \mathcal{U}$. $\mathcal{A}\widehat{\otimes}_\mathcal{U}\mathcal{B}$ is defined to be the quitiont Banach space $\frac{\mathcal{A}\widehat{\otimes}\mathcal{B}}{I}$, that is, $\mathcal{A}\widehat{\otimes}_\mathcal{U}\mathcal{B} \cong \frac{\mathcal{A}\widehat{\otimes}\mathcal{B}}{I}$ [9].

Let $\mathcal{A}, \mathcal{B}$ be commutative Banach $\mathcal{U}$-bimodules and let $\varphi \in \mathcal{HOM}_{\omega^*}(\mathcal{A})$, $\psi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{B})$. Consider $\mathcal{A}\widehat{\otimes}_\mathcal{U}\mathcal{B}$ with the product specified by $(a\otimes b)(c\otimes d) = ac\otimes bd$ $(a, c \in \mathcal{A},\ \ b, d \in \mathcal{B})$. Let $\varphi \otimes \psi$ denotes the elements of $\mathcal{HOM}^b_{\omega^*}(\mathcal{A}\widehat{\otimes}\mathcal{B})$ satisfying $\varphi \otimes \psi(a \otimes b) = \varphi(a) \otimes \psi(b)$ for all $a \in \mathcal{A}, b \in \mathcal{B}$. $\varphi \otimes \psi$ induces a map $\varphi \otimes_\mathcal{U} \psi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{A}\widehat{\otimes}_\mathcal{U}\mathcal{B})$ with $\varphi \otimes_\mathcal{U} \psi(a \otimes b) = \varphi(a) \otimes \psi(b) + I$ [4].
By above details, we obtain the following theorem.

Theorem 2.9. *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{U}$ be dual Banach algebras, let $\mathcal{A}, \mathcal{B}$ be unital dual Banach $\mathcal{U}$- modules and let $\mathcal{A}\widehat{\otimes}_\mathcal{U}\mathcal{B}$ be a dual Banach algebra and $\varphi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{A})$, $\psi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{B})$. If $\mathcal{A}, \mathcal{B}$ are $\varphi, \psi$-Connes module amenable respectively, then $\mathcal{A}\widehat{\otimes}_\mathcal{U}\mathcal{B}$ is $\varphi\widehat{\otimes}_\mathcal{U}\psi$-Connes module amenable.*

Let $S$ be a inverse semigroup. For $s \in S$, we define $L_s, R_s : S \to S$ by $L_s(t) = st$, $R_s(t) = ts$, $(t \in S)$. If for each $s \in S$, $L_s$ and $R_s$ are finite-to-one maps, then we say that $S$ is weakly cancellative. We know that if $S$ is a weakly cancellative semigroup, then $(c_0(S))^* = l^1(S)$ [5].

Theorem 2.10. *Let $S$ be a weakly cancellative semigroup, let $S$ be an inverse semigroup with idempotents $E_S$, $\chi \in \mathcal{HOM}^b_{\omega^*}(l^1(S))$ and let $l^1(S)$ be a Banach $l^1(E_S)$-module. Then $l^1(S)$ is $\chi$-Connes module amenable if and only if the short exact sequences $\Sigma_\chi$ $\chi$-splits.*

Corollary 2.11. *Let $S$ be a weakly cancellative semigroup, let $S$ be an inverse semigroup with idempotents $E_S$ and let $l^1(S)$ be a Banach $l^1(E_S)$-module. Then $l^1(S)$ is Connes module amenable if and only if the short exact sequences $\Sigma_{\chi=id}$ splits.*

In the following, we present an example of above corollary.

Example 2.12. Let $(\mathbb{N}; \vee : \mathbb{N} \to \mathbb{N})$ be the semigroup of natural numbers with maximum operation. We know that $\mathbb{N}$ is weakly cancellative, because

$$L_s : \mathbb{N} \to \mathbb{N}, \;\; L_s(n) = sn \;\; and \;\; R_s : \mathbb{N} \to \mathbb{N}, \;\; R_s(n) = ns; \;\; (n \in \mathbb{N}),$$

are not one to one. Then $l^1(\mathbb{N})$ is a dual Banach algebra that $(c_0(\mathbb{N}))^* = l^1(\mathbb{N})$. By [5, Theorem 5.13], $l^1(\mathbb{N})$ is not Connes amenable. Moreover, $l^1(\mathbb{N})$ is module amenable on $l^1(E_{\mathbb{N}})$, so it is Connes module amenable (See [3]). Suppose that $M$ is a $\chi - \sigma wc$- virtual diagonal for $l^1(\mathbb{N})$. Now if we define $\rho : \sigma wc((l^1(\mathbb{N})\widehat{\otimes} l^1(\mathbb{N}))^*) \to l^1(\mathbb{N})_*$ by

$$\langle \delta_n, \rho(T)\rangle = \langle T.\delta_n, M\rangle, \qquad \big(n \in \mathbb{N}, \delta_n \in l^1(\mathbb{N}), T \in \sigma wc((l^1(\mathbb{N})\widehat{\otimes} l^1(\mathbb{N}))^*)\big).$$

We obtain

$$\langle \delta_n, \rho o \pi_\chi^*(\chi)\rangle = \langle \pi_\chi^*(\chi).\delta_n, M\rangle = \langle \pi_\chi^*(\chi), \delta_n.M\rangle = \chi(\delta_n)\langle \pi_\chi^*(\chi), M\rangle = \chi(\delta_n).$$

Next for $m, n \in \mathbb{N}, \delta_n, \delta_m \in l^1(\mathbb{N})$ we have

$$\begin{aligned} \langle \delta_m, \rho(T.\delta_n)\rangle &= \langle T.\delta_n\delta_m, M\rangle = \langle T, \delta_n\delta_m.M\rangle = \chi(\delta_n\delta_m)\langle T, M\rangle \\ &= \chi(\delta_n)\langle T, \delta_m.M\rangle = \chi(\delta_n)\langle T.\delta_m, M\rangle = \chi(\delta_n)\langle \delta_m, \rho(T)\rangle. \end{aligned}$$

All in all, the short exact sequences $\Sigma_{\chi=id}$ splits.

## References


1. M. Amini and D. Ebrahimi Bagha, *Weak module amenability for semigroup algebras*, Semigroup Forum **71** (2005) 18–26.
2. M. Amini, *Module amenability for semigroup algebras*, Semigroup Forum **69** (2004) 243–254.
3. M. Amini, A. Bodaghi and D. Ebrahimi Bagha, *Module amenability of the second dual and module topological center of semigroup algebras*, Semigroup Forum **80** (2010) 302–312.
4. H. G. Dales, *Banach Algebras and Automatic Continuity*, Clarendon Press, Oxford, 2000.
5. M. Daws, *Connes amenability of bidual and weighted semigroup algebras*, Math. Scand. **99** (2) (2006) 217–246.
6. E. G. Effros, *Amenability and virtual diagonals for von Neumann algebras*, J. Funct. Anal. **78** (1) (1988) 137–153.
7. A. Ghaffari and S. Javadi, *$\varphi$-Connes amenability of dual Banach algebras*, Bull. Iranian Math. Soc. **43** (1) (2017) 25–39.
8. A. Ghaffari, S. Javadi and E. Tammi, *$\varphi$-Connes module amenability of dual Banach algebras*, J. Algebraic Systems. **8** (1) (2020) 69–82.
9. M. A. Rieffel, *Induced Banach representations of Banach algebras and locally compact groups*, J. Func. Analysis. **1** (4) (1967) 443–491.



E-mail: aghaffari@semnan.ac.ir
E-mail: tamimi_ebrahim@semnan.ac.ir

# Locally Solid Vector Lattices with the $AM$-Property

Omid Zabeti*
Department of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran

ABSTRACT. Let $X$ be a locally solid vector lattice. We say that $X$ possesses the $AM$-property provided that for every bounded set $B \subseteq X$, the set of all finite suprema of elements of $B$, denoted by $B^\vee$, is also bounded. This notion extends some properties regarding $AM$-spaces in Banach lattices to the category of all locally solid vector lattices. With the aid of this concept, we investigate some topological and ordered structures for the spaces of all bounded order bounded operators between locally solid vector lattices.
**Keywords:** Locally solid vector lattice, Bounded operator, $AM$-Property, Levi property, Lebesgue property.
**AMS Mathematical Subject Classification [2010]:** 46A40, 47B65, 46A32.

## 1. Introduction

We say that a Banach lattice $E$ is an $AM$-space provided that $\|x \vee y\| = \|x\| \vee \|y\|$ for every $x, y \in E_+$. There are many interesting results regarding $AM$-spaces and operators between them; for a comprehensive context about the results in this talk and related notions, see [5]. Since locally solid vector lattices are a natural extension of normed lattices, it is of independent interest to investigate $AM$-spaces from this point of view. But there are many locally solid vector lattices which are not normed lattices so that we lack the norm structure in these spaces. Thus, we should look for a notion which does not depend on the norm, directly. We see that in an $AM$-space, by the definition, the finite suprema of elements in the closed unit ball are also bounded; more precisely, they lie in the closed unit ball, again. This motivates us to define the following fruitful observation.

Let $X$ be an Archimedean vector lattice. For every subset $A$ of $X$, $A^\vee$ denotes the set of all finite suprema of elements of $A$; that is, $A^\vee = \{a_1 \vee \ldots \vee a_n : n \in \mathbb{N}, a_i \in A\}$. It is easy to see that $A$ is bounded above in $X$ if and only if so is $A^\vee$ and in this case, when the supremum exists, $\sup A = \sup A^\vee$. Moreover, put $A^\wedge = \{a_1 \wedge \ldots \wedge a_n : n \in \mathbb{N}, a_i \in A\}$; then $A$ is bounded below if and only if so is $A^\wedge$ and $\inf A = \inf A^\wedge$ ( when the infimum exists). Note that $A^\vee$ can be viewed as an upward directed set in $X$ and $A^\wedge$ can be considered as a downward directed set.

DEFINITION 1.1. Let $X$ be a locally solid vector lattice. We say that $X$ has the *$AM$-property* provided that for every bounded set $B \subseteq X$, $B^\vee$ is also bounded with the same scalars; suppose $V \subseteq X$ is an arbitrary zero neighborhood. If for any positive scalar $\alpha$ we have $B \subseteq \alpha V$, then, we also have $B^\vee \subseteq \alpha V$.

*Speaker

In the sequel of the talk, we shall show that the $AM$-property is the right extension for the notion "$AM$-spaces" in the category of all locally solid vector lattices. Moreover, with the aid of this concept, we show that some topological and ordered structures such as the Lebesgue or Levi property can be transformed between the space of all bounded order bounded operators between locally solid vector lattices and the underlying space. For undefined terminology and related notions see [**1**, **2**, **5**]. In prior to anything, we state some preliminaries about bounded operators between topological vector spaces. Suppose $X$ and $Y$ are topological vector spaces. A linear operator $T$ from $X$ into $Y$ is called $nb$-bounded if there exists a zero neighborhood $U \subseteq X$ such that $T(U)$ is bounded in $Y$. $T$ is $bb$-bounded if for each bounded set $B \subseteq X$, $T(B)$ is also bounded. These concepts are far from being equivalent; moreover, these notions are not even equivalent to the continuous operators. Nevertheless, in a normed space, these concepts have the same meaning. The class of all $nb$-bounded operators from $X$ into $Y$ is denoted by $B_n(X,Y)$ and is equipped with the topology of uniform convergence on some zero neighborhood, recall that a net $(S_\alpha)$ of $nb$-bounded operators converges to zero on some zero neighborhood $U \subseteq X$ if for any zero neighborhood $V \subseteq Y$ there is an $\alpha_0$ such that $S_\alpha(U) \subseteq V$ for each $\alpha \geq \alpha_0$. The class of all $bb$-bounded operators from $X$ into $Y$ is denoted by $B_b(X,Y)$ and is equipped with the topology of uniform convergence on bounded sets. A net $(S_\alpha)$ of $bb$-bounded operators uniformly converges to zero on a bounded set $B \subseteq X$ if for any zero neighborhood $V \subseteq Y$ there is an $\alpha_0$ with $S_\alpha(B) \subseteq V$ for each $\alpha \geq \alpha_0$.

The class of all continuous operators from $X$ into $Y$ is denoted by $B_c(X,Y)$ and is equipped with the topology of equicontinuous convergence, namely, a net $(S_\alpha)$ of continuous operators converges equicontinuously to zero if for each zero neighborhood $V \subseteq Y$ there is a zero neighborhood $U \subseteq X$ such that for every $\varepsilon > 0$ there exists an $\alpha_0$ with $S_\alpha(U) \subseteq \varepsilon V$ for each $\alpha \geq \alpha_0$. See [**4**] for a detailed exposition on these classes of operators. In general, we have $B_n(X,Y) \subseteq B_c(X,Y) \subseteq B_b(X,Y)$ and when $X$ is locally bounded, they coincide.

## 2. Main Results

Recall that a locally solid vector lattice $X$ possesses the $AM$-property provided that for every bounded set $B \subseteq X$, the set of all finite suprema of $B$, $B^{\vee}$, is also bounded with the same coefficients. On the other hand, observe that a Banach lattice $E$ is called an $AM$-space if for all positive $x, y \in E$, $\|x \vee y\| = \|x\| \vee \|y\|$. First of all, we show that $AM$-property is the "right" extension of the property fulfilled by $AM$-spaces; that is the $AM$-property and being an $AM$-space in a Banach lattice agree. For the proofs of all of the results in this talk, we refer the reader to [**5**].

Proposition 2.1. *Let $E$ be a Banach lattice. Then, $E$ is an $AM$-space if and only if it possesses the $AM$-property.*

Proposition 2.2. *Let $(X_\alpha)_{\alpha \in A}$ be a family of locally solid vector lattices. Put $X = \prod_{\alpha \in A} X_\alpha$ with the product topology and pointwise ordering. If each $X_\alpha$ possesses the $AM$-property, then so is $X$.*

Proposition 2.3. *Let $(X_\alpha)_{\alpha\in A}$ be a family of locally solid vector lattices. Put $X = \prod_{\alpha\in A} X_\alpha$ with the product topology and pointwise ordering. If each $X_\alpha$ possesses the Levi property, then so is $X$.*

By using Proposition 2.2 and Proposition 2.3, we are able to construct some examples of locally solid vector lattices with the *AM* and Levi properties; consider $\mathbb{R}^{\mathbb{N}}$, the space of all real sequences. It is a locally solid vector lattice with the product topology and pointwise ordering. Observe that $\mathbb{R}$ has the *AM* and Levi properties so that by the above propositions, $\mathbb{R}^{\mathbb{N}}$ possesses the *AM* and Levi properties, as well. Morover, consider $\ell_\infty$, the space of all bounded real sequences with the uniform norm topology and pointwise ordering; it is a Banach lattice. It has the *AM* and Levi properties; see [2] for more information. Put $X = {\ell_\infty}^{\mathbb{N}}$, the space of all sequences with values in $\ell_\infty$ with the product topology and pointwise ordering. Again, using the above results, we see that $X$ possesses the *AM* and Levi properties, too.

Remark 2.4. Suppose $E$ is a Banach lattice and $F$ is a normed lattice. It is known that each order bounded operator from $E$ into $F$ is continuous. Nevertheless, when we are dealing with bounded operators between locally solid vector lattices, there is no specific relation between these classes of bounded operators and order bounded operators; see [3] for more information. Therefore, we consider $B_n^b(X,Y)$: the space of all order bounded $nb$-bounded operators, $B_b^b(X,Y)$: the space of all $bb$-bounded order bounded operators, $B_c^b(X,Y)$: the space of all continuous order bounded operators between locally solid vector lattices $X$ and $Y$. It is shown in [3, Lemma 2.2] that these classes of operators under some mild assumptions: order completeness and the Fatou property of the range space, form vector lattices, again. Moreover, with respect to the assumed topology, each class of bounded order bounded operators, is locally solid. Let $X$ be an Archimedean vector lattice and $Y$ be an order complete vector lattice. Recall that every order bounded operator $T : X \to Y$ possesses a modulus which is calculated via the remarkable Riesz-Kantorovich formulae defined by

$$|T|(x) = \sup\{|T(y)| : |y| \le x\},$$

for each $x \in X_+$.

Theorem 2.5. *Let $X$ be an order complete locally solid vector lattice. The following are equivalent.*

i) *$X$ possesses the AM and Levi properties.*
ii) *Every order bounded set in $X$ is bounded and vice versa.*

Observe that order completeness is necessary as an hypothesis for Theorem 2.5 and cannot be dropped. Put $X = C[0,1]$; it has the *AM*-property. Furthermore, boundedness and order boundedness agree in $X$. Nevertheless, it does not have the Levi property; for more details, see [1, 2]. Observe that in a locally solid vector lattice, Levi property implies order completeness; combining this with Theorem 2.5, we have the following useful facts.

Corollary 2.6. *Let $X$ and $Y$ be locally solid vector lattices such that $Y$ possesses the AM and Levi properties. Then every bb-bounded operator $T : X \to Y$ is order bounded; similar results hold for nb-bounded operators as well as continuous operators.*

By using Corollary 2.6 and [3, Lemma 2.2], we have the following observations.

Corollary 2.7. *Let $X$ and $Y$ be locally solid vector lattices such that $Y$ possesses the AM, Fatou, and Levi properties. Then $B_n(X,Y)$, $B_b(X,Y)$, and $B_c(X,Y)$ are vector lattices.*

Corollary 2.8. *Let $X$ be a locally solid vector lattice which possesses the AM and Levi properties and $Y$ be any locally solid vector lattice. Then, every order bounded operator $T: X \to Y$ is bb-bounded.*

Note that by $B^b(X,Y)$, we mean the space of all order bounded operators from a vector lattice $X$ into a vector lattice $Y$.

Corollary 2.9. *Let $X$ be a locally solid vector lattice which possesses the AM and Levi properties and $Y$ be an order complete locally solid vector lattice with the Fatou property. Then, $B_b^b(X,Y) = B^b(X,Y)$.*

Lemma 2.10. *Let $X$ be a locally solid vector lattice and $Y$ be an order complete locally solid vector lattice that possesses the Fatou property. Then $B_n^b(X,Y)$, $B_b^b(X,Y)$, and $B_c^b(X,Y)$ are ideals in $B^b(X,Y)$.*

Corollary 2.11. *Let $X$ be a locally solid vector lattice and $Y$ be an order complete locally solid vector lattice that possesses the Fatou property. Furthermore, assume that $T, S: X \to Y$ are operators such that $0 \leq T \leq S$. Then we have the following.*

i) *If $S \in B_n^b(X,Y)$ then $T \in B_n^b(X,Y)$.*
ii) *If $S \in B_b^b(X,Y)$ then $T \in B_b^b(X,Y)$.*
iii) *If $S \in B_c^b(X,Y)$ then $T \in B_c^b(X,Y)$.*

Theorem 2.12. *Let $X$ be a locally solid-convex vector lattice and $Y$ be an order complete locally solid vector lattice with the Fatou property. Then $B_n^b(X,Y)$ possesses the Levi property if and only if so is $Y$.*

Theorem 2.13. *Let $X$ be a locally solid-convex vector lattice and $Y$ be an order complete locally solid vector lattice with the Fatou property. Then $B_b^b(X,Y)$ possesses the Levi property if and only if so is $Y$.*

Theorem 2.14. *Let $X$ be a locally solid-convex vector lattice and $Y$ be an order complete locally solid vector lattice with the Fatou property. Then $B_c^b(X,Y)$ possesses the Levi property if and only if so is $Y$.*

Proposition 2.15. *Let $X$ be a locally solid-convex vector lattice and $Y$ be an order complete locally solid vector lattice that has the Fatou property. If either $B_n^b(X,Y)$ or $B_b^b(X,Y)$ or $B_c^b(X,Y)$ possesses the Lebesgue property, then so is $Y$.*

For the converse, we have the following.

Theorem 2.16. *Let $X$ be a locally solid vector lattice with the AM and Levi properties and $Y$ be an order complete locally solid vector lattice. If $Y$ possesses the Lebesgue property, then so is $B^b(X,Y)$.*

Furthermore, since $B_c(X,Y)$ can be viewed as a subspace of $B_b(X,Y)$, we are able to consider the induced topology on it. So, we have the following.

Proposition 2.17. *Let $X$ be a locally solid vector lattice with the Heine-Borel property and $Y$ be an order complete locally solid vector lattice. If $Y$ possesses the Lebesgue property, then so is $B_c^b(X,Y)$; while it is considered with the topology of uniform convergence on bounded sets.*

## References


1. C. D. Aliprantis and O. Burkinshaw, *Locally Solid Riesz Spaces with Applications to Economics,* Mathematical Surveys and Monographs, 105, Amer. Math. Soc., Providence, RI, 2003.
2. C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, 2nd ed., Springer, Dordrecht, 2006.
3. N. Erkursun-Ozcan, N. Anl Gezer and O. Zabeti, *Spaces of $u\tau$-Dunford-Pettis and $u\tau$-compact operators on locally solid vector lattices*, Matematiki Vesnik **71** (4) (2019) 351–358.
4. V. G. Troitsky, *Spectral radii of bounded operators on topological vector spaces*, Panamer. Math. J. **11** (3) (2001) 1–35.
5. O. Zabeti, *AM-property in locally solid vector lattices and applications,* Bull. Iran. Math. Soc. (2020). DOI: 10.1007/s41980-020-00458-7


E-mail: o.zabeti@gmail.com

# Contributed Talks

## Geometry and Topology

# Application of Frölicher-Nijenhuis Theory in Geometric Characterization of Metric Legendre Foliations on Contact Manifolds

Fatemeh Ahangari*
Faculty of Mathematical Sciences, Department of Mathematics, Alzahra University, Tehran, Iran

Abstract. In the context of geometry and mathematical physics, the impression of Lagrangian foliations on symplectic manifolds is of specific significance. More recent is the study of the theory of Legendre foliation on contact manifolds which are geometrically reckoned as the analogues of Lagrangian foliations in the odd dimensional circumstances. In this paper, a comprehensive analysis of the geometric organization of metric Legendre foliations on contact manifolds via the Frölicher - Nijenhuis formalism is presented. For this purpose, the global expression of Helmholtz metrizability constraints expressed by an arbitrary semi-basic 1-form is applied in order to induce a metric structure which leads to construction of a Legendre foliation equipped with a bundle-like metric on an arbitrary contact manifold. Moreover, the local framework of metric Legendre foliations is exhaustively analyzed by applying two significant local invariants existing on the tangent bundle of a Legendre foliation of the contact manifold $(M, \varpi)$; One of them is a symmetric 2-form and the other one is a symmetric 3- form. Mainly, it is proved that under some particular circumstances the behaviour of the Legendre foliation on the contact manifold $(M, \varpi)$ is locally the same as the foliation defined by the determined distribution which is fundamentally perpendicular complement in $TTM^\circ$ whose leaves are explicitly the $c$-indicatrix bundle defined on $M$.
**Keywords:** Frölicher-Nijenhuis formalism, Legendre foliation, Semi-basic 1-form, Contact manifolds, $c$-Indicatrix bundle.
**AMS Mathematical Subject Classification [2010]:** 53D35, 53C12, 58E10.

## 1. Introduction

A little while back, an increasing attention has been dedicated on contact geometry mainly due to its considerable applications in modeling of physical phenomena particularly in optics and time-depending mechanical systems. In this paper, we comprehensively focus on structural analysis of Legendre foliations of contact manifolds. The noticeable fact is that there exists a close relationship among the geometry of Legendre foliations and the geometry of Legrangian foliations of symplectic manifolds. In other words, Legendre foliations on contact manifolds are canonically odd-dimensional counterpart of Lagrangian foliations on symplectic manifolds which are of special significance in Geometry and Mathematical Physics. The researches of M. Y. Pang [1] and P. Libermann [2] in 90's

*Speaker

can be reckoned as the first exhaustive and systematic studies regarding Legendre foliations on contact manifolds which are relatively recent in this context.

Assume that $M$ is a real smooth manifold of dimension $(2n+1)$ which carries a 1-form $\varpi$ which all over of $M$ complying with the constraint $\varpi \wedge (d\varpi)^n \neq 0$, where the order of the corresponding exterior power is explicitly denoted by $n$. Then $\big(M,\varpi\big)$ is called a contact manifold designates the characteristic vector field or Reeb vector field on the contact manifold $\big(M,\varpi\big)$ and is expressed on $M$ via these significant provisions: $i_\varsigma \varpi = 1$ and $i_\varsigma d\varpi = 0$. A contact manifold $\big(M,\varpi\big)$ admits a natural $2n$-dimensional distribution $\mathcal{H}$ which is defined by the kernel of $\varpi$. In other words, promptly $\mathcal{H}$ can be considered as the subbundle of $TM$ on which $\varpi = 0$. Hence, we have:

$$\Gamma(\mathcal{H}) = \Big\{X \in \Gamma(TM) : \ \varpi(X) = 0\Big\}.$$

The distribution $\mathcal{H}$ is defined by the contact distribution on $\big(M,\varpi\big)$. In the following, we want to provide a constructive setting in order to structurally connect the concept of contact manifolds with the contact metric manifolds implication. Suppose that $(M,g)$ be an arbitrary $(2n+1)$-dimensional Riemannian manifold which is geometrically equipped with a $\binom{1}{1}$ tensor field which is denoted by $\varphi$, a 1-form $\varpi$ and a vector field $\varsigma$. Then $\big(M,g,\varphi,\varsigma,\varpi\big)$ is denoted by a contact metric manifold if for any $X,Y \in \Gamma(TM)$, the following tensor fields satisfy:

$$(1) \qquad \begin{cases} (\mathbf{a}) : \varphi^2 = -I + \varpi \otimes \varsigma, \\ (\mathbf{b}) : \varpi(X) = g\big(X,\varsigma\big), \\ (\mathbf{c}) : g\big(X,\varphi Y\big) = d\varpi\big(X,Y\big). \end{cases}$$

Taking into account (1) we can easily check that the following important identities hold for any $X,Y \in \Gamma(TM)$.

$$(2) \qquad \begin{cases} (\mathbf{d}) : \varpi(\varsigma) = 1, \qquad (\mathbf{e}) : \varphi(\varsigma) = 0, \\ (\mathbf{f}) : \varpi(\varphi X) = 0, \quad (\mathbf{g}) : g\big(X,\varphi Y\big) + g\big(Y,\varphi X\big) = 0, \\ (\mathbf{h}) : g\big(\varphi X,\varphi Y\big) = g\big(X,Y\big) - \varpi(X)\varpi(Y). \end{cases}$$

It is clear that a contact metric manifold is a contact manifold. In [3] it is proved that the converse is also true i.e. any contact manifold $\big(M,\varpi\big)$ clearly endows with a contact metric structure $\big(g,\varphi,\varsigma,\varpi\big)$. According to relation $(c)$ of (1) the 2-form $\Omega$ on $M$ can be defined by:

$$\Omega\big(X,Y\big) = g\big(X,\varphi Y\big), \quad \forall\ X,Y \in \Gamma(TM),$$

and it is called the fundamental 2-form which diametrically corresponds to the contact metric structure $\big(g,\varphi,\varsigma,\varpi\big)$. Since $\Omega$ is non-degenerate and we have $d\varpi = 0$ on $\Gamma(\mathcal{H})^3$, it is deduced that $\Omega$ characterizes a symplectic structure corresponding to the contact distribution. Furthermore, we intend to analyze the integrability of the contact distribution $\mathcal{H}$. To be specific, it can evidently be explicated that the contact condition $\varpi \wedge \big(d\varpi\big)^n \neq 0$ by declaring that the distribution $\mathcal{H}$ is as far from being integrable as possible.

Taking into account (1.b) it can be inferred that the contact distribution $\mathcal{H}$ identically coincides with the distribution which is transparently a distribution which is perpendicular complement to the characteristic distribution identified by $\text{span}\{\varsigma\}$. Now, assume that $\mathcal{H}$ is integrable. Hence, for any $X, Y \in \Gamma(\mathcal{H})$ we have $[X, Y] \in \Gamma(\mathcal{H})$ i.e. $\varpi([X, Y]) = 0$. Consequently, for any $X, Y \in \Gamma(\mathcal{H})$ we have: $d\varpi(X, Y) = 0$. On the other hand, from (1.c) and (2.e) we have:

$$d\varpi(X, \varsigma) = 0, \quad \forall \ X \in \Gamma(TM). \tag{3}$$

So, it is resulted that $d\varpi = 0$ on $M$, which is totally impossible since $M$ is a contact manifold. Ultimately, the following consequence is achieved.

Proposition 1.1. *Let $(M, \varpi)$ be an arbitrary contact manifold. The contact distribution $\mathcal{H}$ is not an involutive distribution.*

Now according to [4] this significant theorem is asserted:

Theorem 1.2. *Presume that $(M, \varpi)$ be an arbitrary contact manifold of dimension $(2n+1)$. Then the maximal dimension of any integrable subbundle of the contact distribution $\mathcal{H}$ is $n$.*

Proof. The exterior derivative of $\varpi$ is expressed by the following formula:

$$d\varpi(X, Y) = \frac{1}{2}\Big(X\big(\varpi(Y)\big) - Y\big(\varpi(X)\big) - \varpi\big([X, Y]\big)\Big), \ \forall \ X, Y \in \Gamma(TM). \tag{4}$$

Hence, by applying (3), (4), (2.d) and (1.b) it is deduced that:

$$\forall \ X \in \Gamma(\mathcal{H}), \qquad \varpi\big([X, \varsigma]\big) = 0. \tag{5}$$

Now, assume that $P$ is a $k$-dimensional integral manifold of the contact distribution $\mathcal{H}$. Taking into account (4) it is inferred that:

$$\forall \ X, Y \in \Gamma(TN), \qquad d\varpi(X, Y) = 0.$$

Accordingly, considering (1.c) we have $g(X, \varphi Y) = 0$, which explicitly implies that $\varphi(TP) \subset TP^{\perp}$. Thus $P$ can be significantly reckoned as a submanifold of $(M, g, \varphi, \varsigma, \varpi)$ which is of anti-invariant type. Moreover, it is normal to the characteristic vector field $\varsigma$. Considering the point that $\varphi$ is an automorphism of $\Gamma(\mathcal{H})$, it can be inferred that $p < n + 1$. Subsequently, it is illustrated that the maximum dimension of an arbitrary integral manifold of the contact distribution $\mathcal{H}$ is $k = n$. In addition, the existence of the integral manifolds of the maximum dimension can be resulted via the Darboux's theorem. According to this theorem, for an arbitrary $(n + 1)$-dimensional contact manifold $(M, \varpi)$, corresponding to each optional point there is at least one local coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n, z)$ such that the 1-form $\varpi$ on $M$ has the following interpretation:

$$\varpi = dz - \sum_{i=1}^{n} y^i dx^i. \tag{6}$$

As a consequence, due to (6) the $n$-dimensional integral manifold associated to the contact distribution $\mathcal{H}$ is defined by: $x^i = \text{const.}$, $z = \text{const.}$, $i \in \{1, \ldots, n\}$. □

Now, taking into account [1], we can define the notion of Legendre foliation as follows:

Definition 1.3. A Legendre distribution on a $(n+1)$-dimensional contact manifold $(M,\varpi)$ is an $n$-dimensional subbundle $P$ of the contact distribution such that for all $X,\widetilde{X}\in\Gamma(P)$, we have: $d\varpi(X,\widetilde{X})=0$. Whenever $P$ is integrable, it defines a Legendre foliation of $(M,\varpi)$.

Thus due to above definition, a foliation $\mathcal{F}$ of $(M,\varpi)$ is a Legendre foliation if and only if the distribution $\mathcal{D}$ tangent to $\mathcal{F}$ is an $n$-subbundle of the $2n$-distribution $\mathcal{H}$. Some fundamental deductions regarding the geometry of Legendre foliations are presented in [1, 2] and [4].
The main goal of the current research is thoroughgoing study of metric Legendre foliations on contact manifolds via the global Helmholtz conditions, declared in terms of a semi-basic 1-form, which specify when a semispray is locally Lagrangian. The inverse problem of the calculus of variations can be explicitly expressed as follows: Suppose that $M$ be an arbitrary $m$-dimensional manifold. What are the principal circumstances under which the solutions of a system of second order differential equations (SODE)

$$\frac{d^2x^i}{dt^2}+2G^i(x,\dot{x})=0,\quad i\in\{1,\dots,m\},\tag{7}$$

can be deduced from a variational principle? In other words, for the optional Lagrangian function $L$, can be reckoned as the solutions of the following system of differential equations which is denoted by the Euler-Lagrange equations:

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial\dot{x}^i}\Big)\frac{\partial L}{\partial x^i}=0,\quad i\in\{1,\dots,m\}.$$

Literally, one privileged standpoint regarding the problem mentioned above, applies the worthwhile Helmholtz conditions which are basically considered as the necessary and sufficient conditions for the indispensable essence of a specified multiplier matrix $g_{ij}(x,\dot{x})$ such that the following identity holds:

$$g_{ij}(x,\dot{x})\bigg(\frac{d^2x^j}{dt^2}+2G^i(x,\dot{x})\bigg)=\frac{d}{dt}\bigg(\frac{\partial \mathrm{L}}{\partial\dot{x}^i}\bigg)-\frac{\partial \mathrm{L}}{\partial x^i},$$

for appointive Lagrangian function $L(x,\dot{x})$. It is worth mentioning that the multiplier matrix $g_{ij}$ literally creates a symmetric $\binom{0}{2}$-type tensor field $g$ along the tangent bundle projection. Geometric formulation of Helmholtz conditions in terms of $g_{ij}$ has been extensively investigated from various approaches in the last few years. Refer to for example [5] and [6].

In this paper, taking into account [5], we will inquire the inverse problem of calculus of variations when the system of SODE in Eq. (7) arise from a semispray. Moreover, we will apply a global formulation for the Helmholtz conditions in connection with a semi-basic 1-form which is presented in [5]. Based on I. Bucataru and M. F. Dahl's point of view, if for some semi-basic 1-form which fulfills the Helmholtz conditions, the 1-form is explicitly the Poincare-Cartan 1-form of a locally defined Lagrangian function. As a consequence, the original semispray can be precisely reckoned as an Euler-Lagrange vector field for this Lagrangian.

The general framework of the current paper is as follows: In section 2, a quick review of the Frölicher-Nijenhuis formalism is presented. Meanwhile, a brief discussion regarding the correspondence of the Helmholtz conditions for a 1-form and the classic formulation of the Helmholtz conditions in terms of a multiplier matrix is asserted. In addition, according to [5], it is remarked that relying on the degree of homogeneity, one or two of the Helmholtz conditions is obviously derived from the other ones. Hence, a spray $S$ is Lagrangian if and only if based on the degree of homogeneity, two or three of the four Helmholtz conditions are accomplished. The mentioned two specific cases are precisely compatible with two inverse problems in the calculus of variations i.e. Finsler metrizbility for a spray and projective metrizability for a spray. Section 3 is assigned to the detailed enquiry of the notion of metric Legendre foliations on an arbitrary contact manifold $\big(M,\varpi\big)$ via the Frölicher-Nijenhuis formalism of Helmholtz metrizability conditions. Significantly, two local invariants proposed by Pang [1] for classifying the Legendre foliations are applied in order to structural investigation of the constraints for a Legendre foliation to be allocated as the class of foliations equipped with a bundle-like metric. Ultimately, some outstanding achievements regarding the main objectives of the current research are explicitly pointed out.

## 2. Frölicher-Nijenhuis Formalism of Helmholtz Metrizability Conditions

In this section, we present a breif appraisal of Frölicher-Nijenhuis theory on $TM\backslash\{0\}$. Assume that $A$ is a vector valued $l$-form on $TM\backslash\{0\}$ and $\alpha$ is a $k$-form on $TM\backslash\{0\}$ where $k\geq 1$ and $l\geq 0$. Then the inner product of $A$ and $\alpha$ is the $\big(k+1-l\big)$-form $i_A\alpha$ which is expressed by:

$$i_A\alpha\big(X_1,\dots,X_{k+l-1}\big)=$$
$$\frac{1}{l!\big(k-1\big)!}\sum_{\sigma\in S_{k+l-1}}\mathrm{sign}(\sigma)\alpha\Bigg(A\Big(X_{\sigma(1)},\dots,X_{\sigma(l)}\Big),X_{\sigma\left(l+1\right)},\dots,X_{\sigma\left(k+l-1\right)}\Bigg),$$

where $X_1,\dots,X_{k+l-1}\in\mathcal{X}\big(TM\backslash\{0\}\big)$ and $S_p$ denotes the permutation group of the elements $1,\dots,p$. In particular, when $l=0$ then $A$ is a vector field on $TM\backslash\{0\}$ and $i_A\alpha$ is the usual inner product of $k$-form $\alpha$ concerning a vector field $A$. Besides, whenever $l=1$ then $A$ is a $\binom{1}{1}$-type tensor field and $i_A\alpha$ is the following $k$-form:

$$i_A\alpha\big(X_1,\dots,X_k\big)=\sum_{i=1}^{k}\alpha\Big(X_1,\dots,AX_i,\dots,X_k\Big).$$

Suppose that $A$ is a vector valued $l$-form on $TM\backslash\{0\}$. Then for $k,l\geq 0$ the exterior derivative with reference to $A$ is this specified map:

$$\begin{aligned} &d_A:\Lambda^k\big(TM\backslash\{0\}\big)\longrightarrow\Lambda^{k+1}\big(TM\backslash\{0\}\big),\\ &d_A=i_A\circ d-(-1)^{l-1}d\circ i_A, \end{aligned}\tag{8}$$

where the $C^\infty$ module of $k$-forms is represented by $\Lambda^k(M)$. Furthermore, a $k$-form $\omega$ on $TM\backslash\{0\}$ is designated $d_A$-closed if $d_A\omega=0$ and $d_A$-exact whenever there is at least one $\Theta\in\Lambda^{k-1}\Big(TM\backslash\{0\}\Big)$ such that $\omega=d_A\Theta$. Specifically, when $A\in\mathcal{X}\big(TM\backslash\{0\}\big)$ and $k\geq 0$, we acquire $d_A=\mathcal{L}_A$, where $\mathcal{L}_A$ is the common Lie derivative $\mathcal{L}_A:\Lambda^k\big(TM\backslash\{0\}\big)\longrightarrow\Lambda^k\big(TM\backslash\{0\}\big)$. In this case, relation (8) is exactly the Cartan's formula. For the two vector valued forms $A$ and $B$ on $TM\backslash\{0\}$ of degrees respectively $l\geq 0$ and $k\geq 0$ the Frölicher-Nijenhuis bracket of $A$ and $B$ can be regarded explicitly as a vector valued $(k+l)$-form $[A,B]$ on $TM\backslash\{0\}$ which is uniquely determined by:

$$d_{[A,B]} = d_A \circ d_B - (-1)^{kl} d_B \circ d_A,$$

when $A$ and $B$ are vector fields, then the Frölicher-Nijenhuis bracket $[A,B]$ is identically the usual Lie bracket $[A,B] = \mathcal{L}_A B$. Moreover, for a vector field $X \in \mathcal{X}(TM\backslash\{0\})$ and a $\binom{1}{1}$-type tensor field $A$ on $TM\backslash\{0\}$, the Frölicher-Nijenhuis bracket $[X,A] = \mathcal{L}_X A$ is the $\binom{1}{1}$-type tensor field on $TM\backslash\{0\}$ which is expressed by:

$$\mathcal{L}_X A = \mathcal{L}_X \circ A - A \circ \mathcal{L}_X.$$

Furthermore, for $\binom{1}{1}$-type tensor fields $A$ and $B$ the next commutation formula on $\Lambda^k(TM\backslash\{0\}), k \geq 0$ holds:

$$\begin{aligned}
&(a): i_A d_B - d_B i_A = d_{B\circ A} - i_{[A,B]}, \\
&(b): \mathcal{L}_X i_A - i_A \mathcal{L}_X = i_{[X,A]}, \qquad (9)\\
&(c): i_X d_A + d_A i_X = \mathcal{L}_{AX} - i_{[X,A]}.
\end{aligned}$$

It is worth mentioning that formula (9.c) is referred as the generalized Cartan's formula.

A remarkable standpoint to the inverse problem of the calculus of variations applies the Helmholtz conditions which are straightforwardly declared by:

$$g_{ij} = g_{ji}, \qquad \frac{\partial g_{ij}}{\partial y^k} = \frac{\partial g_{ik}}{\partial y^j}, \qquad (10)$$

$$\nabla g_{ij} = 0, \qquad g_{ik} R^k_j = g_{jk} R^k_i. \qquad (11)$$

It is noticeable that conditions (10) are necessary and sufficient conditions for the existence of a Lagrange function which has as Hessian the matrix multiplier $g_{ij}$. Moreover, the conditions (11) represent the compatibility among the multiplier matrix and the analyzed SODE and pertaining geometric constructions such as: The Douglas tensor (Jacobi endomorphism) $\Phi$ and the dynamical covariant derivative.

## 3. Identification of Metric Legendre Foliations on Contact Manifolds via Semi-Basic 1-forms

Suppose that $M$ be an $m$-dimensional arbitrary manifold and $(TM, \pi, M)$ with local coordinates $(x^i, y^i)$ represents its corresponding tangent bundle. Furthermore, the distribution $VTM$ indicates the corresponding vertical subbundle. In the local coordinate system, the tangent structure $\mathcal{J}$ is fundamentally demonstrated by $\mathcal{J} = \dfrac{\partial}{\partial y^i} \otimes dx^i$ and the vector field $\mathbb{C} \in \mathcal{X}(TM)$ defined by $\mathbb{C} = y^i \dfrac{\partial}{\partial y^i}$ is identified as the Liouville vector field. In addition, a $k-$form $\omega$ is called semi-basic if $\omega(X_1, X_2, \ldots, X_k) = 0$ whenever one of the vector fields $X_i$ is vertical for $i \in \{1, \ldots, k\}$. Moreover, the module of semi-basic $k$-forms is denoted by $\mathrm{Sec}(\Lambda^k T^*_V)$. Also, a vector valued $k$-form $A$ on $TM\backslash\{0\}$ is said to be semi-basic if it admits values in the vertical bundle and specifically when one of the vectors $X_i$, $i \in \{1, \ldots, k\}$ is vertical the following relation holds: $A(X_1, X_2, \ldots, X_k) = 0$.

Hence according to Frölicher Nijenhuis theory a semispray (spray) on $M$ is a vector field $\mathcal{S} \in \mathcal{X}(TM\backslash\{0\})$ such that $\mathcal{J}\mathcal{S} = \mathbb{C}$ (and $[\mathbb{C}, \mathcal{S}] = \mathcal{S}$). Now consider the almost tangent structure $\Gamma = -\mathcal{L}_{\mathcal{S}}\mathcal{J} = h - v$ where $h$ and $v$ are the horizontal and vertical projectors determined by $\mathcal{S}$ respectively. Then the Jacobi endomorphism (or Douglas tensor) $\Phi$ is characterized as the following $(1,1)$-type tensor field

$$\Phi = vo\mathcal{L}_{\mathcal{S}}h = -vo\mathcal{L}_{\mathcal{S}}v = R^i_j \frac{\partial}{\partial y^i} \otimes dx^j.$$

The dynamical covariant derivative $\nabla$ is defined by:

$$\begin{aligned}&\nabla X = h[\mathcal{S}, hX] + v[\mathcal{S}, vX], \quad \forall\ X \in \mathcal{X}(TM\backslash\{0\}),\\ &\nabla = \mathcal{L}_{\mathcal{S}} + ho\mathcal{L}_{\mathcal{S}}h + vo\mathcal{L}_{\mathcal{S}}v = \mathcal{L}_{\mathcal{S}} + \Psi.\end{aligned}$$

Taking into account that $\nabla$ is a zero-degree derivation on $\Lambda^k(TM\backslash\{0\})$ it can be distinctively decomposed into the sum of a Lie derivation $\mathcal{L}_{\mathcal{S}}$ and an algebraic derivation $i_\Psi$ as follows: $\nabla = \mathcal{L}_{\mathcal{S}} - i_\Psi$. According to [5] the following significant relations hold:

$$\begin{aligned}&(a): \ \nabla\mathcal{S} = 0, \quad \nabla\mathbb{C} = 0, \quad \nabla i_{\mathcal{S}} = i_{\mathcal{S}}\nabla, \quad \nabla i_{\mathbb{C}} = i_{\mathbb{C}}\nabla;\\ &(b): \ \nabla h = 0, \quad \nabla v = 0, \quad \nabla\mathcal{J} = 0, \quad \nabla\mathbb{F} = 0;\\ &(c): \ d\nabla - \nabla d = d_\Psi, \quad \nabla i_h = i_h\nabla = 0, \quad \nabla i_{\mathcal{J}} - i_{\mathcal{J}}\nabla = 0.\end{aligned}$$

So a semispray $\mathcal{S}$ on $M$ is denoted by a Lagrangian vector field if there is at least one $\mathrm{L} \in C^\infty(TM\backslash\{0\})$ such that $\mathcal{L}_{\mathcal{S}}d_{\mathcal{J}}\mathrm{L} = d\mathrm{L}$. Mainly due to [6] we have:

Theorem 3.1. *Suppose that $\mathcal{S}$ be an arbitrary semispray defined on the manifold $M$. The necessary and sufficient conditions for the semispray $\mathcal{S}$ to be a Lagrangian vector field is that there is at least one semi-basic 1-form $\Theta$ on $TM\backslash\{0\}$ that fulfills the following reformulations of the Helmholtz metrizability conditions:*

$$d_h\Theta = 0, \quad d_{\mathcal{J}}\Theta = 0, \quad \nabla d\Theta = 0, \quad d_\Phi\Theta = 0.$$

Overall, considering above discussion a spray $\mathcal{S}$ is projectively metrizable if there exists a 1-homogeneous function $F \in C^\infty(TM\backslash\{0\})$ such that $\mathcal{L}_{\mathcal{S}}d_{\mathcal{J}}F = dF$; Moreover a spray $\mathcal{S}$ is Finsler metrizable if for some 2-homogeneous function $\mathrm{L} \in C^\infty(TM)\backslash\{0\}$ the following identity satisfied: $\mathcal{L}_{\mathcal{S}}d_{\mathcal{J}}\mathrm{L} = \mathrm{L}$. Equivalently relating to the notion of projective metrizabilty from the Frölicher Nijenhuis theory approach we have [5]:

Proposition 3.2. *Assume that $\mathcal{S}$ be an optional spray existing on the manifold $M$. The necessary and sufficient conditions for the spray $\mathcal{S}$ in order to being metrizable in the projective sense is that for some semi-basic 1-form $\Theta$ on $TM\backslash\{0\}$ the following main identities are accomplished:*

$$\mathcal{L}_{\mathbb{C}}\Theta = 0, \quad d_{\mathcal{J}}\Theta = 0, \quad d_h\Theta = 0.$$

Now, according to [1] and [5] this theorem is derived:

Theorem 3.3. *Let $\big(M, \varpi, \mathcal{F}\big)$ be an arbitrary $\big(2n+1\big)$-dimensional contact manifold equipped with an $n$-dimensional Legendre foliation $\mathcal{F}$. Assume that $\mathcal{S} \in \mathcal{X}\big(TM\backslash\{0\}\big)$ be a semispray which is characterized by: $\mathcal{S} = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ in the local coordinate system. We denote by $\mathcal{D}$ the tangent distribution to $\mathcal{F}$.*

*Meanwhile, consider that $\mathcal{L}$ denotes the Lie derivative on $M$. Then the symmetric $F(M)$-bilinear form $\Pi$ on $\Gamma(\mathcal{D})$ which is determined by:*

$$\Pi\big(X,Y\big) = -\big(\mathcal{L}_X\mathcal{L}_Y\varpi\big)(\varsigma), \quad \forall\ X,Y\in\Gamma(\mathcal{D}),$$

*is positive definite if and only if for some 1-homogeneous semi-basic 1-form $\Theta\in\Lambda^1\big(TM\backslash\{0\}\big)$ the following relations are fulfilled :*

$$d_{\mathcal{J}}\Theta = 0\ , \quad d_h\Theta = 0\ , \quad \nabla d\Theta = 0. \tag{12}$$

Proof. In [5] this significant point illustrated that the inverse problem has solutions if and only if there exists a semi-basic 1-form $\Theta$ on $TM\backslash\{0\}$ which literally satisfies Helmholtz-type conditions. Thus, the number of these conditions directly depends on the degree of homogeneity of $\Theta$. Consequently, the Lagrangian function $L$ will be determined as the potential of the homogeneous semi-basic 1-form. Accordingly, for the semi-basic 1-form $\Theta$ on $TM\backslash\{0\}$, the following two specific cases are totally analyzed:

**Case (1):** If $\Theta$ is 0-homogeneous and satisfies the following two Helmholtz conditions: $d_{\mathcal{J}}\Theta = 0$ and $d_{\langle}\Theta = 0$, then the corresponding potential $i_{\mathcal{S}}\Theta$ is a 1-homogeneous function which lead to the projective metrizability of the spray $\mathcal{S}$.

**Case (2):** Provided that $\Theta$ is 1-homogeneous and complies with these three Helmholtz conditions: $d_{\mathcal{J}}\Theta = 0$ , $d_h\Theta = 0$ and $\nabla d\Theta = 0$, then its associated potential $i_{\mathcal{S}}\Theta$ is a 2-homogeneous function that prompts the Finsler metrizability of the spray $\mathcal{S}$.

As a result, taking into account [5] these remarkable points can be explicitly pointed out:

**(i):** A spray $\mathcal{S}$ is metrizable in the projective sense if for some 1-homogemeous function $F\in C^\infty\big(TM\backslash\{0\}\big)$ the following identity: $\mathcal{L}_{\mathcal{S}}d_{\mathcal{J}}F = dF$ holds.

**(ii):** A spray $\mathcal{S}$ is Finsler metrizable if there is at least one 2-homogemeous function $L\in C^\infty\big(TM\backslash\{0\}\big)$ such that $\mathcal{L}_{\mathcal{S}}d_{\mathcal{J}}L = dL$.

Summing up the points asserted above, it is deduced that: taking into account the fact that $d_\Phi\Theta = 0$ is identically the same as $g_{ik}R^k_j = g_{jk}R^k_i$, one of the Helmholtz conditions (10) and (11) is obviously redundant. Besides, if $\mathcal{L}$ is 2-homogeneous we have: $2L = i_{\mathcal{S}}\Theta$ which explicitly implies that $d_{\mathcal{J}}\Theta = 0$ if and only if $\Theta = d_{\mathcal{J}}L$. In addition, whenever $d_{\mathcal{J}}\Theta = 0$, the two conditions $d_h\Theta = 0$ and $\nabla d\Theta = 0$ are thoroughly equivalent to $\mathcal{L}_{\mathcal{S}}d_{\mathcal{J}}L = dL \longleftrightarrow i_{\mathcal{S}}dd_{\mathcal{J}}L = -dL \longleftrightarrow d_hL = 0$.

Ultimately, the necessary and sufficient conditions for a selective spray $\mathcal{S}$ to be metrizable in the Finsler sense is that for some 1-homogeneous semi-basic 1-form $\Theta\in\Lambda^1\big(TM\backslash\{0\}\big)$ the identities (12) hold.

On the other hand, by elementary computations applying (2.d) and (5), it is inferred that:

$$\Pi\big(X,Y\big) = \varpi\big(\big[Y,[\varsigma,X]\big]\big). \tag{13}$$

It is worth noticing that $\Pi$ depends neither on the Riemannian metric $g$ nor the tensor field $\varphi$ of an arbitrary contact metric structure denoted by $\big(g,\varphi,\varsigma,\varpi\big)$. Nevertheless, considering (13), (4), (5), (2.e) and (2.g) it is resulted that:

$$\Pi\big(X,Y\big) = 2g\big([\varsigma,X],\varphi Y\big).$$

As a consequence, according to [1] the proof completes. □

According to above theorem, it is deduced that:

Corollary 3.4. *Suppose that* $(M, \varpi)$ *be an arbitrary contact manifold. If all the conditions of Theorem 3.3 are satisfied, then the Legendre foliation* $\mathcal{F}$ *on* $(M, \varpi)$ *is identically equivalent to the foliations constructed via the c-indicatrices of the Finsler function denoted by* $F$ *and resulted from the metrizability of the spray* $\mathcal{S}$.

## References


1. M. Y. Pang, *The structure of Legendre foliations*, Trans. Amer. Math. Soc. **320** (2) (1990) 417–455.
2. P. Libermann, *Legendre foliations on contact manifolds*, Differential Geom. Appl. **1** (1) (1991) 57–76.
3. D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1976.
4. A. Bejancu and H. R. Farran, *Foliations and Geometric Structures*, Springer, Dordrecht, 2006.
5. I. Bucataru and M. F. Dahl, *Semi-basic* 1*-forms and Helmholtz conditions for the inverse problem of the calculus of variations*, J. Geom. Mech. **(1)** (2) (2009) 159–180.
6. I. Bucataru and Z. Muzsnay, *Projective metrizability and formal integrability*, SIGMA Symmetry Integrability Geom. Methods Appl. **7** (2011) 114–135.


E-mail: f.ahangari@alzahra.ac.ir; fa.ahangari@gmail.com

# Smooth Quasifibrations

Alireza Ahmadi*
Department of Mathematical Sciences, Yazd University, Yazd, Iran

Abstract. As a homotopical extension of diffeological fiber bundles and fibrations, we study a version of quasifibrations, called smooth quasifibrations, in the context of diffeology based on smooth homotopy. Some characterizations of smooth quasifibrations are given and a few basic results are obtained.
**Keywords:** Diffeological spaces, Quasifibrations, Smooth homotopy.
**AMS Mathematical Subject Classification [2010]:** 55R99, 55P99, 57P99.

## 1. Introduction

This article is aimed to investigate a smooth version of quasifibrations in the context of diffeology. Quasifibrations are a generalization of fibrations and fiber bundles, introduced by Dold and Thom [1], where the fibers are weakly homotopy equivalent to the homotopy fibers (See [3, p. 479]). On the other hand, diffeology extends ordinary differential geometry by diffeological spaces, established by J.-M. Souriau [7] in the early 1980s. These spaces and smooth maps between them constitute a complete, cocomplete, and cartesian closed quasitopos, which include smooth manifolds and orbifolds as full subcategories. The main reference for diffeology is the book [5].

One of the most significant features of diffeological spaces is their smooth homotopy groups, initiated by P. Iglesias-Zemmour in his Ph.D. thesis [4]. Furthermore, attempts have taken place to provide a model category to do smooth homotopy theory on diffeological spaces (See [2, 6, 8]). We consider smooth quasifibrations as the counterpart of the classical notion of quasifibrations in smooth homotopy of diffeological spaces. We characterize smooth quasifibrations, where fibers are weak homotopy equivalent to the homotopy fibers (in the diffeological sense), and that any quasifibration induces a long exact homotopy sequence.

## 2. Preliminaries

In this section, we briefly recall some elementary definitions of diffeology from [5]. For the details of the smooth homotopy of diffeological spaces, see [5, Chapter 5].

Definition 2.1. A *domain* is an open subset of Euclidean space $\mathbb{R}^n$ with the standard topology, for all non-negative integer $n$. Any map from a domain to a set $X$ is said to be a *parametrization* in $X$.

Definition 2.2. A *diffeological space* $(X, \mathcal{D})$ is an underlying set $X$ equipped with a *diffeology* $\mathcal{D}$ on it, which is a set of parametrizations in $X$ called *plots*, satisfying the following axioms:

*Speaker

**D1.** The union of the images of plots covers $X$.
**D2.** For every plot $P : U \to X$ and every smooth map $F : V \to U$ between domains, the parametrization $P \circ F$ is a plot.
**D3.** If $P : U \to X$ is a parametrization and for every point $r$ of $U$, there exists an open neighborhood $V$ of $r$ such that $P|_V$ is a plot, then $P$ is a plot.

A diffeological space is just denoted by the underlying set, when the diffeology is understood.

Example 2.3. Any smooth manifold is a diffeological space in which usual smooth parameterizations are plots. In particular, domains are diffeological spaces.

Definition 2.4. Let $X$ be a diffeological space. A *diffeological subspace* of $X$ is a subset $X' \subseteq X$ equipped with the *subspace diffeology*, which is the set of all plots in $X$ with values in $X'$.

Definition 2.5. Let $\{X_i\}_{i\in J}$ be a family of diffeological spaces. The *product diffeology* on $X = \prod_{i\in J} X_i$ is given by the parametrizations $P$ in $X$ for which $\pi_i \circ P$ is a plot in $X_i$ for all $i \in J$, where $\pi_i : X \to X_i$ is the natural projection. If $J = \{1, \ldots, n\}$ is a finite set, then the plots in the product $X = X_1 \times \cdots \times X_n$ are $n$-tuples $(P_1, \ldots, P_n)$ where each $P_i$ is a plot in $X_i$.

Definition 2.6. Let $X$ and $Y$ be two diffeological spaces. A map $f : X \to Y$ is *smooth* if for every plot $P$ in $X$, the composition $f \circ P$ is a plot in the space $Y$. The set of all smooth maps from $X$ to $Y$ is denoted by $\mathrm{C}^\infty(X, Y)$. Diffeological spaces together with smooth maps form a category denoted by **Diff**. Isomorphisms of **Diff** are called *diffeomorphisms*.

One advantage of **Diff** is cartesian closedness. Indeed, there exists a so-called *functional diffeology* on $\mathrm{C}^\infty(X, Y)$ such that the natural map $\mathrm{C}^\infty(X, \mathrm{C}^\infty(Y, Z)) \longrightarrow \mathrm{C}^\infty(X \times Y, Z)$ taking $f \mapsto \tilde{f}$ with the property $\tilde{f}(x, y) = f(x)(y)$, is a diffeomorphism (See [5, art. 1.60]).

Definition 2.7. Every smooth map from $\mathbb{R}$ to a diffeological space $X$ is called a *path* or *homotopy* in $X$. The space of all paths in $X$ equipped with the functional diffeology is denote by $\mathrm{Paths}(X) = \mathrm{C}^\infty(\mathbb{R}, X)$.

Definition 2.8. Two points $x, x'$ of a diffeological space $X$ are said to be *connected* or *homotopic* if there exists a path $\gamma$ in $X$ such that $\gamma(0) = x$ and $\gamma(1) = x'$. To be connected by a path is an equivalence relation on $X$. The classes of this relation are called the pathwise-connected components or simply the *connected components* of $X$. The set of the connected components of $X$ is denoted by $\pi_0(X)$.

Definition 2.9. A *loop* in a diffeological space $X$, based at $x \in X$, is a path $\ell$ with $\ell(0) = x = \ell(1)$. Denote by $\mathrm{Loops}(X, x)$ the space of loops in $X$, based at $x$ equipped with the functional diffeology. More generally, for all $k \geq 1$, denote by $\mathrm{Loops}_k(X, x)$ the space $k$-loops in $X$, based at $x$, which are defined recursively by $\mathrm{Loops}_k(X, x) = \mathrm{Loops}(\mathrm{Loops}_{k-1}(X, x), \mathbf{x}_{k-1})$, where $\mathbf{x}_{k-1} : \mathbb{R}^{k-1} \to X$ is the constant map with the value $x$, $\mathrm{Loops}_0(X, x) = X$ and $\mathbf{x}_0 = x$.

Definition 2.10. The *smooth homotopy groups* of $X$, based at $x$, are defined by $\pi_k(X, x) = \pi_0(\mathrm{Loops}_k(X, x), \mathbf{x}_k)$, for all $k \geq 0$.

Definition 2.11. Let $A$ be a subspace of a diffeological space $X$, and let $\mathrm{a} \in A$. Let $\mathrm{Paths}_k(X, A, \mathrm{a}) := \mathrm{Paths}(\mathrm{Loops}_{k-1}(X, \mathrm{a}), \mathrm{Loops}_{k-1}(A, \mathrm{a}), \mathbf{a}_{k-1})$, for $k \geq 1$. The *relative smooth homotopy groups* of the pair $(X, A)$ are defined by $\pi_k(X, A, \mathrm{a}) = \pi_0(\mathrm{Paths}_k(X, A, \mathrm{a}), \mathbf{a}_{k-1})$, for all $k \geq 0$.

Any smooth map $f : X \to X'$ induces group homomorphisms $f_\# : \pi_k(X, x) \to \pi_k(X', x')$ and $\mathbf{f}_{k_\#} : \pi_k(X, f^{-1}(x'), x) \longrightarrow \pi_k(X', x')$ with $f(x) = x'$, for all $k \geq 1$, and a map of pointed spaces for $k = 0$ (See [5, art. 5.17]).

Definition 2.12. Every diffeological space admits an intrinsic topology called the D-*topology* with this definition that a subset of $X$ is D-*open* if its preimage by any plot is open. Endowing diffeological spaces with the D-topology, any smooth map is continuous.

Remark 2.13. A diffeological space $X$ as a topological space has homotopy groups as well. However, in general, smooth homotopy groups and usual homotopy groups do not coincide, see [8, Example 1.7.20].

Definition 2.14. [5, art. 8.9] A *diffeological fiber bundle* of fiber type $F$ is a smooth surjective map $p : E \to X$ locally trivial along the plots in $X$, that is, the pullback of $p$ by every plot in $X$ is locally trivial with the fiber $F$.

## 3. Main Results

We begin with smooth quasifibrations and give some characterizations of this notion in the sequel.

Definition 3.1. A smooth surjective map $f : X \to B$ is called a *smooth quasifibration* if for every point $b \in B$ and $x \in f^{-1}(b)$, the induced map $\mathbf{f}_{k_\#} : \pi_k(X, f^{-1}(b), x) \longrightarrow \pi_k(B, b)$ is an isomorphism, for all $k \geq 1$, and the sequence $\pi_0(f^{-1}(b), x) \xrightarrow{i_\#} \pi_0(X, x) \xrightarrow{f_\#} \pi_0(B, b)$ is exact.

Example 3.2. Any diffeological fiber bundle is a smooth quasifibration by [5, art. 8.21].

Before we state the following theorem, we remark that in general for a smooth surjective map $f : X \to B$, one has the long exact relative homotopy sequence

$$\begin{aligned} (1) \quad & \cdots \longrightarrow \pi_{k+1}(X, f^{-1}(b), x) \xrightarrow{\hat{O}_\#} \pi_k(f^{-1}(b), x) \xrightarrow{i_\#} \pi_k(X, x) \xrightarrow{j_\#} \pi_k(X, f^{-1}(b), x) \longrightarrow \\ & \cdots \longrightarrow \pi_1(X, f^{-1}(b), x) \xrightarrow{\hat{O}_\#} \pi_0(f^{-1}(b), x) \xrightarrow{i_\#} \pi_0(X, x), \end{aligned}$$

for all $b \in B$ and $x \in f^{-1}(b)$, by [5, art. 5.19].

Theorem 3.3. *A smooth surjective map $f : X \to B$ is a smooth quasifibration if and only if for all $b \in B$ and $x \in f^{-1}(b)$, we have an exact homotopy sequence*

$$\begin{aligned} & \cdots \longrightarrow \pi_{k+1}(B, b) \xrightarrow{\Delta} \pi_k(f^{-1}(b), x) \xrightarrow{i_\#} \pi_k(X, x) \xrightarrow{f_\#} \pi_k(B, b) \longrightarrow \\ & \cdots \longrightarrow \pi_1(B, b) \xrightarrow{\Delta} \pi_0(f^{-1}(b), x) \xrightarrow{i_\#} \pi_0(X, x) \xrightarrow{f_\#} \pi_0(B, b), \end{aligned}$$

*such that* $\hat{O}_\# = \Delta \circ \mathbf{f}_{k+1_\#}$.

Proof. If $f : X \to B$ is a smooth quasifibration, a long exact sequence is obtained from the sequence (1) by replacing $\pi_k(X, f^{-1}(b), x)$ with $\pi_k(B, b)$ under isomorphism $\mathbf{f}_{k_\#}$ and setting $\Delta := \hat{O}_\# \circ (\mathbf{f}_{k+1_\#})^{-1}$. Conversely, assume we are given a smooth surjective map $f : X \to B$. Consider the commutative diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \pi_{k+1}(X, f^{-1}(b), x) & \xrightarrow{\hat{O}_\#} & \pi_k(f^{-1}(b), x) & \xrightarrow{i_\#} & \pi_k(X, x) & \xrightarrow{j_\#} & \pi_k(X, f^{-1}(b), x) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \mathbf{f}_{k+1_\#}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathbf{f}_{k_\#}} & & \\
\cdots & \longrightarrow & \pi_{k+1}(B, b) & \xrightarrow{\Delta} & \pi_k(f^{-1}(b), x) & \xrightarrow{i_\#} & \pi_k(X, x) & \xrightarrow{f_\#} & \pi_k(B, b) & \longrightarrow & \cdots
\end{array}$$

where the bottom row is exact, too. It follows from the five lemma that $\mathbf{f}_{k_\#} : \pi_k(X, f^{-1}(b), x) \longrightarrow \pi_k(B, b)$ is an isomorphism, for $k > 1$. One can also verify that $\mathbf{f}_{1_\#} : \pi_1(X, f^{-1}(b), x) \longrightarrow \pi_1(B, b)$ is an isomorphism. Hence $f : X \to B$ is a smooth quasifibration. $\square$

Corollary 3.4. *For a smooth quasifibration $f : X \to B$, the fiber $f^{-1}(b)$ is week equivalent to $X$ if and only if $\pi_k(B, b) = 0$, for all $k$.*

Definition 3.5. Given a smooth map $f : X \to B$, define the *mapping path space* of $f$, and denote it by $P_f$, to be $P_f = \{(x, \gamma) \mid x \in X, \gamma \in \text{Paths}(B), \gamma(0) = f(x)\} \subseteq X \times \text{Paths}(B)$, equipped with the subspace diffeology.

Consider $\phi : P_f \to B$ given by $\phi = ev_1 \circ \text{Pr}_2$, where $ev_1 : \text{Paths}(B) \to B, \gamma \mapsto \gamma(1)$, and $\imath : X \to P_f$ defined by $\imath(x) = (x, \mathbf{c}_{f(x)})$, where $\mathbf{c}_{f(x)}$ is the constant path with the value $f(x)$. Then $f : X \to B$ factors as

$$\begin{array}{ccc}
X & \xrightarrow{\quad f \quad} & B \\
 & {\scriptstyle \imath}\searrow \quad \nearrow{\scriptstyle \phi} & \\
 & P_f &
\end{array}$$

Proposition 3.6. *The inclusion map $\imath : X \to P_f$ is a homotopy equivalence. More precisely, $\imath(X)$ is a smooth deformation retract of $P_f$.*

Proposition 3.7. *$\phi : P_f \to B$ is a smooth quasifibration.*

Proof. We first prove that the induced map $\Phi_{k_\#} : \pi_k\big(P_f, \phi^{-1}(b), (x, \gamma)\big) \longrightarrow \pi_k(B, b)$ is an isomorphism, for all $k \geq 1$, for all basepoints $b \in B$ and $(x, \gamma) \in \phi^{-1}(b)$. Let $\ell \in \text{Loops}_k(B, b)$ and define $\alpha : \mathbb{R}^{k-1} \times \mathbb{R} \to P_f$ by $\alpha(r, s) = (x, \beta(r, s))$, where $\beta(r, s) \in \text{Paths}(B)$ is given by

$$\beta(r,s)(t) = \begin{cases} \gamma\Big(\lambda(t) + \big(1 - \lambda(s)\big)\ \lambda(t)\Big), & 0 \leq \lambda(t) \leq \frac{1}{2-\lambda(s)}, \\ \ell\Big(r, \big(\lambda(t) + \big(1 - \lambda(s)\big)\ \lambda(t) - 1\big)\Big), & \frac{1}{2-\lambda(s)} \leq \lambda(t) \leq 1, \end{cases}$$

and $\lambda : \mathbb{R} \to \mathbb{R}$ is an increasing smooth function $\lambda : \mathbb{R} \to \mathbb{R}$ with $\lambda|_{(-\infty, \epsilon)} = 0$ and $\lambda|_{(1-\epsilon, \infty)} = 1$, where $0 < \epsilon < 1$ is a fixed real number, (See [2, p. 31] for a construction of such a function). Then $\alpha \in \text{Paths}_k\big(P_f, \phi^{-1}(b), (x, \gamma \circ \lambda)\big)$. But by [5, art. 5.5], $(x, \gamma \circ \lambda)$ is connected to $(x, \gamma)$ in $P_f$ so that there is a $k$-path $\alpha' \in \text{Paths}_k\big(P_f, \phi^{-1}(b), (x, \gamma)\big)$ with $\Phi_{k_\#}([\alpha']) = [\ell]$. This implies that $\Phi_{k_\#}$ is surjective. To see the injectivity of $\Phi_{k_\#}$, let $\alpha_0, \alpha_1 \in \text{Paths}_k\big(P_f, \phi^{-1}(b), (x, \gamma)\big)$ such that $\Phi_{k_\#}([\alpha_0]) = \Phi_{k_\#}([\alpha_1])$ and that $h : \mathbb{R} \to \text{Loops}_k(B, b)$ be a path from $\phi \circ \alpha_0$ to $\phi \circ \alpha_1$. Similar to the above, one can construct a path $\widetilde{h} : \mathbb{R} \to \mathrm{Paths}_k\big(P_f, \phi^{-1}(b), (x, \gamma)\big)$ from $\alpha_0$ to $\alpha_1$. Finally, to check that the sequence

$$\pi_0\big(\phi^{-1}(b), (x, \gamma)\big) \xrightarrow{i_\#} \pi_0\big(P_f, (x, \gamma)\big) \xrightarrow{f_\#} \pi_0(B, b),$$

is exact, let $(x', \gamma') \in P_f$, with $\phi((x', \gamma')) = \gamma'(1)$ connected to $b$ by some path $\xi$ in $B$. Then there exists a path $\eta : \mathbb{R} \to P_f$ connecting $(x', \gamma' \circ \lambda)$ (and so $(x', \gamma')$) to $\phi^{-1}(b)$, defined by $\eta(s) = (x', \theta(s))$, where $\theta(s) \in \mathrm{Paths}(B)$ is given by

$$\theta(s)(t) = \begin{cases} \gamma'\Big(\lambda(t) + \lambda(s)\ \lambda(t)\Big), & 0 \le \lambda(t) \le \frac{1}{1+\lambda(s)}, \\ \xi\Big(\lambda(t) + \lambda(s)\ \lambda(t) - 1\Big), & \frac{1}{1+\lambda(s)} \le \lambda(t) \le 1. \end{cases}$$

□

We denote the homotopy fiber over $b \in B$ by $\mathrm{hofiber}_b(f) := \phi^{-1}(b) = \{(x, \gamma) \in P_f \mid \gamma(1) = b\}$. As a consequence of Theorem 3.3 and Proposition 3.7, we can state:

THEOREM 3.8. *A surjective smooth map $f : X \to B$ is a smooth quasifibration if and only if for all $b \in B$, the canonical map*

$$f^{-1}(b) \longrightarrow \mathrm{hofiber}_b(f),$$
$$x \longmapsto (x, \boldsymbol{b}),$$

*is a weak equivalence, where $\boldsymbol{b}$ is the constant path with the value $b$.*

## References


1. A. Dold and R. Thom, *Quasifaserungen und unendliche symmetrische Produkte*, Ann. Math. **67** (1958) 239–281.
2. T. Haraguchi, *A Homotopy Theory of Diffeological and Numerically Generated Spaces,* Ph.D. Thesis, Okayama University, Japan, 2013.
3. A. Hatcher, *Algebraic Topology,* Cambridge University Press, Cambridge, 2002.
4. P. Iglesias-Zemmour, *Fibrations Difféologiques Et Homotopie,* Thése de Doctorat Es-sciences, L' Université de Provence, 1985.
5. P. Iglesias-Zemmour, *Diffeology,* Mathematical Surveys and Monographs, Vol. 185, American Mathematical Society, Providence, RI, USA, 2013.
6. H. Kihara, *Model category of diffeological spaces*, J. Homotopy Relat. Struct. **14** (1) (2019) 51–90.
7. J. -M. Souriau, *Groupes différentiels*, Differential geometrical methods in mathematical physics (Proc. Conf., Aix-en-Provence/Salamanca, 1979), Lecture Notes in Math. 836, Springer Verlag, Berlin, (1980) pp. 91–128.
8. E. Wu, *A Homotopy Theory for Diffeological Spaces,* Ph.D. Thesis, The University of Western Ontario, Canada, 2012.


E-mail: ahmadi@stu.yazd.ac.ir; alirezaahmadi13@yahoo.com

# The Problem of Toroidalization of Morphisms: A Step Forward

Razieh Ahmadian*
School of Mathematics, Institute for Research in Fundamental Sciences (IPM),
P. O. Box 19395-5746, Tehran, Iran

Abstract. Toroidal varieties are algebraic varieties that are locally (formally) toric in structure, and toroidal morphisms are those morphisms of varieties which are locally determined by toric morphisms. The problem of toroidalization, proposed first in [1], is to construct a toroidal lifting of a dominant morphism $\varphi : X \to Y$ of algebraic varieties by blowing up nonsingular subvarieties in the target and domain. This problem is evidently very difficult, and it has been solved only when Y is a curve, or when $\varphi$ is dominant and $X, Y$ are of dimension $\leqslant 3$ – see [7]. This article provides a comprehensive survey of the toroidalization problem. In addition, we discuss some recent results in toroidalization of locally toroidal morphisms [2], which is among patching type problems.
**Keywords:** Toroidalization, Resolution of morphisms, Principalization.
**AMS Mathematical Subject Classification [2010]:** 14M99, 14B25, 14B05.

## 1. Introduction

In algebraic geometry, studying the structure of morphisms of varieties has led to fundamental and challenging types of problems such as factorization of birational morphisms, monomialization and toroidalization of morphisms. Of interest to us is the problem of toroidalization, which can be read as resolution of singularities of morphisms in logarithmic category.

A normal variety $X$ is *toroidal* if it contains a nonsingular Zariski open subset $U \subset X$ with the property that for each $p \in X$, there exists a neighborhood $U_p$ of $p$, and an affine toric variety $X_\sigma$ with an étale morphism $\pi : U_p \to X_\sigma$ such that $\pi^{-1}(T) = U \cap U_p$ where $T$ is the algebraic torus in $X_\sigma$. When $X$ is nonsingular any simple normal crossings (SNC) divisor $D \subset X$, letting $U = X \setminus D$, specifies a toroidal structure on $X$. A dominant morphism $\varphi : X \to Y$ of nonsingular varieties is *toroidal* if there exists a SNC divisor $D_Y$ on $Y$ such that $D_X := \varphi^*(D_Y)_{red}$ is a SNC divisor on $X$ which contains the non smooth locus of $\varphi$, and $\varphi$ is locally given by monomials in appropriate étale local parameters on $X$. The precise definitions of toroidal embeddings and their morphisms are in [7].

*Speaker

The problem of toroidalization of a dominant morphism $\varphi : X \to Y$ of algebraic varieties is to construct a commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{\varphi}} & \widetilde{Y} \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow[\varphi]{} & Y \end{array} \tag{1}$$

where $\lambda : \widetilde{X} \to X$ and $\pi : \widetilde{Y} \to Y$ are sequences of blowups with non-singular centers, $\widetilde{X}$ and $\widetilde{Y}$ are non-singular, and there exist simple normal crossing (SNC) divisors $D_{\widetilde{Y}}$ and $D_{\widetilde{X}} = \tilde{\varphi}^*(D_{\widetilde{Y}})_{\text{red}}$ on $\widetilde{Y}$ and $\widetilde{X}$ respectively, such that $\tilde{\varphi} : \widetilde{X} \to \widetilde{Y}$ is toroidal with respect to $D_{\widetilde{X}}$ and $D_{\widetilde{Y}}$.

The idea of toroidalization, which is fundamental in studying the structure of birational morphisms, is first proposed in Problem 6.2.1 of [1]. This problem has been addressed in many research articles such as [3, 7, 9]. We note that toroidalization does not exist in positive characteristic $p > 0$, even for maps of curves, for instance, $y = x^p + x^{p+1}$ [7]. In addition, due to the existence of resolution of singularities in characteristic zero [10], the problem of toroidalization can be reduced to the case of morphisms of nonsingular varieties. The following conjecture, considered by Cutkosky in [7], is the strongest structure theorem which could be true for general morphisms of varieties.

**Conjecture 1.1.** *Suppose $\varphi : X \to Y$ is a dominant morphism of non-singular varieties over a field $\mathcal{K}$ of characteristic zero, $D_Y$ is a SNC divisor on $Y$ and $D_X = \varphi^{-1}(D_Y)$ is a SNC divisor on $X$ such that* $\mathrm{Sing}(\varphi) \subset D_X$. *Then there are sequences of blowups $\lambda : \widetilde{X} \to X$ and $\pi : \widetilde{Y} \to Y$ of non-singular subvarieties which are supported in the preimages of $D_X$ and $D_Y$ respectively, and make SNCs with them such that the diagram (1) commutes and $\tilde{\varphi}$ is toroidal with respect to $D_{\widetilde{X}} = \lambda^{-1}(D_X)$ and $D_{\widetilde{Y}} = \pi^{-1}(D_Y)$.*

Cutkosky has proven strong toroidalization for dominant morphisms of 3-folds in [7], where he has also outlined some proofs of the problem in lower dimensions. He also has given a significantly simpler and more conceptual proof of toroidalization of morphisms of 3-folds to surfaces. In general, specially when $\dim Y > 2$, the problem of toroidalization seems hard enough to be considered in rather restricted classes of morphisms such as strongly prepared, or locally toroidal morphisms. The latter notion is originated from Cutkosky's proof of toroidalization, locally along a fixed valuation, in all dimensions in [6]. A form of local toroidalization, which one can hope to reduce to in general, is that of a locally toroidal morphism.

Definition 1.2. [2, Definition 1.1] Let $\varphi : X \to Y$ be a dominant morphism of nonsingular varieties. Suppose that there exist finite open covers $\{U_\alpha\}_{\alpha\in I}$ of $X$ and $\{V_\alpha\}_{\alpha\in I}$ of $Y$, and SNC divisors $D_\alpha \subset U_\alpha$ and $E_\alpha \subset V_\alpha$ for each $\alpha \in I$, such that

(1) $\varphi_\alpha := \varphi|_{U_\alpha} : U_\alpha \to V_\alpha$,
(2) $D_\alpha = \varphi_\alpha^{-1}(E_\alpha)$, and
(3) $\varphi_\alpha : U_\alpha \setminus D_\alpha \to V_\alpha \setminus E_\alpha$ is smooth,

for all $\alpha \in I$. We say that $\varphi : (X, U_\alpha, D_\alpha)_{\alpha\in I} \to (Y, V_\alpha, E_\alpha)_{\alpha\in I}$ is ***locally toroidal*** if for each $\alpha \in I$, $\varphi_\alpha : U_\alpha \to V_\alpha$ is toroidal with respect to $D_\alpha$ and $E_\alpha$ (c.f. [9, Definition 1.3]).

Toroidalization of locally toroidal morphisms has been proved when $Y$ is a surface by Hanumanthu [9], and in [4, 5] when $X$ and $Y$ are 3-folds. The strategy and most of the methods used in those papers can be extended to solve the problem in higher dimensions. However, some important parts of the proofs is specific to dimension three – c.f. [4, Example 2.7]. In [2], we overcome all of those technical hurdles, and we give a solution to the toroidalization of locally toroidal morphisms in arbitrary dimensions of $X$ and $Y$.

## 2. Main Sections and Results

In this section, we review our recent results in [2]. Throughout this paper, $\mathcal{K}$ is an algebraically closed field of characteristic zero, and a variety is a quasi–projective variety over $\mathcal{K}$. Suppose that $\varphi : (X, U_\alpha, D_\alpha)_{\alpha\in I} \to (Y, V_\alpha, E_\alpha)_{\alpha\in I}$ is a locally toroidal morphism. The key fact is that if there exists a global toroidal structure, i.e., a SNC divisor $E$, on $Y$ with the property that $E_\alpha \subset E$, for all $\alpha \in I$, then $\varphi$ is toroidal with respect to $E$ and $D := \varphi^*(E)_{\mathrm{red}}$[1]. Such a toroidal structure on $Y$ can be constructed by using the algorithm of embedded resolution of singularities [10]. We have observed that the result of this algorithm, and the permissible blowing ups have our required geometric properties in the following theorem. This theorem was fist proved in dimension three [3, Theorem 3.4], and then we have generalized it to the case when $Y$ has arbitrary dimension. We note that if $\dim Y = 2$, the resolution algorithm only consists of point blowing ups, and this theorem is not necessary in this case.

Theorem 2.1. [2, Theorem 3.5] *Let $(V_\alpha, E_\alpha)_{\alpha\in I}$ be local toroidal data of a nonsingular variety $Y$, and consider the hypersurface $\widetilde{\mathcal{E}}_0 = \sum_{\alpha\in I} \overline{E}_\alpha \subset Y$, where $\overline{E}_\alpha$ is the Zariski closure of $E_\alpha$ in $Y$. There exists a finite sequence*

$$\pi : \widetilde{Y} = Y_{n_0} \xrightarrow{\pi_{n_0}} Y_{n_0-1} \to \cdots \to Y_k \xrightarrow{\pi_k} Y_{k-1} \to \cdots \to Y_1 \xrightarrow{\pi_1} Y_0 = Y$$

*of monoidal transforms centered in the closed sets of points of maximum order such that $\widetilde{Y}$ is nonsingular and the total transform $\pi^{-1}(\widetilde{\mathcal{E}}_0)$ is a SNC divisor on $\widetilde{Y}$, i.e., $\pi$ is an embedded resolution of singularities of $\widetilde{\mathcal{E}}_0$. For $0 \leqslant k \leqslant n_0$ and $\alpha \in I$, let:*

| | |
|---|---|
| $\Pi_k = \pi_0 \circ \cdots \circ \pi_k$, | $V_{k,\alpha} = \Pi_k^{-1}(V_\alpha)$, |
| $\pi_{k,\alpha} = \pi_k\|_{V_{k,\alpha}} : V_{k,\alpha} \to V_{k-1,\alpha}$, | $\Pi_{k,\alpha} = \Pi_k\|_{V_{k,\alpha}} : V_{k,\alpha} \to V_\alpha$, |
| $Z_k$*: the center of* $\pi_{k+1}$, | $Z_{k,\alpha} = Z_k \cap V_{k,\alpha}$, |
| $E_{k,\alpha} = \Pi_{k,\alpha}^{-1}(E_\alpha) = \Pi_{k,\alpha}^*(E_\alpha)_{\mathrm{red}}$, | $\widetilde{\mathcal{E}}_k = (\sum_{\alpha\in I} \overline{E}_{k,\alpha})_{\mathrm{red}}$, |

*where $\overline{E}_{k,\alpha}$ is the Zariski closure of $E_{k,\alpha}$ in $Y_k$. We further have that*

1) *$E_{k,\alpha}$ is a SNC divisor on $V_{k,\alpha}$ for all $k, \alpha$, and $Z_{k,\alpha}$ makes SNCs with $E_{k,\alpha}$ on $V_{k,\alpha}$ for all $k, \alpha$. (Although possibly $Z_{k,\alpha} \cap E_{k,\alpha} \neq \emptyset$ but $Z_{k,\alpha} \not\subset E_{k,\alpha}$).*

[1]This has been shown in the proof of [2, Theorem 1.2].

2) $\widetilde{\mathcal{E}}_k \subseteq \Pi_k^{-1}(\widetilde{\mathcal{E}}_0)$ *for all* $k$.[2]

Suppose $\pi : Y_1 \to Y$ is a permissible blowup in the ERS of $\widetilde{\mathcal{E}}_0 = \sum_{\alpha \in I} \overline{E}_\alpha$ with center $Z \subset Y$. The key observation is that indeterminacy of the rational map $\pi^{-1} \circ \varphi : X \dashrightarrow Y_1$ coincides with the locus of points where $\mathcal{I}_Z \mathcal{O}_X$ is not locally principal, i.e.,

$$W_Z(X) := \{p \in X \mid \mathcal{I}_Z \mathcal{O}_{X,p} \text{ is not principal}\}.$$

We have applied a specific algorithm for principalization of an ideal sheaves in order to resolve the indeterminacy of $\pi^{-1} \circ \varphi$. The effect of a principalization sequence on toroidal morphisms has been studied carefully in [2, Lemma 3.12] using the following characterization of toroidal morphisms. This is a generalization of [8, Lemma 4.2] and [3, Lemma 3.4] to arbitrary dimensions of $X, Y$.

Theorem 2.2. [2, Theorem 2.7] *Suppose that* $\varphi : X \to Y$ *is a dominant morphism of nonsingular* $\mathcal{K}$*–varieties where* $\dim X = d$ *and* $\dim Y = m$*. Further suppose that there is a simple normal crossings (SNC) divisor* $D_Y$ *on* $Y$ *such that* $D_X = \varphi^{-1}(D_Y)$ *is a SNC divisor on* $X$ *which contains the non smooth locus of the map* $\varphi$*. Then the morphism* $\varphi$ *is toroidal if and only if for each* $n$*–point* $p \in D_X$ *and* $l$*–point* $q = \varphi(p) \in D_Y$*, there exist permissible parameters* $\mathbf{x} = (x_1, \dots, x_d)$ *at* $p$ *for* $D_X$*, and permissible parameters* $\mathbf{y} = (y_1, \dots, y_m)$ *at* $q$ *for* $D_Y$*, and there exist* $(a_{ij})_{\substack{1\leqslant i\leqslant \ell \\ 1\leqslant j\leqslant n}} \in \mathbb{N}^{\ell \times n}$ *such that* $\varphi$ *is given by a system of equations of the following form:*

$$y_i = \begin{cases} x_1^{a_{i1}} \dots x_n^{a_{in}}, & 1 \leqslant i \leqslant r, \\ x_1^{a_{i1}} \dots x_n^{a_{in}} (x_{n-r+i} + \alpha_{n-r+i}), & r < i \leqslant \ell, \\ x_{n-r+i}, & \ell < i \leqslant m, \end{cases}$$

*where* $r = \operatorname{rank}(a_{ij})_{\substack{1\leqslant i\leqslant \ell \\ 1\leqslant j\leqslant n}} = \operatorname{rank}(a_{ij})_{\substack{1\leqslant i\leqslant r \\ 1\leqslant j\leqslant r}}$*, and* $\alpha_{n+1}, \dots, \alpha_{n+\ell-r} \in \mathcal{K}^\times$*. In addition,*

$$\text{for all } j \in [n], \sum_{i=1}^{\ell} a_{ij} > 0, \text{ and for all } i \in [\ell], \sum_{j=1}^{n} a_{ij} > 0.$$

Applying embedded resolution of singularities, i.e., blowing up permissible centers above $Y$, and resolving indeterminacy at each steps, we con construct the following commutative diagram inductively.

Theorem 2.3. [2, Theorem 4.2] *Suppose that*

$$\varphi : (X, U_\alpha, D_\alpha)_{\alpha \in I} \to (Y, V_\alpha, E_\alpha)_{\alpha \in I},$$

[2]See [4, Example 2.5] for an example which shows that the inclusion of (2.1) is not in general an equality.

*is a locally toroidal morphism of nonsingular varieties, and let* $\mathcal{E}_0 = \sum_{\alpha \in I} \overline{E}_\alpha$, *where* $\overline{E}_\alpha$ *is the Zariski closure of* $E_\alpha$ *in* $Y$. *There exists a commutative diagram*

$$\begin{array}{ccccccccc}
\lambda : \widetilde{X} = X_{n_0} & \xrightarrow{\lambda_{n_0}} & X_{n_0-1} & \longrightarrow & \cdots & \longrightarrow & X_1 & \xrightarrow{\lambda_1} & X = X_0 \\
\downarrow{\scriptstyle \widetilde{\varphi}=\phi_{n_0}} & & \downarrow{\scriptstyle \phi_{n_0-1}} & & & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0=\varphi} \\
\pi : \widetilde{Y} = Y_{n_0} & \xrightarrow[\pi_{n_0}]{} & Y_{n_0-1} & \longrightarrow & \cdots & \longrightarrow & Y_1 & \xrightarrow[\pi_1]{} & Y = Y_0
\end{array}$$

*with the following properties. For* $\alpha \in I$ *and* $0 \leqslant k \leqslant n_0$, *let*

$$\begin{array}{ll}
\Pi_k = \pi_0 \circ \cdots \circ \pi_k, & \Lambda_k = \lambda_0 \circ \cdots \circ \lambda_k, \\
V_{k,\alpha} = \Pi_k^{-1}(V_\alpha), & U_{k,\alpha} = \Lambda_k^{-1}(U_\alpha), \\
\pi_{k,\alpha} = \pi_k|_{V_{k,\alpha}} : V_{k,\alpha} \to V_{k-1,\alpha}, & \lambda_{k,\alpha} = \lambda_k|_{U_{k,\alpha}} : U_{k,\alpha} \to U_{k-1,\alpha}, \\
\Pi_{k,\alpha} = \Pi_k|_{V_{k,\alpha}} : V_{k,\alpha} \to V_\alpha, & \Lambda_{k,\alpha} = \Lambda_k|_{U_{k,\alpha}} : U_{k,\alpha} \to U_\alpha, \\
E_{k,\alpha} = \Pi_{k,\alpha}^{-1}(E_\alpha) = \Pi_{k,\alpha}^*(E_\alpha)_{\mathrm{red}}, & D_{k,\alpha} = \Lambda_{k,\alpha}^{-1}(D_\alpha) = \Lambda_{k,\alpha}^*(D_\alpha)_{\mathrm{red}}.
\end{array}$$

i) *The morphisms* $\lambda : \widetilde{X} \to X$ *and* $\pi : \widetilde{Y} \to Y$ *are sequences of monoidal transforms.*

ii) *For all* $\alpha \in I$ *and* $0 \leqslant k \leqslant n_0$, $D_{k,\alpha}$ *is a SNC divisor on* $U_{k,\alpha}$, $E_{k,\alpha}$ *is a SNC divisor on* $V_{k,\alpha}$, *and* $\phi_k : (X_k, U_{k,\alpha}, D_{k,\alpha})_{\alpha \in I} \to (Y_k, V_{k,\alpha}, E_{k,\alpha})_{\alpha \in I}$ *is a locally toroidal morphism of nonsingular varieties.*

iii) *The divisor* $\widetilde{\mathcal{E}} := \pi^{-1}(\mathcal{E}_0)$ *is SNC on* $\widetilde{Y}$, *and for all* $\alpha \in I$, $\pi^{-1}(E_\alpha) \subset \widetilde{\mathcal{E}}$.

Finally, in our main result we have proved that the locally toroidal morphism $\phi_{n_0}$, constructed above, is actually toroidal.

Theorem 2.4. [2, Theorem 1.2] *Suppose that*

$$\varphi : (X, U_\alpha, D_\alpha)_{\alpha \in I} \to (Y, V_\alpha, E_\alpha)_{\alpha \in I},$$

*is a locally toroidal morphism of nonsingular varieties. There exists a commutative diagram*

$$\begin{array}{ccc}
\widetilde{X} & \xrightarrow{\widetilde{\varphi}} & \widetilde{Y} \\
\lambda\downarrow & & \downarrow\pi \\
X & \xrightarrow[\varphi]{} & Y
\end{array}$$

*such that* $\lambda : \widetilde{X} \to X$ *and* $\pi : \widetilde{Y} \to Y$ *are sequences of blowups with nonsingular centers,* $\widetilde{X}$ *and* $\widetilde{Y}$ *are nonsingular, and there exists SNC divisor* $\widetilde{E}$ *on* $\widetilde{Y}$ *such that* $\widetilde{D} := \widetilde{\varphi}^{-1}(\widetilde{E})$ *is a SNC divisor on* $\widetilde{X}$, *and* $\widetilde{\varphi}$ *is toroidal with respect to* $\widetilde{E}$ *and* $\widetilde{D}$.

## References


1. D. Abramovich, K. Karu, K. Matsuki and J. Wlodarczyk, *Torification and factorization of birational maps*, J. Amer. Math. Soc. **15** (3) (2002) 531–572.
2. R. Ahmadian, *Toroidalization of locally toroidal morphisms*, (2020). arXiv:2012.04499
3. R. Ahmadian, *Toroidalization of locally toroidal morphisms from n-folds to 3-folds*, J. Pure Appl. Algebra **224** (11) (2020) 106406.
4. R. Ahmadian, *A principalization algorithm for locally monomial ideal sheaves on 3-folds with an application to toroidalization*, J. Algebra **454** (2016) 139–180.

5. R. Ahmadian, *Toroidalization of locally toroidal morphisms of* 3*-folds*, Bull. Iranian Math. Soc. **42** (2) (2016) 371–405.
6. S. D. Cutkosky, *Local monomialization and factorization of morphisms*, Astérisque **260** (1999).
7. S. D. Cutkosky, *Toroidalization of Dominant Morphisms of* 3*-Folds*, Memoirs of the AMS 890, Amer. Math. Soc, Providence, 2007.
8. S. D. Cutkosky and O. Kascheyeva, *Monomialization of strongly prepared morphisms from nonsingular $n$-folds to surfaces*, J. Algebra **275** (2004) 275–320.
9. K. Hanumanthu, *Toroidalization of locally toroidal morphisms from $n$-folds to surfaces*, J. Pure Appl. Algebra **213** (3) (2009) 349–359.
10. H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. of Math. **79** (1) (1964) 109–203.

E-mail: ahmadian@ipm.ir

# Characterization of the Killing and Homothetic Vector Fields on Lorentzian PP-Wave Four-Manifolds

Parvaneh Atashpeykar*
Department of Mathematics, Basic Sciences Faculty, University of Bonab, Bonab 5551761167, Iran
and Ali Haji-Badali
Department of Mathematics, Basic Sciences Faculty, University of Bonab, Bonab 5551761167, Iran

Abstract. We consider the Lorentzian pp-wave four-manifolds. We obtain a full classification of the Killing and homothetic vector fields of these spaces. We also provide an example of killing vector fields on these manifolds.
**Keywords:** PP-wave manifold, Killing vector field, Homothetic vector field, Lorentzian.
**AMS Mathematical Subject Classification [2010]:** 53C43, 53B30.

## 1. Introduction

A Lorentzian manifold with a parallel light-like vector field is called Brinkmann-wave, due to [1]. A Brinkmann-wave manifold $(M, g)$ is called pp-wave if its curvature tensor $R$ satisfies the trace condition $tr_{(3,5)(4,6)}(R \otimes R) = 0$. In [2], Schimming proved that an $(n+2)$-dimensional pp-wave manifold admits coordinates $(x, y_1, \dots, y_n, z)$ such that $g$ has the form

$$g = 2dxdz + \sum_{k=1,\dots,n} (dy_k)^2 + f(dz)^2, \ with \ \partial_x f = 0.$$

In [3], Leistner gave another equivalence for pp-wave manifold. More precisely, he proved that a Brinkmann-wave manifolds $(M, g)$ with parallel light-like vector field $X$ and induced parallel distributions $\Xi$ and $\Xi^{\perp}$ is a pp-wave if and only if its curvature tensor satisfies

$$R(U, V) : \Xi^{\perp} \to \Xi, \ for \ all \ U, V \in TM,$$

or equivalently $R(Y_1, Y_2) = 0 \ for \ all \ Y_1, Y_2 \in \Xi^{\perp}$. From this description, it follows that a pp-wave manifold is Ricci-isotropic, which means that the image of the Ricci operator is totally light-like, and has vanishing scalar curvature [3].

In this paper, we shall investigate killing and homothetic vector fields on the Lorentzian pr-wave four-manifolds. If $(M, g)$ denotes a Lorentzian manifold and $T$ a tensor on $(M, g)$, codifying some either mathematical or physical quantity, a symmetry of $T$ is a one-parameter group of diffeomorphisms of $(M, g)$, leaving $T$ invariant. As such, it corresponds to a vector field $X$ satisfying $\mathcal{L}_X T = 0$, where $\mathcal{L}$ denotes the Lie derivative. Isometries are a well known example of symmetries,

*Speaker

for which $T = g$ is the metric tensor. The corresponding vector field $X$ is then a Killing vector field. Homotheties and conformal motions on $(M, g)$ are again examples of symmetries. (See, for example, [4, 5, 6, 7, 8] and references therein). All calculations have also been checked using $Maple16^{©}$.

## 2. Killing and Homothetic Vector Fields of PP-Wave Four-Manifolds

We first classify Killing and homothetic and affine vector fields of $(M, g)$. The classifications we obtain are summarized in the following theorem.

Theorem 2.1. *Let $X = X^1\partial_1 + X^2\partial_2 + X^3\partial_3 + X^4\partial_4$ be an arbitrary vector field on the pp-wave four-manifold $(M, g)$. Then*

i) *$X$ is a Killing vector field if and only if*

$$X^1 = -c_1x_1 - x_2x_3f_1^{'}(x_4) - x_2f_2(x_4), \qquad X^2 = x_3f_1(x_4) + f_2(x_4),$$
$$X^3 = -x_2f_1(x_4) + f_3(x_4), \qquad X^4 = c_1z + c_2,$$

*where $f_1, f_2, f_3$ are smooth functions on $M$, satisfying*

$$(1)\quad 2c_1f - 2f_1^{''}(x_4)x_2x_3 + (f_1(x_4)x_3 + f_2(x_4))\partial_2f + (-f_1(x_4)x_2 + f_3(x_4))\partial_3f + (c_1x_4 + c_2)\partial_4f = 0.$$

ii) *A homothetic, non-Killing vector field if and only if*

$$X^1 = \eta x_1 - c_1x_1 - x_2x_3f_1^{'}(x_4) - x_2f_2(x_4), \qquad X^2 = \frac{\eta}{2}x_2 + x_3f_1(x_4) + f_2(x_4),$$
$$X^3 = \frac{\eta}{2}x_3 - x_2f_1(x_4) + f_3(x_4), \qquad X^4 = c_1x_4 + c_2,$$

*where $\eta \neq 0$ is a real constant and*

$$(2c_1 - \eta)f - 2f_1^{''}(x_4)x_2x_3 + (\frac{\eta}{2}x_2 + f_1(x_4)x_3 + f_2(x_4))\partial_2f + (\frac{\eta}{2}x_3 - f_1(x_4)x_2 + f_3(x_4))\partial_3f + (c_1x_4 + c_2)\partial_4f = 0.$$

Proof. We start from an arbitrary smooth vector field $X = X^1\partial_1 + X^2\partial_2 + X^3\partial_3 + X^4\partial_4$ on the four-dimensional pp-wave manifold $(M, g)$, and calculate $\mathcal{L}_Xg$. we have

$$\begin{aligned}\mathcal{L}_Xg = &\, 2\partial_1X^4dx_1dx_1 + 2(\partial_1X^2 + \partial_2X^4)dx_1dx_2 + 2(\partial_1X^3 \\ &+ \partial_3X^4)dx_1dx_3 + 2(\partial_1X^1 + f\partial_1X^4 + \partial_4X^4)dx_1dx_4 \\ &+ 2\partial_2X^2dx_2dx_2 + 2(\partial_2X^3 + \partial_3X^2)dx_2dx_3 + 2(\partial_2X^1 + f\partial_2X^4 + \partial_4X^2)dx_2dx_4 \\ &+ 2\partial_3X^3dx_3dx_3 + 2(\partial_3X^1 + \partial_4X^3 + f\partial_3X^4)dx_3dx_4 \\ &+ (2\partial_4X^1 + X^2\partial_2f + X^3\partial_3f + 2f\partial_4X^4 + f\partial_4X^4)dx_4dx_4.\end{aligned}$$

Then, $X$ satisfies $\mathcal{L}_Xg = \eta g$ for some real constant $\eta$ if and only if the following system of partial differential equations is satisfied:

$$\partial_1X^4 = 0, \quad \partial_1X^2 + \partial_2X^4 = 0, \quad \partial_1X^3 + \partial_3X^4 = 0, \quad \partial_2X^2 = \frac{\eta}{2},$$
$$\partial_2X^3 + \partial_3X^2 = 0, \quad \partial_3X^3 = \frac{\eta}{2}, \quad \partial_1X^1 + f\partial_1X^4 + \partial_4X^4 = \eta,$$
$$\partial_2X^1 + f\partial_2X^4 + \partial_4X^2 = 0, \quad \partial_3X^1 + \partial_4X^3 + f\partial_3X^4 = 0,$$
$$(2)\qquad 2\partial_4X^1 + X^2\partial_2f + X^3\partial_3f + 2f\partial_4X^4 + f\partial_4X^4 = \eta f,$$

We then proceed to integrate (2). From the first six equations in (2) we get

$$X^2 = \frac{\eta}{2}x_2 - f_1(x_4)x_1 + f_4(x_4)x_3 + f_5(x_4),$$
$$X^3 = \frac{\eta}{2}x_3 - f_2(x_4)x_1 - f_4(x_4)x_2 + f_7(x_4),$$
$$X^4 = f_1(x_4)x_2 + f_2(x_4)x_3 + f_3(x_4).$$

Then, the seventh equation in (2) yields

$$X^2 = \eta x_1 - f_1^{'}(x_4)x_1x_2 + f_2^{'}(x_4)x_1x_3 - f_3^{'}(x_4)x_1 + f_7(x_2, x_3, x_4).$$

So, the eighth equation in (2) yields

$$2f_1^{'}(x_4)x_1 = f_1(x_4)f + f_4^{'}(x_4)x_3 + f_5^{'}(x_4) + \partial_2 f_7(x_2, x_3, x_4),$$

which must hold for all values of $x_1$, implying that $F_1(x_4) = c_1$ is a constant. Also, the ninth equation in (2) yields

$$2f_2^{'}(x_4)x_1 = f_2(x_4)f - f_4^{'}(x_4)x_2 + f_6^{'}(x_4) + \partial_3 f_7(x_2, x_3, x_4),$$

which must hold for all values of $x_1$, implying that $f_2(x_4) = c_2$ is a constant. Now, the last equation in (2) gives

$$\begin{aligned}
-(c_1\partial_2 f + c_2\partial_3 f + 2f_3^{''}(x_4))x_3 &= (2f_3^{'}(x_4) - \eta)f + 2\partial_2 f_7(x_2, x_3, x_4)\\
&+ (\frac{\eta}{2}x_2 + f_4(x_4)x_3 + f_5(x_4))\partial_2 f\\
&+ (\frac{\eta}{2}x_2 - f_4(x_4)x_2 + f_6(x_4))\partial_3 f\\
&+ (c_1x_2 + c_2x_3 + f_3(x_4))\partial_4 f,\\
&\Rightarrow c_1\partial_2 f + c_2\partial_3 f + 2f_3^{''}(x_4) = 0,\\
&\Rightarrow c_1\partial_{22}^2 f + c_2\partial_{32}^2 f = 0,\\
&\Rightarrow c_1 = c_2 = 0,\\
&\Rightarrow f_3^{''}(x_4) = 0,\\
&\Rightarrow f_3(x_4) = c_3x_4 + c_4,
\end{aligned}$$

and the last equation gives

$$\begin{aligned}
&(2c_1 - \eta)f - 2f_1^{''}(x_4)x_2x_3 + (\frac{\eta}{2}x_2 + f_1(x_4)x_3 + f_2(x_4))\partial_2 f\\
&+ (\frac{\eta}{2}x_3 - f_1(x_4)x_2 + f_3(x_4))\partial_3 f + (c_1x_4 + c_2)\partial_4 f = 0.
\end{aligned}$$

This proves the statement (i) in the case $\eta = 0$ and the statement (ii) if we assume $\eta \neq 0$. □

EXAMPLE 2.2. The functions in Eq. (1) for the killing vector fields on the pp-wave four-manifolds produce a various family of killing vector fields on the pp-wave four-manifolds. for example, let $f(x, y, z) = \cos x_2 + \sin x_3$, we have

$$\begin{aligned}
&2c_1(\cos x_2 + \sin x_3) - 2f_1^{''}(x_4)x_2x_3 - (f_1(x_4)x_3 + f_2(x_4))\sin x_2\\
&+ (-f_1(x_4)x_2 + f_3(x_4))\cos x_3 = 0.
\end{aligned}$$

In a special case, it can be assumed $c_1 = 0$. Therefore,

$$f_3(x_4) = \frac{1}{\cos x_3}(2f_1^{''}(x_4)x_2x_3 + (f_1(x_4)x_3 + f_2(x_4))\sin x_2 + f_1(x_4)x_2\cos x_3).$$

Now, with the arbitrary selection for function $f_1(z)$ and $f_2(z)$, killing vector fields are generated, which is a special example as follows:

$$f_1(x_4) = x_4,\ \ f_2(x_4) = \cos x_4.$$

So, we have

$$f_3(x_4) = \frac{1}{\cos x_3}((x_3x_4 + \cos x_4)\sin x_2 + x_2x_4\cos x_3).$$

In a special case, it can be assumed $c_2 = 0$. Hence,

$$\begin{aligned}
X^1 &= -x_2x_3 - x_2\cos x_4,\\
X^2 &= x_3x_4 + \cos x_4',\\
X^3 &= -x_2x_4 + \frac{1}{\cos x_3}((x_3x_4 + \cos x_4)\sin x_2 + x_2x_4\cos x_3,\\
X^4 &= 0.
\end{aligned}$$

## References

[1] H. W. Brinkmann, *Einstein spaces which are mapped conformally on each other,* Math. Ann. **94** (1925) 119–145.

[2] R. Schimming, *Riemannsche Räume mit ebenfrontiger und mit ebener Symmetrie,* Math. Nachr. **59** (1974) 128–162.

[3] T. Leistner, *Conformal holonomy of C-spaces, Ricci-flat, and Lorentzian manifolds,* Differential Geom. Appl. **24** (5) (2006) 458–478.

[4] P. C. Aichelburg, *Curvature collineations for gravitational pp-waves,* J. Math. Phys. **11** (8) (1970) 2458–2462.

[5] G. Calvaruso and A. Zaeim, *Invariant symmetries on non-reductive homogeneous pseudo-Riemannian four manifolds,* Rev. Mat. Complut. **28** (2015) 599–622.

[6] E. Calviño-Louzao, J. Seoane-Bascoy, M. E. Vázquez-Abal and R. Vázquez-Lorenzo, *Invariant Ricci collineations on three-dimensional Lie groups,* J. Geom. Phys. **96** (2015) 59–71.

[7] G. S. Hall, *Symmetries and Curvature Structure in General Relativity,* World Scientific, Lecture Notes in Physics, Vol. 46, World Scientific Publishing Co., Inc., River Edge, NJ, 2004.

[8] G. S. Hall and M. S. Capocci, *Classification and conformal symmetry in three-dimensional space-times,* J. Math. Phys. **40** (3) (1999) 1466–1478.

E-mail: p.atashpeykar@ubonab.ac.ir
E-mail: haji.badali@ubonab.ac.ir

# Hom-Lie Algebroid Structures on Double Vector Bundles and Representation up to Homotopy

Mohammad Reza Farhangdoost
Department of Mathematics, College of Sciences, Shiraz University, P. O. Box 71457-44776, Shiraz, Iran
Sadegh Merati
Department of Mathematics, College of Sciences, Shiraz University, P. O. Box 71457-44776, Shiraz, Iran
and Ahmad Reza Attari Polsangi*
Department of Mathematics, College of Sciences, Shiraz University, P. O. Box 71457-44776, Shiraz, Iran

Abstract. In this paper we show that, there exists a correspondence between the VB-hom algebroids, which is basically defined as a hom-Lie algebroid object in the category of vector bundles and two term representations up to homotopy of hom-Lie algebroid.

**Keywords:** Hom-Lie algebroid, Representation up to homotopy, VB hom-Lie algebroid.
**AMS Mathematical Subject Classification [2010]:** 13F55, 05E40, 05C65.

## 1. Introduction

The lack of a well-defined adjoint representation is a serious issue with the usual notion of hom-Lie algebroid representation, same as Lie algebroid representations. The effort to resolve this problem has led to a number of proposed generalizations of the notion of Lie algebroid representation [1, 3, 7, 8, 9, 11, 13, 14, 15]. The notion of representation up to homotopy is the most popular of these generalizations [10, 11].

We will show that, there exists a correspondence between the VB-hom algebroids and two term representation up to homotopy of hom-Lie algebroids.

We recall the notion of hom-Lie algebroids.

Definition 1.1. [5] A hom-Lie algebroid is a quintuple $(U \to N, \psi, [\cdot,\cdot]_U, \tau, \Psi)$, where $U \to N$ is a vector bundle over a manifold $N$, $\psi : N \to N$ is a smooth map, $[\cdot,\cdot]_U : \Gamma(U) \otimes \Gamma(U) \to \Gamma(U)$ is a bilinear map, called bracket, $\tau : U \to TN$ is a vector bundle morphism, called anchor, and $\Psi : \Gamma(U) \to \Gamma(U)$ is a linear endomorphism of $\Gamma(U)$ such that

(1) $\Psi(fW) = \psi^*(f)\Psi(W)$, for all $W \in \Gamma(U), f \in C^\infty(N)$,
(2) the triple $(\Gamma(U), [\cdot,\cdot]_U, \Psi)$ is a hom-Lie algebra,
(3) the following hom-Leibniz identity holds:

$$[W, fZ]_U = \psi^*(f)[W,Z]_U + \mathcal{L}_{\tau(W)}(f)\Psi(Z), \quad \text{for all } W, Z \in \Gamma(U), f \in C^\infty(N),$$

*Speaker

(4) $(\Psi, \psi^*)$ is a representation of $(\Gamma(U), [\cdot,\cdot]_U, \Psi)$ on $C^\infty(N)$.

A vector bundle morphism is a map $\varphi$ between two hom-Lie algebroids $(U \to N, \psi, [\cdot,\cdot]_U, \tau, \Psi)$ and $(U' \to N, \psi', [\cdot,\cdot]_{U'}, \tau', \Psi')$ such that

(1) $\tau' \circ \varphi = \tau$,
(2) $\Psi' \circ \varphi^* = \varphi^* \circ \Psi$ and
(3) $\varphi([W, Z]_U) = [\varphi(W), \varphi(Z)]_{U'}$,

for all $W, Z \in \Gamma(U)$.

Let $\epsilon$ be a graded vector bundle with respect to degree preserving operator $\beta$ on $\epsilon$. A representation up to homotopy of a hom-Lie algebroid $U$ is a degree 1 operator $R_\beta$ on $\Omega_\beta(U;\epsilon)$ such that $R_\beta^2 = 0$, $\Psi^* R_\beta = \beta \circ R_\beta$, and $R_\beta(\omega\eta) = R_\beta\omega\Psi^*(\eta) + (-1)^p \Psi^*\omega R_\beta(\eta)$, for any $\omega \in \Omega^p(U)$ and $\eta \in \Omega_\beta(U;\epsilon)$.

We mean a representation up to homotopy with respect to $\beta$, By an $\beta$-representation up to homotopy.

A degree zero $\Omega(U)$-linear map $\varphi : \epsilon_1 \to \epsilon_2$ is a morphism between $\beta$-representation up to homotopy $(\epsilon_1, R_\beta)$ and $\gamma$-representation up to homotopy $(\epsilon_2, R_\gamma)$ of hom-Lie algebroid $U$, if commutes with $\beta$ and $\gamma$ and the structure differentials $R_\beta$ and $R_\gamma$.

In [12], a double vector bundle was introduced and was more studied by [2, 4, 6, 7].

Definition 1.2. [4] A double vector bundle is a commutative square

$$\begin{array}{ccc} S & \xrightarrow{q_V^S} & V \\ {\scriptstyle q_U^S}\downarrow & & \downarrow{\scriptstyle q_V} \\ U & \xrightarrow[q_U]{} & N \end{array}$$

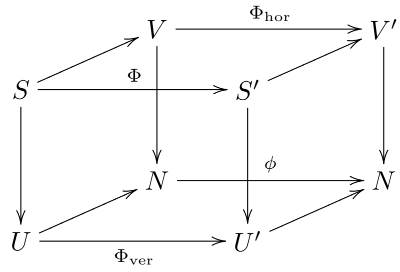
where all four sides are vector bundles and $q_V^S$ and $+_V$ are vector bundle morphisms over $q^U$ and additional map $+ : U \times_N U \to U$, respectively.

Let $(S; U, V; N)$ be a double vector bundle, two vector bundles $U$ and $V$ are called the side bundles. We define the zero sections that are denoted by $0^U : N \to U$, $0^V : N \to V$, ${}^U0 : U \to S$ and ${}^V0 : V \to S$. We write $(s; u, v; n)$ as elements of $S$, where $s \in S$, $n \in N$ and $u = q_U^S(s) \in U_n$, $v = q_V^S(s) \in V_n$.

The intersection of the kernels of $q_U^S$ and $q_V^S$ is the *core* of a double vector bundle $S$, which is denoted by $C$. It has a natural vector bundle structure over $N$, the projection of which we call $q_C : C \to N$. The inclusion $C \hookrightarrow S$ is denoted by

$$C_n \ni c \longmapsto \overline{c} \in (q_U^S)^{-1}(0_n^U) \cap (q_V^S)^{-1}(0_n^V).$$

Definition 1.3. A double vector bundle morphism $(\Phi; \Phi_{\rm ver}, \Phi_{\rm hor}; \phi)$ between two double vector bundle $(S; U, V; N)$ and $(S'; U', V'; N)$ is a commutative cube




where $\Phi$ over $\Phi_{\rm ver}$ and $\Phi_{\rm hor}$, also $\Phi_{\rm ver}$ and $\Phi_{\rm hor}$ over $\phi$ are vector bundle morphisms.

Let $(\Phi; \Phi_{\rm ver}, \Phi_{\rm hor}; \phi)$ be a double vector bundle morphism from $S$ to $S'$. We find a vector bundle morphism $\Phi_c : C \to C'$ by its restriction to the core bundles.

The *core section* $\hat{c} : V \to S$ is defined as

$$\hat{c}(v_n) = {}^V0_{v_n} \underset{U}{+} \overline{c(n)},\ n \in N,\ v_n \in V_n, core_section$$

where, $c : N \to C$ is a section of core bundle $C$. The space of core sections is denoted by $\Gamma_c(V, S)$. A section $\mathcal{W} : V \to S$ is a linear section, where it is a bundle morphism from $V \to N$ to $S \to U$, $\Gamma_\ell(V, S)$ denotes the space of linear sections.

Let $(S; U, V; N)$ be a double vector bundle, the $C^\infty(V)$-module $\Gamma(V, S)$ is generated by two distinguished classes of sections, the *linear* and the *core sections* [6]. There exists a vector bundle $\widehat{U}$ over $N$ such that $\Gamma_\ell(V, S)$ is isomorphic to $\Gamma(\widehat{U})$ as $C^\infty(N)$-modules, since the space of linear sections is a locally free $C^\infty(N)$-module [4]. Hence, if $\mathcal{W}$ be a linear section, then there exists a section $\mathcal{W}_0 : N \to U$ such that $q^S_U \circ \mathcal{W} = \mathcal{W}_0 \circ q_V$. The map $\mathcal{W} \mapsto \mathcal{W}_0$ induces a short exact sequence of vector bundles

$$0 \longrightarrow V^* \otimes C \hookrightarrow \widehat{U} \longrightarrow U \longrightarrow 0, \tag{1}$$

where for $T \in \Gamma(V^* \otimes C)$, the corresponding section $\widehat{T} \in \Gamma_\ell(V, S)$ is given by

$$\widehat{T}(v_n) = {}^V0_{v_n} \underset{U}{+} \overline{T(v_n)}. \tag{2}$$

Splitting $h : U \to \widehat{U}$ of the short exact sequence (1) is called *horizontal lifts.*

Let $U$, $V$, $C$ be vector bundles over $N$, then there is a natural double vector bundle structure on $S = U \oplus V \oplus C$. By using vector bundle structures $S = q^!_U(V \oplus C) \to U$ and $S = q^!_V(U \oplus C) \to V$, double vector bundle $(S; U, V; N)$ is said to be *decomposed* with core $C$. If $(S, U, V, N)$ be a double vector bundle, then a decomposition of $S$ is an isomorphism inducing the identity map on $U$, $V$ and $C$, between $S$ and the decomposed double vector bundle $U \oplus V \oplus C$. The space of decompositions for $S$ is denoted by $\mathrm{Dec}(S)$.

## 2. Main Results

Now, we want to state and prove the main theorem of this paper.

Definition 2.1. A VB-hom algebroid is a double vector bundle as in (1.2), equipped with a hom-Lie algebroid structure on $S \to V$ such that the anchor map

$\tau_S : S \to TV$ is a bundle morphism over $U \to TN$ and where the bracket $[\cdot,\cdot]_S$ is such that

(1) $[\Gamma_\ell(V,S), \Gamma_\ell(V,S)]_S \subseteq \Gamma_\ell(V,S)$,
(2) $[\Gamma_\ell(V,S), \Gamma_c(V,S)]_S \subseteq \Gamma_c(V,S)$,
(3) $[\Gamma_c(V,S), \Gamma_c(V,S)]_S = 0$.

Let double vector bundle $(S;U,V;N)$ be a VB-hom algebroid. There exists a induced hom-Lie algebroid structure on $U$ by taking the anchor to be $\tau_U$, hom map $\Psi_U$ and the hom-Lie bracket $[\cdot,\cdot]_U$ are defined as follows: if $\mathcal{W},\mathcal{Z} \in \Gamma_\ell(V,S)$ cover $\mathcal{W}_0,\mathcal{Z}_0 \in \Gamma(U)$ respectively, then $[\mathcal{W},\mathcal{Z}]_S \in \Gamma_\ell(V,S)$ covers $[\mathcal{W}_0,\mathcal{Z}_0]_U \in \Gamma(U)$ and $\Psi_S(\mathcal{W})$ covers $\Psi_U(\mathcal{W}_0)$. We call $U$ the base hom-Lie algebroid of $S$.

Example 2.2. Let $(U,\tau_U,[\cdot,\cdot]_U,\Psi_U)$ be a hom-Lie algebroid over $N$ and $V \to N$ and $C \to N$ be vector bundles. There exists a VB-hom algebroid structures on the decomposed double vector bundle $U \oplus v \oplus C$.

In the next proposition, we want to state the relation between VB-hom algebroid structures on decomposed double vector bundles and representations up to homotopy of hom-Lie algebroids.

Proposition 2.3. *Let $(U,\tau_U,[\cdot,\cdot]_U,\Psi_U)$ be a hom-Lie algebroid over $N$. Let $V \to N$ and $C \to N$ be vector bundles. There is a one-to-one correspondence between VB-hom algebroid structures on the decomposed double vector bundle $U \oplus V \oplus C$ with core $C$ and $U$ as side hom-Lie algebroid, and 2-term representations up to homotopy of $U$ on $L = C_{[0]} \oplus V_{[1]}$, with respect to $\beta \in \mathcal{S}(L)$.*

Proof. Let us give an explicit description of the VB-hom algebroid structure on $S = U \oplus V \oplus C$ corresponding to a 2-term representation $(\partial,\nabla,K)$ of $U$ on $C_{[0]} \oplus V_{[1]}$, with respect to $\beta$. For $u \in \Gamma(U)$, let $h : \Gamma(U) \hookrightarrow \Gamma_\ell(V,S)$ be the canonical inclusion of decomposed double vector bundle $S$. Define as follows the anchor of $S$, $\tau_S : S \to V$, on linear and core sections:

$$\tau_S(h(u)) = W_{\nabla^1_u},\ \ \tau_S(\widehat{c}) = \partial(c)^\uparrow,$$

where $W_{\nabla^1_u}$, $\partial(c)^\uparrow \in \mathfrak{W}(V)$ are, respectively, the linear vector fields corresponding to the derivation ${\nabla^1_u}^* : \Gamma(V^*) \to \Gamma(V^*)$ and the vertical vector field corresponding to $\partial(c) \in \Gamma(V)$ (See Example 2.2). The hom map $\Psi_S$ on $\Gamma(S)$ is define as follows

$$\Psi_S(h(u)) = h(\Psi_U(u)),$$

and

$$\Psi(\hat{c}) = 0,$$

for $u \in \Gamma(U)$ and $c \in \Gamma(C)$. The hom Lie bracket $[\cdot,\cdot]_S$ on $\Gamma(S)$ is given by the formulas below:

$$[\widehat{c}_1,\widehat{c}_2]_S = 0,$$

$$[h(u),\widehat{c}]_S = \widehat{\nabla^0_u c},$$

and

$$[h(u_1),h(u_2)]_S = h([u_1,u_2]_U) + \widehat{K}(u_1,u_2),$$

where $u,u_1,u_2 \in \Gamma(U)$ and $c,c_1,c_2 \in \Gamma(C)$ and $\widehat{K}(u_1,u_2) \in \Gamma_\ell(V,S)$ is the linear section given by (2). □

## References

1. C. A. Abad and M. Crainic, *Representations up to homotopy of Lie algebroids*, J. Reine Angew. Math. **663** (2012) 91–126.
2. Z. Chen, Z. Liu and Y. Sheng, *On double vector bundles*, Acta. Math. Sin. (Engl. Ser.) **30** (10) (2014) 1655–1673.
3. M. Crainic and R. L. Fernandes, *Lectures on integrability of Lie brackets*, Geom. Topol. Monogr. **17** (2011) 1–107.
4. A. Gracia-Saz and R. A. Mehta, *Lie algebroid structures on double vector bundles and representation theory of Lie algebroids*, Adv. Math. **223** (4) (2010) 1236–1275.
5. C. Laurent-Gengoux and J. Teles, *Hom-Lie algebroids*, J. Geom. Phys. **68** (2013) 69–75.
6. K. C. H. Mackenzie, *Ehresmann doubles and Drinfel'd doubles for Lie algebroids and Lie bialgebroids*, J. Reine Angew. Math. **658** (2011) 193–245.
7. K. C. H. Mackenzie, *Double Lie algebroids and second-order geometry I*, Adv. Math. **94** (2) (1992) 180–239.
8. K. C. H. Mackenzie, *Double Lie algebroids and second-order geometry II*, Adv. Math. **154** (2000) 46–75.
9. R. A. Mehta, *Lie algebroid modules and representations up to homotopy*, Indag. Math. **25** (2014) 1122–1134.
10. S. Merati and M. R. Farhangdoost, *Representation and central extension of hom-Lie algebroids*, J. Algebra Appl. **17** (11) (2018) 1850219.
11. S. Merati and M. R. Farhangdoost, *Representation up to homotopy of hom-Lie algebroids*, Int. J. Geom. Methods Mod. Phys. **15** (5) (2018) 1850074.
12. J. Pradines, *Géométrie differentielle au-dessus d'un grupoïde*, C. R. Acad. Sci. Paris série A **266** (1968) 1194–1196.
13. Y. Sheng, *On deformations of Lie algebroids*, Results. Math. **62** (2012) 103–120.
14. Y. Sheng and C. Zhu, *Semidirect products of representations up to homotopy*, Pacific J. Math. **249** (1) (2011) 211–236.
15. A. Yu. Vaĭntrob, *Lie algebroids and homological vector fields*, Uspekhi Mat. Nauk. **52** (2(314)) (1997) 161–162.

E-mail: farhang@shirazu.ac.ir
E-mail: s.merati@shirazu.ac.ir
E-mail: a.attari@shirazu.ac.ir

# Characterization of Osculating and Rectifying Curves in Semi-Euclidean Space of Index 2

Neda Ebrahimi*
Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran

Abstract. Osculating and rectifying curves in Euclidean space and Minkowski space were investigated in several articles. In this paper the concept of osculating and rectifying null and partially null curves are generalized in four dimensional semi-Euclidean space of index two and the coefficients of their position vector in each case by using of Frenet equations, are given. Partially null curves with constant second and third curvature are classified and it is shown that partially null curves with zero second curvature are planer. In addition, a characterization for rectifying null curves is given and it is shown that any null rectifying curve with constant second and third curvature is spherical.

**Keywords:** Ferenet equation, Semi-Euclidean space, Curve, Spherical curve.
**AMS Mathematical Subject Classification [2010]:** 53C40, 53C50.

## 1. Introduction

In analogy with the Euclidean curves, the Ferenet frame and Ferenet equations for causal curves can be defined in semi-Euclidean spaces. In this paper we investigate the properties of null and partially null curves in $E^4_2$, semi-Euclidean four dimensional space of index two [3]. The semiEuclidean space $E^4_2$ is the standard vector space $R^4$ equipped with an indefinite flat metric $g$ given by

$$g = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2,$$

where $(x_1, x_2, x_3, x_4)$ is a rectangular coordinate for $E^2_4$. We recall that $w \in E^4_2$ is called a spacelike, a timelike or a null vector if $g(w,w) > 0$, $g(w,w) < 0$ or $g(w,w) = 0$ and $w \neq 0$, respectively. The norm of a vector $w$ is given by $\|w\| = \sqrt{|g(w,w)|}$ and $w$ is a unit vetor if $g(w,w) = \pm 1$. An arbitrary curve $\alpha : I \to E^4_2$ is called spacelike, timelike or null, if respectively $\alpha'(t)$, for all $t \in I$, be spacelike, timelike or null. A pseudo-sphere $S^3_2$ and pseudo-hyperbolic space $H^3_2$ are hyperquadrics in $E^4_2$ defined respectively by $S^3_2(r) = \{x : g(x,x) = r^2\}$ and $H^3_2(-r) = \{x : g(x,x) = -r^2\}$. The Ferenet frame, of causal curves in $E^4_2$ are given in [2, 3]. Ferenet frame for a causal curve $\alpha$ in $E^4_2$ consists of four orthogonal non-zero vector fields $\{T, N, B_1, B_2\}$ which are called respectively, the tangent, the principal normal, the first binormal and the second binormal vector field.

*Speaker

## 2. Osculating Partially Null Curves

$\alpha : I \to E_2^4$ is called partially null if it is timelike or spacelike and satisfies:

- $g(T,T) = \epsilon_1 = 1$,
- $g(N,N) = \epsilon_2 = 1$,
- $g(B_1,B_1) = 1$,
- $g(B_1,B_1) = g(B_2,B_2) = 0$,
- $g(T,N) = g(T,B_1) = g(T,B_2) = g(N,B_1) = g(N,B_2) = 0$, $\epsilon_1\epsilon_2 = -1$.

Then the position vector of $\alpha$ is given by:

$$\alpha = g(\alpha,T)\epsilon_1 T + g(\alpha,N)\epsilon_2 N + g(\alpha,B_2)B_1 + g(\alpha,B_1)B_2.$$

And the Ferenet equations are given by [2]:

- $T' = k_1 N$,
- $N' = k_1 T + k_2 B_1$,
- $B_1' = k_3 B_1$,
- $B_2' = -\epsilon_2 k_2 N - k_3 B_2$.

It is trivial that $k_3 = 0$.
In this case if $k_1 = 0$ then $\alpha$ is a line. Hence in this section we suppose that $k_1 \neq 0$.

Remark 2.1. $k_3 = 0$ implies that $B_1$ is constant and consequently $g(\alpha,B_1)' = g(T,B_1) + g(\alpha,0) = 0$. Hence $g(\alpha,B_1)$ is constant.

(i) If $\alpha$ is a partially null curve then it is called first kind osculating curve if its position curve is given by $\alpha = aT + bN + cB_2$, where $a, b$ and $c$ are differentiable functions. It can easily be checked that:

Lemma 2.2. *If $\alpha$ be a partially null osculating curve then $\alpha$ is first kind osculating if and only if $k_2 = 0$ or $g(\alpha,N) = 0$.*

(ii) If $\alpha$ is partially null then it is called second kind osculating curve if its position vector is given by $\alpha = aT + bN + cB_1$, where $a, b$ and $c$ are differentiable functions.

Remark 2.1 and Ferenet equation, $B_2' = -\epsilon_2 k_2 N$, implies the following lemma.

Theorem 2.3. *$\alpha$ is a second kind osculating curve if and only if $k_2 = 0$ or $g(\alpha,B_1) = 0$.*

Remark 2.4. If $\alpha$ be a partially null curve with $g(\alpha,B_1), g(\alpha,N) \neq 0$ then it is first kind osculating if and only if it is second kind osculating. Every partially null curve with $k_2 = 0$ is planer.

Theorem 2.5. *Let $\alpha$ be a partially null curve in $E_2^4$ with $k_2 = 0$ then $\alpha$ satisfies:*

$$\alpha''' = (k_1')/k_1 \alpha'' k_1^2 \alpha'. \tag{1}$$

Theorem 2.6. [3] *$\alpha$ is a partially null unit curve with constant $k_1$ and $k_2$ and with $g(T,T) = \epsilon$ if and only if under an isomorphism we have*

$$\alpha(s) = As + 1/k_1(E\cosh(k_1 s) + F\sinh(k_1 s)),$$

*where $A$, $E$ and $F$ are orthogonal and $g(A,A) = 0$, $g(E,E) = -g(F,F) = -\epsilon$.*

Substituting Eq. (1) in Theorem 2.6 implies the following corollary.

Corollary 2.7. *Let $\alpha$ be a partially null curve in $E_2^4$ with constant $k_1$ that $k_2 = 0$ and $g(T,T) = \epsilon$. Then there are orthogonal vectors $E$ and $F$ such that*

$$\alpha(s) = (1/k_1(E\cosh(k_1 s) + F\sinh(k_1 s)),$$

*and $g(E,E) = -g(F,F) = -\epsilon$.*

## 3. Osculating and Rectifying Null Curves

Let $\alpha$ be a null curve in $E_2^4$ and $\{T, N, B_1, B_2\}$ be the moving Ferenet frame along it [3]. Position vector of $\alpha$ is as follows:

$$\alpha = g(\alpha, B_1)T + g(\alpha, N)N + g(\alpha, T)B_1 - g(\alpha, B_2)B_2.$$

Its Ferenet equations are given by

- $T' = N,$
- $N' = k_2 T - B_1,$
- $B_1' = -k_2 T - B_1,$
- $B_2' = -k_3 T.$

**3.1. Osculating Curves.** The notion of osculating curves can be generalized for null curves.

i) $\alpha$ is called a first kind osculating curve if its position vector be of the form $\alpha = aT + bN + cB_1$, where $a$, $b$ and $c$ are differentiable functions. It can easily be checked that $\alpha$ is first kind osculating if and only if $k_3 = 0$ or $g(\alpha, B_2) = 0$.

ii) $\alpha$ is called a second kind osculating curve if its position vector satisfies $\alpha = aN + bB_1 + cB_2$, where $a$, $b$ and $c$ are differentiable functions. In this case, $T = (ak_2 - ck_3)T + (a' - bk_2)N + (b' - ck_1)B_1 + (c' - bk_3)B_2$. Hence we have

$$ak_2 - ck_3 = 1, a' - bk_2 = 0, b' - c = 0, c' - bk_3 = 0.$$

Remark 3.1. The above equations implies that $\alpha$ satisfies the following equations:

$$(a^2 - c^2)' = 2b, \rho^2 = \int 2b ds, \tag{2}$$

where $\rho$ is distance function.

Theorem 3.2. *Let $\alpha$ be a second kind null osculating curve with $k_2, k_3 \neq 0$ and $b \neq 0$. Then we have $a = 1/k_2(c+1)$. In addition, if $k_3 > 0$, then*

- $c = A\sqrt{k_3}e^{(\sqrt{k_3}s)} - B\sqrt{k_3}e^{-\sqrt{k_3}s},$
- $b = Ae^{\sqrt{k_3)}}s) + Be^{-\sqrt{k_3}s}.$

*And if $k_3 < 0$, then*

- $b = A\cos(\sqrt{-k_3}s) + B\sin(\sqrt{-k_3}s),$
- $c = -A\sqrt{-k_3}\sin(\sqrt{-k_3}s) + B\sqrt{-k_3}\sin(\sqrt{-k_3}s).$

Proof. By differentiation from $ak_2 - ck_3 = 1$ we have $a'k_2 - c'k_3 = 0$. Equatin (2) implies that $b(k_3^2 - k_2^2) = 0$. Since $b \neq 0$, $k_2 = \pm k_3$. In addition $b' = c$ and consequently $b'' = c' = k_3 b$. □

**3.2. Rectifying Curves.** $\alpha$ is called a rectifying null curve if it has a position vector of the form $\alpha = aT+bB_1+cB_2$, where $a$, $b$ and $c$ are differentiable functions.

LEMMA 3.3. *$\alpha$ is a ratifying null curve if and only if $g(\alpha, N) = 0$.*

PROOF. Since $N' = k_2T - B_1$ is not zero, $g(\alpha, N) = 0$. □

THEOREM 3.4. *Let $\alpha$ be a null rectifying curve with $k_2 \neq 0$. Then*
i) $g(\alpha, T) = b$,
ii) $|\rho^2|' = |bs + b_0|$,
iii) $g(\alpha, B_1)' = 1 - k_3(\int k_3 ds + b_0)$ *and* $g(\alpha, B_2) = -b \int k_3 ds + b_0$.
*For constant values $b$, $b_0$and $d_0$. Conversely if any of these conditions satisfies then $\alpha$ is null rectifying.*

PROOF. (i) is obtained by using of Lemma 3.3. Since $\rho^2 = g(\alpha', \alpha')$ (i) implies (ii). In addition, $g(\alpha, B_1)' = 1 - k_2g(\alpha, N) - k_3g(\alpha, B_2)$, $g(\alpha, B_2)' = -k_3g(\alpha, T)$ hence we have (iii). To prove the converse of (iii) note that $g(\alpha, B_1)' = 1 - k_3g(\alpha, B_2)$ and consequently $g(\alpha, N) = 0$. □

THEOREM 3.5. *Let $\alpha$ be a null rectifying curve with constant second and third curvature then $\alpha$ is spherical.*

PROOF. By differentiation of position vector we have:

$$1 = a' - k_3c, 0 = a - bk_2, 0 = b', c' - bk_3 = 0.$$

Using the above equations $b$ is constant and $k_3 \neq 0$. If $b = 0$ then $a = 0$ and $c = 1/k_3$. Hence $\rho(\alpha, \alpha) = 1/k_3^2$ and $\alpha$ is spherical. If $b \neq 0$ then $c = bk_3s + c_0$. First equation implies that $a' = 1+k_3(bk_3s+c_0)$. Hence $a = (1+k_3c_0)s+(k_3^2/2)bs^2+a_0$. Coffitient of $s^2$ is non zero and $a = bk_2$ which is constant by assumption. This is a contradiction. □

The following theorem gives a characterization for rectifying null curves. It is similar to characterization of rectifying curves in Euclidean space [1].

THEOREM 3.6. *Let $\alpha$ be a null rectifying curve in $E_2^4$. The position vector of $\alpha$ is spacelike or timelike if and only if under a parametrization we have $\alpha(t) = e^t y(t)$, where $y(t)$ is a unit speed spacelike (timelike) in $S_2^3(1)$ ($H_2^3(1)$).*

PROOF. Let $\alpha$ be a null rectifying curve. If its position vector be spacelike then $g(\alpha, \alpha) > 0$, by theorem $\rho^2 = bs + a_0$. Let $y(s) = \alpha(s)/\rho(s)$. Hence $\alpha(s) = y(s)\sqrt{(bs + a_0)}$. This implies that $T(s) = b/(\sqrt{(bs + a_0)})y(s) + \sqrt{(bs + a_0)}y'(s)$, and $0 = g(T(s), T(s)) = (as + a_0)g(y'(s), y'(s)) + b^2(4(bs + a_0))$. This implies that $g(y(s)', y(s)') = -b^2/(4(bs + a_0)^2)$. Hence $y(s)$ is timelike. Let $t = \int_0^s \|y'(u)\|du = \int_0^s b(2(bu+a_0)))du = 1/2 \ln (bs+a_0)$. Using this parametrization we have $e^{2t} = bs+a_0$. Hence $\alpha(t) = e^t y(t)$. Conversely if $\alpha(t) = e^t y(t)$ and $y(t)$ be a timelike position vector which is on a semi-sphere of radius one then we can reparemetrization $\alpha$ with $t = 1/2 \ln (bs + a_0)$ such that $s$ is a semi arc-length and $bs + a_0 > 0$ and $b \neq 0$. hence $\alpha(s) = y(s)\sqrt{bs + a_0}$ and consequently$\rho^2 = g(\alpha, \alpha) = as + a_0$ and $\alpha$ is rectifying. The theorem is proved for the case that position vector is timelike in a similar way. □

## References

1. B. Y. Chen, *When does the position vector of a space curve always lie in its rectifying plane?*, Amer. Math. Monthly **110** (2) (2003) 147–152.
2. K. Ilarslan, *Some Special Curves on Non–Euclidean Manifolds*, Ph.D. Thesis, Ankara University, Graduate School of Natural and Applied Sciences, 2002.
3. M. P. Torgašev, K. Ilarslan and E. Nešović, *A partially null and pseudo null curves in the semi-euclidean space $R_2^4$*, J. Geom **84** (2005) 106–116.

E-mail: n_ebrahimi@uk.ac.ir

# On a Weighted Asymptotic Expansion Concerning Prime Counting Function and Applications to Landau's and Ramanujan's Inequalities

Mehdi Hassani*
Department of Mathematics, University of Zanjan, Zanjan, Iran

Abstract. Landau's inequality and Ramanujan's inequality concerning prime counting function assert that $\pi(2x) < 2\pi(x)$ and $\pi(x)^2 < \frac{\mathrm{e}\,x}{\log x}\,\pi(\frac{x}{\mathrm{e}})$, respectively, for sufficiently large $x$. In this paper we give an asymptotic expansion for $\pi(\alpha\,x)$ as the common key to study Landau's inequality and Ramanujan's inequality. Then, we give several refinements and generalizations of these inequalities.

**Keywords:** Prime counting function, Landau's inequality, Ramanujan's inequality, The Riemann hypothesis.
**AMS Mathematical Subject Classification [2010]:** 11A41, 11N05.

## 1. Introduction and Summary of the Results

Let $\pi(x)$ denote the number of primes not exceeding $x$. Several inequalities concerning the prime counting function $\pi(x)$ have been studied in the literature. To motivate present work, we recall two of them.

Motivated by comparing the portion of primes in the intervals $(0, x]$ and $(x, 2x]$ for sufficiently large $x$, Landau showed that [8, page 216]

$$\pi(2x) - 2\pi(x) = -2\log 2\,\frac{x}{\log^2 x} + o\left(\frac{1}{\log^2 x}\right).$$

Hence, the interval $(0, x]$ has more primes than the interval $(x, 2x]$ for sufficiently large $x$. This may read as $\pi(x) > \pi(2x) - \pi(x)$, or equivalently

$$\pi(2x) < 2\pi(x), \tag{1}$$

which is known as Landau's inequality concerning prime counting function. This inequality has been studied by the author in [6].

Among several conjectures and results concerning distribution of prime numbers, Ramanujan [10, page 310, line -4 and -3] asserts that as $N \to \infty$, the number of primes less than $N$ is less than $\sqrt{\frac{\mathrm{e}N}{\log N}}$ the number of primes less than $\frac{N}{\mathrm{e}}$. Berndt [2, page 112] rewrites this inequality as follows.

$$\pi(x)^2 < \frac{\mathrm{e}\,x}{\log x}\,\pi\left(\frac{x}{\mathrm{e}}\right), \qquad \text{(for } x \text{ sufficiently large)}. \tag{2}$$

This inequality is known as Ramanujan's inequality concerning prime counting function in the literature [1, 2, 3, 4, 7, 9, 11]. To confirm (2), for sufficiently

*Speaker

large $x$, we note that the prime number theorem with error term [8] gives the expansion

$$\pi(x) = x \sum_{k=0}^{n} \frac{k!}{\log^{k+1} x} + O\left(\frac{x}{\log^{n+2} x}\right), \tag{3}$$

for any integer $n \geqslant 0$. Using (3) with $n = 4$, implies

$$\pi(x)^2 - \frac{\mathrm{e}\, x}{\log x}\, \pi\left(\frac{x}{\mathrm{e}}\right) = -\frac{x^2}{\log^6 x} + O\left(\frac{x^2}{\log^7 x}\right),$$

and so the inequality (2) holds for sufficiently large $x$. This inequality has been studied by the author in [4, 5] and [7].

The common idea to study Landau's inequality and Ramanujan's inequality is to find asymptotic expansion for $\pi(\alpha\, x)$, similar to (3), with $\alpha = 2$ in the case of Landau's inequality and with $\alpha = \frac{1}{\mathrm{e}}$ in the case of Ramanujan's inequality. More precisely, we prove the following widely applicable expansion.

THEOREM 1.1. *Let $\alpha > 0$. For a given integer $n \geqslant 0$, as $x \to \infty$, we have*

$$\pi(\alpha\, x) = \alpha\, x \sum_{k=0}^{n} \frac{(-1)^k P_k(\log \alpha)}{\log^{k+1} x} + O\left(\frac{x}{\log^{n+2} x}\right), \tag{4}$$

*where $P_k$ is a polynomial with degree $k$ given by*

$$P_k(t) = \sum_{j=0}^{k} (-1)^j \frac{k!}{(k-j)!}\, t^{k-j}.$$

As some applications of the above weighted expansion to Landau's inequality, we prove the following generalization.

COROLLARY 1.2. *For a given $\lambda \in (0,1)$, the inequality*

$$\pi(x) < \pi(\lambda\, x) + \pi\left((1-\lambda)x\right), \tag{5}$$

*holds for sufficiently large $x$.*

Note that the inequality (5), with $\lambda = \frac{1}{2}$ and replacing $x$ by $2x$, gives the inequality (1). Also, the following two corollaries provide refinements of Landau's inequality (1).

COROLLARY 1.3. *For a given $\lambda \in (0,1)$, the inequality*

$$\pi(2x) < \pi\left((1-\lambda)x\right) + \pi\left((1+\lambda)x\right) < 2\pi(x),$$

*holds for sufficiently large $x$.*

COROLLARY 1.4. *For a given $\lambda \in (0,1)$, the inequality*

$$\begin{aligned} 2\pi(x) + \pi(2x) &< \pi\left(\lambda\, x\right) + \pi\left((1-\lambda)x\right) + \pi\left((1+\lambda)x\right) + \pi\left((2-\lambda)x\right) \\ &< 4\pi(x), \end{aligned} \tag{6}$$

*holds for sufficiently large $x$.*

## 2. Proofs

Proof of Theorem 1.1. The prime number theorem with error term asserts that

$$\pi(x) = \mathrm{li}(x) + O\left(x\,\mathrm{e}^{-c\sqrt{\log x}}\right), \tag{7}$$

where $c > 0$ is a computable constant and

$$\mathrm{li}(x) = CPV \int_0^x \frac{1}{\log t}\, dt,$$

denotes the logarithmic integral function defined by the Cauchy principal value of integral. Integrating by parts gives

$$\mathrm{li}(x) = x \sum_{k=0}^{n} \frac{k!}{\log^{k+1} x} + O\left(\frac{x}{\log^{n+2} x}\right),$$

for any integer $n \geqslant 0$. One may write $x\,\mathrm{e}^{-c\sqrt{\log x}} = o(\frac{x}{\log^{n+2} x})$, as $x \to \infty$, for any integer $n \geqslant 0$. Thus, by using (7), we get the expansion (3). Therefore, we have

$$\pi(\alpha\, x) = \alpha\, x \sum_{k=0}^{n} \frac{k!}{(\log x + \log \alpha)^{k+1}} + O\left(\frac{x}{\log^{n+2} x}\right).$$

The binomial expansion asserts that

$$(1+t)^{-(k+1)} = \sum_{m=0}^{n} (-1)^m c_m\, t^m + O(t^{n+1}),$$

as $t \to 0$, where $c_0 = 1$ and $c_m = \frac{1}{m!}\prod_{i=1}^{m}(k+i)$ for $m \geqslant 1$. Thus

$$\begin{aligned}\sum_{k=0}^{n} \frac{k!}{(\log x + \log \alpha)^{k+1}} &= \sum_{k=0}^{n} \frac{k!}{\log^{k+1} x}\left(1 + \frac{\log \alpha}{\log x}\right)^{-(k+1)} \\ &= \sum_{k=0}^{n} \frac{k!}{\log^{k+1} x} \sum_{m=0}^{n} \frac{(-1)^m c_m \log^m \alpha}{\log^m x} + O\left(\frac{1}{\log^{n+2} x}\right).\end{aligned}$$

Diagonal collecting terms of the above double sum gives

$$\sum_{k=0}^{n} \frac{k!}{\log^{k+1} x} \sum_{m=0}^{n} \frac{(-1)^m c_m \log^m \alpha}{\log^m x} = \sum_{k=0}^{n} \frac{r_k}{\log^{k+1} x} + O\left(\frac{1}{\log^{n+2} x}\right),$$

where

$$r_k = \sum_{j=0}^{k} (-1)^{k-j} j! c_{k-j} \log^{k-j} \alpha.$$

This completes the proof. □

Proof of Corollary 1.2. Note that $P_0(t) = 1$ and $P_1(t) = t - 1$. Thus, (4) implies that

$$\pi(\alpha\, x) = x\left(\frac{\alpha}{\log x} + \frac{\alpha - \alpha \log \alpha}{\log^2 x}\right) + O\Big(\frac{x}{\log^3 x}\Big), \tag{8}$$

for any $\alpha > 0$, as $x \to \infty$. The expansion (8) with $\alpha = \lambda$ and $\alpha = 1 - \lambda$ gives

$$\pi(\lambda x) + \pi((1-\lambda)x) = x\left(\frac{1}{\log x} + \frac{C_2(\lambda)}{\log^2 x}\right) + O\Big(\frac{x}{\log^3 x}\Big), \tag{9}$$

where $C_2(\lambda) = 1 - \lambda \log \lambda - (1-\lambda)\log(1-\lambda)$. The function $C_2(\lambda)$ is strictly increasing for $\lambda \in (0, \frac{1}{2})$ and admits limit conditions $\lim_{\lambda\to 0^+} C_2(\lambda) = \lim_{\lambda \to 1^-} C_2(\lambda) = 1$. Thus, symmetry of $C_2(\lambda)$ with respect to $\lambda = \frac{1}{2}$ implies $1 < C_2(\lambda) \leqslant C_2(\frac{1}{2}) = 1 + \log 2$. Comparing the coefficients of (9) and the expansion (3) with $n = 1$, completes the proof. □

Proof of Corollary 1.3. The expansion (3) with $n = 1$ gives

$$2\pi(x) = x\left(\frac{2}{\log x} + \frac{2}{\log^2 x}\right) + O\Big(\frac{x}{\log^3 x}\Big). \tag{10}$$

Also, we use the expansion (8) with $\alpha = 2$, $\alpha = 1 - \lambda$ and $\alpha = 1 + \lambda$ to obtain

$$\pi(2x) = x\left(\frac{2}{\log x} + \frac{2 - 2\log 2}{\log^2 x}\right) + O\Big(\frac{x}{\log^3 x}\Big), \tag{11}$$

and

$$\pi((1-\lambda)x) + \pi((1+\lambda)x) = x\left(\frac{2}{\log x} + \frac{D_2(\lambda)}{\log^2 x}\right) + O\Big(\frac{x}{\log^3 x}\Big),$$

where $D_2(\lambda) = 2 - (1-\lambda)\log(1-\lambda) - (1+\lambda)\log(1+\lambda)$. The function $D_2(\lambda)$ admits limit values $\lim_{\lambda\to 0^+} D_2(\lambda) = 2$ and $\lim_{\lambda\to 1^-} D_2(\lambda) = 2 - 2\log 2$. Also, for $\lambda \in (0,1)$, we observe that $\frac{d}{d\lambda} D_2(\lambda) = \log \frac{1-\lambda}{1+\lambda} < 0$. Hence, $2 - 2\log 2 < D_2(\lambda) < 2$. Comparing the coefficients, completes the proof. □

Proof of Corollary 1.4. The expansion (3) with $n = 1$ implies

$$4\pi(x) = x\left(\frac{4}{\log x} + \frac{4}{\log^2 x}\right) + O\Big(\frac{x}{\log^3 x}\Big).$$

Also, the expansions (10) and (11) give

$$2\pi(x) + \pi(2x) = x\left(\frac{4}{\log x} + \frac{4 - 2\log 2}{\log^2 x}\right) + O\Big(\frac{x}{\log^3 x}\Big).$$

We use the expansion (8) with $\alpha = \lambda$, $\alpha = 1 - \lambda$, $\alpha = 1 + \lambda$ and $\alpha = 2 - \lambda$ to get

$$\pi(\lambda x) + \pi((1-\lambda)x) + \pi((1+\lambda)x) + \pi((2-\lambda)x) = x\left(\frac{4}{\log x} + \frac{E_2(\lambda)}{\log^2 x}\right) + O\Big(\frac{x}{\log^3 x}\Big),$$

where $E_2(\lambda) = 4 - \lambda\log\lambda - (1-\lambda)\log(1-\lambda) - (1+\lambda)\log(1+\lambda) - (2-\lambda)\log(2-\lambda)$. Note that $E_2(\lambda) = E_2(1-\lambda)$ and $\lim_{\lambda\to 0^+} E_2(\lambda) = \lim_{\lambda\to 1^-} E_2(\lambda) = 4 - 2\log 2$. For $\lambda \in (0, \frac{1}{2})$, we observe that $\frac{d}{d\lambda} E_2(\lambda) = \log \frac{(1-\lambda)(2-\lambda)}{\lambda(1+\lambda)} > 0$. Thus,

$$4 - 2\log 2 < E_2(\lambda) \leqslant E_2\Big(\frac{1}{2}\Big) = 4 + 4\log 2 - 3\log 3 < 4.$$

Comparing the coefficients, completes the proof. □

### 3. Application to Ramanujan's Inequality

In this section, we recall some applications of the expansion (4) (See [7] for more details). The key to study Ramanujan's inequality (2) is full asymptotic expansions of its left and right hand sides expressions, as follows.

Theorem 3.1. *Let* $\ell_k = \sum_{j=0}^{k} j!(k-j)!$ *and* $r_k = \sum_{j=0}^{k} j!\binom{k}{j}$. *Then, for a given integer* $n \geqslant 0$, *we have*

$$\pi(x)^2 = x^2 \sum_{k=0}^{n} \frac{\ell_k}{\log^{k+2} x} + O\left(\frac{x^2}{\log^{n+3} x}\right),$$

*and*

$$\frac{\mathrm{e}\, x}{\log x}\pi\left(\frac{x}{\mathrm{e}}\right) = x^2 \sum_{k=0}^{n} \frac{r_k}{\log^{k+2} x} + O\left(\frac{x^2}{\log^{n+3} x}\right).$$

In the following corollary, we obtain full asymptotic expansions of $\pi(x)^2 - \frac{\mathrm{e}\, x}{\log x}\pi\left(\frac{x}{\mathrm{e}}\right)$ as $x \to \infty$.

Corollary 3.2. *For a given integer* $n \geqslant 4$, *we have*

$$\pi(x)^2 - \frac{\mathrm{e}\, x}{\log x}\, \pi\left(\frac{x}{\mathrm{e}}\right) = x^2 \sum_{k=4}^{n} \frac{d_k}{\log^{k+2} x} + O\left(\frac{x^2}{\log^{n+3} x}\right),$$

*where* $d_k = \ell_k - r_k = \sum_{j=0}^{k} j!\big((k-j)! - \binom{k}{j}\big)$.

Note that $d_0 = d_1 = d_2 = d_3 = 0$ and some more initial values of $d_k$ are $d_4 = -1$, $d_5 = -14$, $d_6 = -145$, $d_7 = -1412$, $d_8 = -13985$, etc. As $d_k < 0$ for any $k \geqslant 4$, we obtain the following refinement of Ramanujan's inequality (2).

Corollary 3.3. *Let* $m \geqslant 4$ *be an integer. Then, for sufficiently large* $x$, *we get*

$$\pi(x)^2 < \frac{\mathrm{e}\, x}{\log x}\, \pi\left(\frac{x}{\mathrm{e}}\right) + x^2 \sum_{k=4}^{m} \frac{d_k}{\log^{k+2} x}.$$

Concerning the sharpness and the structure of Ramanujan's inequality, we obtain the following corollaries.

Corollary 3.4. *Let* $h$ *be a real number. If* $h \geqslant 0$, *then for sufficiently large* $x$

$$\pi(x)^2 < \frac{\mathrm{e}\, x}{\log x - h}\, \pi\left(\frac{x}{\mathrm{e}}\right).$$

*If* $h < 0$, *then the above inequality reverses.*

Corollary 3.5. *If* $\alpha \geqslant \mathrm{e}$, *then for* $x$ *sufficiently large*

$$\pi(x)^2 < \frac{\alpha\, x}{\log x}\, \pi\left(\frac{x}{\alpha}\right).$$

*If* $0 < \alpha < \mathrm{e}$, *then the above inequality reverses.*

Corollary 3.6. *Let* $h$ *be a real number. Then we have*

$$\begin{cases} \pi(\mathrm{e}\, x)^2 < \frac{\mathrm{e}^2 x}{h + \log x}\, \pi(x), & \text{if } h \leqslant 1, \\ \pi(\mathrm{e}\, x)^2 > \frac{\mathrm{e}^2 x}{h + \log x}\, \pi(x), & \text{if } h > 1, \end{cases}$$

*for sufficiently large* $x$.

Remark 3.7. The most important studies regarding Ramanujan's inequality (2) ask about the positive integer $x_{\mathcal{R}}$ for which (2) holds if $x \geqslant x_{\mathcal{R}}$ and fails for $x < x_{\mathcal{R}}$. In 2012, the author [5] approximated $x_{\mathcal{R}}$ under assumption of the existence of some very good bounds for the function $\pi(x)$. In 2015, Dudek and Platt [3], based on the sharp bounds due to Trudgian which appeared some months after their work in [11], obtained such a very good bounds for $\pi(x)$ implying that $x_{\mathcal{R}} \leqslant \mathrm{e}^{9658}$. In fact, by using a result of Mossinghoff and Trudgian [9], Dudek and Platt, on page 292, showed that $x_{\mathcal{R}} \leqslant \mathrm{e}^{9394}$. In 2018, Axler [1] proved that $x_{\mathcal{R}} \leqslant \mathrm{e}^{9032}$ and also showed that (2) holds unconditionally for every $x$ satisfying $38,358,837,683 \leqslant x \leqslant 10^{19}$.

By assuming the Riemann hypothesis, the author proved that $x_{\mathcal{R}} \leqslant 138,766,146,692,471,228$ [5]. Dudek and Platt [3] refined this conditional result, by showing that $x_{\mathcal{R}} \leqslant 1.15 \times 10^{16}$. Furthermore, they proved, by assuming the Riemann hypothesis, that the largest integer counterexample to Ramanujan's inequality (2) is at $x = 38,358,837,682$.

## Acknowledgement

The author wishes to express his thanks to the anonymous referee(s) for careful reading of the manuscript and giving the many valuable suggestions and corrections, which improved the presentation of the paper.

## References

1. C. Axler, *Estimates for $\pi(x)$ for large values of $x$ and Ramanujan's prime counting inequality*, Integers **18** (2018) A61.
2. B. C. Berndt, *Ramanujan's Notebooks*, Part IV, Springer-Verlag, New York, 1994.
3. A. W. Dudek and D. J. Platt, *On solving a curious inequality of Ramanujan*, Exp. Math. **24** (3) (2015) 289–294.
4. M. Hassani, *Generalizations of an inequality of Ramanujan concerning prime counting function*, Appl. Math. E-Notes **13** (2013) 148–154.
5. M. Hassani, *On an inequality of Ramanujan concerning the prime counting function*, Ramanujan J. **28** (3) (2012) 435–442.
6. M. Hassani, *Portion of primes in short intervals and Landau's inequality*, J. Zankoy Sulaimani **22** (2) (2020) 345–352.
7. M. Hassani, *Remarks on Ramanujan's inequality concerning the prime counting function*, Commun. Math. to appear.
8. E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, 3rd ed., Chelsea Pub Co., New York, 1974.
9. M. J. Mossinghoff and T. S. Trudgian, *Nonnegative trigonometric polynomials and a zero-free region for the Riemann zeta-function*, J. Number Theory **157** (2015) 329–349.
10. S. Ramanujan, *Notebooks*, Vol. II, Tata Institute of Fundamental Research, Bombay, 1957.
11. T. Trudgian, *Updating the error term in the prime number theorem*, Ramanujan J. **39** (2) (2016) 225–234.

E-mail: mehdi.hassani@znu.ac.ir

# On Pseudo Slant Submanifolds of 3-Cosymplectic Manifolds

Fatemeh Raei
Department of Mathematics, Farhangian University, Tehran, I. R. Iran
and Mohammad Bagher Kazemi*
Department of Mathematics, University of Zanjan, Zanjan, I. R. Iran

Abstract. In this paper, we study pointwise pseudo 3-slant submanifolds of a 3-cosymplectic manifold. We give a necessary and sufficient condition for such submanifolds to be pointwise pseudo 3-slant and then construct an example of this type of submanifolds. Also, we prove integrability of some distributions of these submanifolds.
**Keywords:** Almost contact 3-structure, Pseudo slant, 3-Cosymplectic manifold.
**AMS Mathematical Subject Classification [2010]:** 53C25, 53C50.

## 1. Introduction

The notion of slant submanifolds became an interesting concept in Riemannian manifolds, after introducing slant submanifolds of almost Hermitian manifolds by Chen. Since then, many important and interesting results have been obtained about slant, semi-slant, bi-slant and pseudo slant submanifolds such that the ambient manifolds were equipped by almost complex and almost contact structures [1, 6, 7].

Later, Etayo [4] has extended these submanifolds by defining quasi-slant submanifolds. On such submanifolds, the slant angle between the image of the structure (1,1)-tensor field and the tangent space is independent of the choice of vector fields of the submanifold. On the other hand, Chen and Garay [3] investigated and characterized this type of submanifolds under the name of point-wise slant submanifolds.

Pseudo slant submanifolds are a special type of bi-slant submanifolds [1] which are generalization of invariant, anti-invariant and slant submanifolds. In the present paper, we study this notion in the pointwise case such that the ambient manifold admits 3-cosymplectic structure.

Let $M$ be a Riemannian manifold and $\phi$, $\xi$, $\eta$ be a tensor field of type (1,1), a vector field and a $1-$form on $M$, respectively. If $\phi$, $\xi$ and $\eta$ satisfy

$$\begin{aligned} \eta(\xi) &= 1, \\ \phi^2(X) &= -X + \eta(X)\xi, \end{aligned}$$

for any vector field $X$ on $M$, then $(M, \xi, \eta, \phi)$ is called an almost contact manifold [2].

*Speaker

$(M, \xi_i, \eta_i, \phi_i)_{i\in\{1,2,3\}}$ is called an almost contact 3-structure manifold [6] if there exist 3 almost contact structures $(\xi_i, \eta_i, \phi_i)$, $i = 1, 2, 3$, on $M$ such that

$$\eta_i(\xi_j) = 0, \ \phi_i\xi_j = -\phi_j\xi_i = \xi_k, \ \eta_i(\phi_j) = -\eta_j(\phi_i) = \eta_k,$$

$$\phi_i o\phi_j - \eta_j \otimes \xi_i = -\phi_j o\phi_i + \eta_i \otimes \xi_j = \phi_k,$$

for a cyclic permutation $(i, j, k)$ of $(1, 2, 3)$.

The vector fields $\xi_1, \xi_2, \xi_3$, are named structure vector fields. Moreover, if there exist a Riemannian metric $g$ on $M$ such that

$$g(\phi_i X, \phi_i Y) = g(X, Y) - \eta_i(X)\eta_i(Y), \ \forall X, Y \in TM, \tag{1}$$

then $(M, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ is said to be an almost contact metric 3-structure manifold. One can easily see that (1) implies

$$g(\phi_i X, Y) = -g(X, \phi_i Y).$$

An almost contact metric 3-structure $(M, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ is a 3-cosymplectic manifold if

$$\tilde{\nabla}\phi_i = 0, \tag{2}$$

and that is a 3-Sasakian manifold if

$$(\tilde{\nabla}_X \phi_i)Y = g(X, Y)\xi_i - \eta_i(Y)X, \ \forall X, Y \in TM, \tag{3}$$

where $\tilde{\nabla}$ is the Levi-Civita connection of $M$. By using (2) and (3), one can obtain

$$\tilde{\nabla}\xi_i = 0 \text{ and } \tilde{\nabla}\xi_i = -\phi_i,$$

in 3-cosymplectic and 3-Sasakian manifolds, respectively.

## 2. Main Results

For an isometrically immersed submanifold $N$ of a Riemannian manifold $M$, we denote its induced Riemannian metric by the same symbol $g$ and the Levi-Civita connection of $N$ by $\nabla$. Let $TN$ and $(TN)^\perp$ be the tangent bundle and normal bundle of $N$, respectively. Then the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X, Y) \text{ and } \tilde{\nabla}_X V = D_X V - A_V X,$$

for $X, Y \in TN$ and $V \in (TN)^\perp$, where $D$ is the connection in the normal bundle, and $B$ is the second fundamental form related to $A$ by the following equation:

$$g(A_V X, Y) = g(B(X, Y), V).$$

$N$ is called totally geodesic if and only if $B$ vanishes identically on $TN$.

Moreover, for any $X \in TN$ and $V \in (TN)^\perp$ we decompose the $\phi_i X$ and $\phi_i V$ as following equations:

$$\phi_i X = T_i X + N_i X \text{ and } \phi_i V = t_i V + n_i V,$$

where $T_i$ and $t_i$ are tangential components of $\phi_i$, $N_i$ and $n_i$ are normal components of $\phi_i$.

DEFINITION 2.1. [5] Let $N$ be a submanifold of a 3-structure manifold $(M, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$. $N$ is a point-wise 3-slant submanifold if at any point $p \in N$ and for each non-zero $X \in T_pN$ linearly independent of $\xi_i$, the Wirtinger angle between $\phi_i X$ and $T_pN$ is constant for all $i \in \{1,2,3\}$. In fact, the angle $\Theta_p(X)$ between $\phi_i X$ and $T_j X$ only depends on the choice of $p$ and it is independent of choosing of $X$ and $i, j$.

DEFINITION 2.2. Let $N$ be a submanifold of a 3-structure manifold $(M, g, \xi_i, \eta_i, \phi_i)$. $N$ is said to be a pointwise pseudo 3-slant if $N$ admits 3 distributions $\mathcal{D}_\theta, \mathcal{D}^\perp$ and $\Xi = Span\{\xi_1, \xi_2, \xi_3\}$ such that

(a) $TN = \mathcal{D}_\theta \oplus \mathcal{D}^\perp \oplus \Xi$;
(b) $\phi_i(\mathcal{D}^\perp) \subset T^\perp N$, for all $i \in \{1,2,3\}$;
(c) For each $Y \in \mathcal{D}_\theta$ the angle function between $\phi_i(Y)$ and $\mathcal{D}_\theta$ dose not depend on choice of $Y$.

EXAMPLE 2.3. Let $(M = \mathbb{R}^{15}, g)$ be the 15-dimensional Euclidean space. We define (1,1)-tensor fields $\phi_1, \phi_2, \phi_3$ as follows

$$\phi_1((x_i)_{i=\overline{1,15}}) = (-x_3, x_4, x_1, -x_2, -x_7, x_8, x_5, -x_6, \ldots, 0, -x_{15}, x_{14}),$$

$$\phi_2((x_i)_{i=\overline{1,15}}) = (-x_4, -x_3, x_2, x_1, -x_8, -x_7, x_6, x_5, \ldots, x_{15}, 0, -x_{13}),$$

$$\phi_3((x_i)_{i=\overline{1,15}}) = (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7, \ldots, -x_{14}, x_{13}, 0).$$

In addition, we put $\xi_1 = \partial_{13}, \xi_2 = \partial_{14}, \xi_3 = \partial_{15}$ and $\eta_1 = dx_{13}, \eta_2 = dx_{14}, \eta_3 = dx_{15}$. One can verify that $(M, g, \xi_i, \eta_i, \phi_i)_{i\in\{1,2,3\}}$ is a 3-cosymplectic manifold.

Now, for real-valued functions $u, v \in C^\infty(\mathbb{R}^{15})$, we suppose a 6-dimensional submanifold $N$ given by the immersion

$$\psi(t_1, t_2, t_3, t_4, t_5, t_6) = (t_1 u, t_2 v, t_2 v, t_2 v, t_3, 0, 0, 0, t_1 v, 0, 0, 0, t_4, t_5, t_6).$$

We assume $\mathcal{D}_\theta = \mathrm{Span}\{\mathrm{X}_1 = \mathrm{u}\partial_1 + \mathrm{v}\partial_9, \mathrm{X}_2 = \mathrm{v}(\partial_2 + \partial_3 + \partial_4)\}$, $\mathcal{D}^\perp = \mathrm{Span}\{\mathrm{X}_3 = \partial_5\}$ and $\Xi = \mathrm{Span}\{\mathrm{X}_4 = \partial_{13}, \mathrm{X}_5 = \partial_{14}, \mathrm{X}_6 = \partial_{15}\}$. By direct computation we conclude $\mathcal{D}_\theta$ is a pointwise 3-slant distribution with slant function $\Theta = cos^{-1}(\frac{v}{\sqrt{3}\sqrt{v^2+u^2}})$ and $\mathcal{D}^\perp$ is an anti-invariant distribution. Therefore, $N$ is a pointwise pseudo 3-slant submanifold of $\mathbb{R}^{15}$.

By using the approach of the proof of in [6, Theorem 2], we have the following characterization.

THEOREM 2.4. *Let $N$ be a isometrically immersed submanifold of a 3-cosymplectic manifold $(M, g, \xi_i, \eta_i, \phi_i)$ and $\xi_i \in TN$ for $i = 1, 2, 3$. $N$ is pointwise pseudo 3-slant if and only if $\forall i, j \in \{1,2,3\}$, we have*

(a) $\mathcal{D} = \{Y \in TN \setminus < \xi_1, \xi_2, \xi_3 > | T_i T_j Y = \mu Y\}$ *is a distribution on $N$ for a function $\mu \in [-1, 0)$;*
(b) $\forall Y \in TN$ *orthogonal to distribution* $\mathcal{D} \oplus < \xi_1, \xi_2, \xi_3 >$, $T_i Y = 0$.

*Also, if $\Theta$ be the slant function, then $\mu = -\cos^2\Theta$.*

PROPOSITION 2.5. *Let $(M, g, \xi_i, \eta_i, \phi_i)$ be a 3-cosymplectic manifold and $N$ be a pointwise pseudo 3-slant submanifold of $M$. Then the distribution spanned by the structure vector fields is a integrable distribution.*

Proof. From Eq. (2) on 3-cosymplectic manifolds $\tilde{\nabla}_{\xi_i}\xi_j = 0$. Moreover, the Levi-Civita connection is torsion free. So, we get $[\xi_i, \xi_j] = 0 \in \Xi$. Therefore, $\Xi = span\{\xi_1, \xi_2, \xi_3\}$ is integrable. □

Theorem 2.6. *Let $(M, g, \xi_i, \eta_i, \phi_i)$ be a 3-cosymplectic manifold and $N$ be a pointwise pseudo 3-slant submanifold of $M$. Then, the anti-invariant distribution $\mathcal{D}^\perp$ is integrable.*

Proof. For any $X, Y \in \mathcal{D}^\perp$ and $i = 1, 2, 3$, we have

$$\phi_i[X, Y] = T_i[X, Y] + N_i[X, Y] = T_i\nabla_Y X - T_i\nabla_X Y + N_i[X, Y].$$

Furthermore, $(M, g, \xi_i, \eta_i, \phi_i)$ is a 3-cosymplectic manifold and $N_iZ = 0$, thus we get

$$(\tilde{\nabla}_X T_i)Y = T_i\nabla_X Y - A_{N_iY}X = 0.$$

So, $\phi_i[X, Y] = -A_{N_iY}X + A_{N_iX}Y + N_i[X, Y]$. By some calculations we conclude

$$\phi_i[X, Y] = N_i[X, Y].$$

That implies $[Y, Z] \in \mathcal{D}^\perp$, So $\mathcal{D}^\perp$ is integrable. □

## References

1. M. Atceken, S. Dirik and U. Yildirim, *Pseudo-slant submanifolds of a locally decomposable Riemannian manifold*, J. Adv. Math. **11** (2015) 5587–5597.
2. D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Math. Vol. 203, Birkhauser, Boston-Basel-Berlin, 2002.
3. B. -Y. Chen and O. J. Garay, *Pointwise slant submanifolds in almost Hermitian manifolds*, Turk. J. Math. **36** (2012) 630–640.
4. F. Etayo, *On quasi-slant submanifolds of an almost Hermitian manifold*, Publ. Math. Debrecen **53** (1998) 217–223.
5. M. B. Kazemi Balgeshir, *Pointwise slant submanifolds in almost contact geometry*, Turk. J. Math. **40** (2016) 657–664.
6. F. Malek and M. B. Kazemi Balgeshir, *Semi-slant and bi-slant submanifolds of almost contact metric 3-structure manifolds*, Turk. J. Math. **37** (2013) 1030–1039.
7. S. Uddin and F. R. Al-Solamy, *Warped product pseudo-slant immersions in Sasakian manifolds*, Publ. Math. Debrecen **91** (2) (2017) 1–14.

E-mail: f.raei@cfu.ac.ir
E-mail: mbkazemi@znu.ac.ir

# Stable Exponential Harmonic Maps with Potential

Seyed Mehdi Kazemi Torbaghan*
Faculty of Basic Sciences, Univesity of Bojnord, Bojnord, Iran,
Salman Babayi
Department of Mathematics, Faculty of Science, Urmia University, Urmia, Iran
and Keyvan Salehi
Central of Theoretical Physic and Chemistry (CTPC), Massey University, Auckland, Newzealand

Abstract. In this paper, the exponential energy functional for a smooth map between Finsler manifolds and Riemannian manifolds is introduced and the first and second variation formulas are obtained. Then, the stability of exponential harmonic maps from a Finsler manifold to the standard unit sphere $S^n(n > 2)$ is investigated.

**Keywords:** Exponential harmonic maps, Stability, Riemannian manifolds, Calculus of variations.
**AMS Mathematical Subject Classification [2010]:** 53C43, 58E20.

## 1. Introduction

Mo introduced harmonic maps from a Finsler manifold, see [5]. In 2000, Professor S. S. Chern speculate that the existence theorem of harmonic map on Finsler manifold is true. In [6], the scholars have done research on this this presumption and proven it. Let $(M, F)$ be a compact Finsler space and $(N, h)$ be a compact Riemannian manifold of non-positive sectional curvature. Then, any smooth map $\phi : (M, F) \longrightarrow (N, h)$ can be deformed into harmonic map.

Harmonic maps with potential, was initially suggested by Ratto in [7] and recently developed by several authors: V. Branding [1], Y. Chu [2], A. Fardoun and all [4] and other. Let $\phi : (M, g) \longrightarrow (N, h)$ be a smooth map between Riemannian manifolds, and let $H$ be a smooth function on $N$. The $H-$energy function of $\phi$ is denoted by $E_H(\phi)$ and defined by

$$E_H(\phi) = \int_M [e(\phi) - H(\phi)]\nu_g,$$

where $e(\phi)$ is the energy density of $\phi$ defined by $e(\phi) := \frac{1}{2} \mid d\phi \mid^2$. Any critical point of $E_H$ is called harmonic with potential $H$.

Eells and Lemaire [3] extended the notion of harmonicc maps to exponential harmonic maps, and studied the stability of these maps under the curvature conditions on the target manifold. They defined the exponential energy functional of

*Speaker

$\phi : (M, g) \longrightarrow (N, h)$ as follows:

$$E_e(\phi) = \int_M \exp(\frac{|\, d\phi\, |^2}{2})\nu_g.$$

A map $\phi$ is called *exponential harmonic* if $\phi$ is a critical point of the exponential energy functional. By calculating the first variation formula, it can be seen that any exponential harmonic $\phi$ satisfies the following equation

$$\tau_e(\phi) = \tau(\phi) + d\phi(grad\ \exp(e(\phi))) = 0.$$

The section $\tau_e(\phi) \in \Gamma(\phi^{-1}TN)$ is called exponential tension field of $\phi$, [3].

In this manuscript, first, the first and second variation formulas for exponential harmonic maps with potential from a Finsler manifold to a Riemannian manifold are calculated. Then, the stability of exponential harmonic maps from a Finsler manifold to the standard unit sphere $S^n(n > 2)$ is investigated.

## 2. Main Results

Let $\phi : (M^m, F) \longrightarrow (N^n, h)$ be a smooth map and let $H$ be a smooth function on $N$. from now on, we denote the pull-back connection on $p^*(\phi^{-1}TN)$, the Levi-Civita connection on $(N, h)$ and the Chern connection on $p^*$TM by $\nabla, {}^N\nabla$ and $\nabla^c$, respectively.

The *energy density* of $\phi$ is a function $e(\phi) : SM \longrightarrow \mathbb{R}$ defined by

$$e(\phi)(x, y) := \frac{1}{2} Tr_g h(d\phi, d\phi).$$

Definition 2.1. Let $\phi : (M, F) \longrightarrow (N, h)$ be a smooth map. the exponential energy functional of $\phi$ is defined as follows

$$E_{e,H}(\phi) := \frac{1}{c_{m-1}} \int_{SM} (\exp(e(\phi)) - H \circ \phi) dV_{SM}.$$

Any critical points of $E_{e,H}$ is called *exponential harmonic* with potential $H$. Here volume of $S^{m-1}$ and the volume element of $SM$ by $c_{m-1}$ and $dV_{SM}$, respectively.

Lemma 2.2. (The first variation formula) *Let $\phi : (M, g) \longrightarrow (N, h)$, then*

$$\frac{d}{dt} E_{e,H}(\phi_t)\big|_{t=0} = -\frac{1}{c_{m-1}} \int_{SM} h(\tau_{e,H}(\phi), V) dV_{SM},$$

*where $\phi_t : M \longrightarrow N$ be a smooth variation of $\phi$ such that $\phi_0 = \phi$ and*

$$\begin{aligned}\tau_{e,H}(\phi) :&= \exp(F'(e(\phi)))Tr_g \nabla d\phi + d\phi \circ p(grad^H F'(e(\phi))) \\ &- F'(e(\phi)) d\phi \circ p(K^H) \in \Gamma((\phi \circ p)^*TN),\end{aligned}$$

*here $V = \frac{\partial \phi_t}{\partial t}\big|_{t=0} := V^\alpha \frac{\partial}{\partial \tilde{x}^\alpha} \circ \phi$, and $K$ is defined by $K := \sum_{a,b} \dot{A}_{bba} e_a \in \Gamma(p^*TM)$. The field $\tau_{e,H}(\phi)$ is said to be the f-tension field of $\phi$.*

Definition 2.3. A map $\phi$ is said to be exponential harmonic with potential $H$ if $\tau_{e,H}(\phi) = 0$.

Definition 2.4. Let $\phi : (M, g) \longrightarrow (N, h)$ be an exponential harmonic map with potential $H$. Setting

$$I(V) = \frac{d^2}{dt^2} E_{e,H}(\phi_t)\ |_{t=0}\ .$$

The map $\phi$ is called stable if $I(V) \geq 0$ for any compactly supported vector field $V$ along $\phi$.

THEOREM 2.5. *Let $\phi : (M, g) \longrightarrow (N, h)$ be an exponential harmonic map with potential $H$. Then*

$$I(V) = \frac{1}{c_{m-1}} \int_{SM} \exp(\frac{| d\phi |^2}{2})\Big\{-TR_g R(d\phi, V, V, d\phi) + \|\nabla V\|\Big\} dV_{SM}$$
$$+ \frac{1}{c_{m-1}} \int_{SM} \exp(\frac{| d\phi |^2}{2})\Big\{<\nabla V, d\phi> - (\nabla_V^N grad^N H) \circ \phi, V)\Big\} dV_{SM},$$

*where $V = \frac{\partial \phi_t}{\partial t} |_{t=0}$, and $| \nabla V |$ denotes the Hilbert-Schmidt norm of the $\hat{\nabla} V \in \Gamma(T^* M \times \phi^{-1} TN)$.*

THEOREM 2.6. *Let $\phi : (M, F) \longrightarrow \mathbb{S}^n$ be a stable exponential harmonic map with potential $H$ from a Riemannian manifold $(M, g)$ to $\mathbb{S}^n (n > 2)$, and let $\triangle^{\mathbb{S}^n} H \circ \phi \geq 0$. Then $\phi$ is constant.*

COROLLARY 2.7. *Let $\phi : (M, F) \longrightarrow \mathbb{S}^n$ be a stable exponential harmonic map with potential $H$ from a Riemannian manifold $(M, F)$ to $\mathbb{S}^n (n > 2)$. Suppose that $H$ is an affine function. Then $\phi$ is constant.*

## References


1. V. Branding, *The heat flow for the full bosonic string*, Ann. Global Anal. Geom. **50** (2016) 347–365.
2. Y. Chu and X. Liu, *Regularity of harmonic maps with the potential*, Sci. China Ser. A. **49** (2006) 599–610.
3. J. Eells and L. Lemaire, *Some properties of exponentially harmonic maps*, In: Partial Differential Equations, Part 1, 2 (Warsaw, 1990), 129–136, Banach Center Pulb. 27, Polish Acad. Sci, Warsaw (1992).
4. A. Fardoun, A. Ratto and R. Regbaoui, *On the heat flow for harmonic maps with potential*, Ann. Global Anal. Geom. **18** (2000) 555–567.
5. X. Mo, *Harmonic maps from Finsler manifolds*, Illinois J. Math. **45** (2001) 1331–1345.
6. X. Mo and Y. Yang, *The existence of harmonic maps from Finsler manifolds to Riemannian manifolds*, Sci. China Ser. A. **48** (2005) 115–130.
7. A. Ratto, *Harmonic maps with potential*, Proceedings of the Workshop on Differential Geometry and Topology (Palermo, 1996). Rend. Circ. Mat. Palermo (2) Suppl. **49** (1997) 229–242.



E-mail: m.kazemi@ub.ac.ir
E-mail: S.babayi@urmia.ac.ir
E-mail: K.salehi@massey.ac.nz

# On $(G, H)$-(Semi) Covering Map

Majid Kowkabi*
Department of Pure Mathematics, University of Gonabad, Gonabad, Iran
and Hamid Torabi
Department of Pure Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159-91775, Mashhad, Iran

Abstract. In this paper, by reviewing the concept of covering maps and semicovering maps, we define and motivate $(G, H)$-(semi) covering map. Also we investigate the properties of $(G, H)$-(semi) covering map. For example, if $p : \tilde{X} \longrightarrow X$ is a $(G, H)$-(semi) covering map and $\alpha$ is a path in $\tilde{X}$ with starting at $\tilde{x_0}$ and $\alpha(1) = x$, then $p$ is an $(\alpha^{-1}G\alpha, (p \circ \alpha)^{-1}H(p \circ \alpha))$-(semi) covering map.

**Keywords:** Fundamental group, Covering map, Semicovering map.
**AMS Mathematical Subject Classification [2010]:** 57M10, 57M12, 57M05.

## 1. Introduction

Recall that a continuous map $p : \tilde{X} \longrightarrow X$ is called a covering of $X$, if for every $x \in X$ there is an open subset $U$ of $X$ with $x \in U$ such that $U$ is evenly covered by $p$ i.e. $p^{-1}(U)$ is a disjoint union of open subsets of $\tilde{X}$ each of which is mapped homeomorphically onto $U$ by $p$.

Assume that $X$ and $\tilde{X}$ are topological spaces and $p : \tilde{X} \longrightarrow X$ is a continuous map. Let $f : (Y, y_0) \longrightarrow (X, x_0)$ be a continuous map and $\tilde{x}_0 \in p^{-1}(x_0)$. If there exists a continuous map $\tilde{f} : (Y, y_0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ such that $p \circ \tilde{f} = f$, then $\tilde{f}$ is called a *lifting* of $f$.

The map $p$ has *path lifting property* if for every path $f$ in $X$, there exists a lifting $\tilde{f} : (I, 0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ of $f$. Also, the map $p$ has *unique path lifting property* if for every path $f$ in $X$, there is at most one lifting $\tilde{f} : (I, 0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ of $f$ (See [3]).

Brazas [1, Definition 3.1] generalized the concept of covering map by the phrase *"A semicovering map is a local homeomorphism with continuous lifting of paths and homotopies"*. Note that a map $p : Y \to X$ has *continuous lifting of paths* if $\rho_p : (\rho Y)_y \to (\rho X)_{p(y)}$ defined by $\rho_p(\alpha) = p \circ \alpha$ is a homeomorphism, for all $y \in Y$, where $(\rho Y)_y = \{\alpha : I = [0, 1] \to Y | \alpha(0) = y\}$. Also, a map $p : Y \to X$ *has continuous lifting of homotopies* if $\Phi_p : (\Phi Y)_y \to (\Phi X)_{p(y)}$ defined by $\Phi_p(\phi) = p \circ \phi$ is a homeomorphism, for all $y \in Y$, where elements of $(\Phi Y)_y$ are endpoint preserving homotopies of paths starting at $y$. He also simplified the definition of semicovering maps by showing that having continuous lifting of paths implies having continuous lifting of homotopies.

*Speaker

Theorem 1.1. (Local Homeomorphism Homotopy Theorem for Paths) *Let* $(\tilde{X}, p)$ *be a local homeomorphism of* $X$ *with unique path lifting and path lifting properties. Consider the diagram of continuous maps*

$$\begin{array}{ccc} I & \xrightarrow{\tilde{f}} & (\tilde{X}, \tilde{x}_0) \\ \downarrow{\scriptstyle j} & \overset{\tilde{F}}{\nearrow} & \downarrow{\scriptstyle p} \\ I \times I & \xrightarrow{F} & (X, x_0) \end{array}$$

*where* $j(t) = (t, 0)$ *for all* $t \in I$*. Then there exists a unique continuous map* $\tilde{F} : I \times I \to \tilde{X}$ *making the diagram commute.*

The following corollary is a consequence of the above theorem.

Corollary 1.2. *Let* $p : \tilde{X} \to X$ *be a local homeomorphism with unique path lifting and path lifting properties. Let* $x_0, x_1 \in X$ *and* $f, g : I \to X$ *be paths such that* $f(0) = g(0) = x_0$*,* $f(1) = g(1) = x_1$ *and* $\tilde{x}_0 \in p^{-1}(x_0)$*. If* $F : f \simeq g$ *rel* $\dot{I}$ *and* $\tilde{f}, \tilde{g}$ *are the lifting of* $f$ *and* $g$*, respectively, with* $\tilde{f}(0) = \tilde{x}_0 = \tilde{g}(0)$*, then* $\tilde{F} : \tilde{f} \simeq \tilde{g}$ *rel* $\dot{I}$*.*

The following theorem can be found in [1, Corollary 2.6 and Proposition 6.2].

Theorem 1.3. (Lifting Criterion Theorem for Semicovering Maps) *If* $Y$ *is connected and locally path connected,* $f : (Y, y_0) \to (X, x_0)$ *is continuous and* $p : \tilde{X} \to X$ *is a semicovering map where* $\tilde{X}$ *is path connected, then there exists a unique* $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ *such that* $p \circ \tilde{f} = f$ *if and only if* $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$*.*

The following theorem can be concluded from [2, Theorem 2.4].

Theorem 1.4. *A map* $p : \tilde{X} \to X$ *is a semicovering map if and only if it is a local homeomorphism with unique path lifting and path lifting properties.*

Since every (semi) covering map $p : \tilde{X} \longrightarrow X$ has Homotopy lifting property, every path $\alpha$ in $\tilde{X}$ such that $[p \circ \alpha] = 1$ i.e. $p \circ \alpha$ is null, $\alpha$ is a null homotopic loop. This fact motivated us to explore the $(G, H)$-(semi) covering map.

In this paper, we introduce the $(G, H)$-(semi) covering map. Also we investigate the properties of $(G, H)$-(semi) covering map. For example, if $p : \tilde{X} \longrightarrow X$ is a $(G, H)$-(semi) covering map and $\lambda$ is a path in $X$ with starting at $x$ and $\tilde{\lambda}$ is lifting of $\lambda$ with starting at $\tilde{x_0}$, then $p$ is a $(\tilde{\lambda}^{-1} G \tilde{\lambda}, \lambda^{-1} H \lambda)$-(semi) covering map. All of the spaces in this paper are path connected.

## 2. $(G, H)$-(Semi) Covering Map

Definition 2.1. Let $p : \tilde{X} \longrightarrow X$ is a (semi) covering map with $p(\tilde{x_0}) = x_0$. If $G \leq \pi_1(\tilde{X}, \tilde{x_0})$ and $H \leq \pi_1(X, x_0)$, we call that $p$ is a $(G, H)$-(semi) covering map if for every path $\alpha$ in $\tilde{X}$ with starting at $\tilde{x_0}$ such that $[p \circ \alpha] \in H$, then $[\alpha] \in G$.

Since every (semi) covering map $p:\tilde{X}\longrightarrow X$ has Homotopy lifting property, every path $\alpha$ in $\tilde{X}$ such that $[p\circ\alpha]=1$ i.e. $p\circ\alpha$ is null, $\alpha$ is a null homotopic loop. So every (semi) covering map is $(1,1)$-(semi) covering map. Also, every (semi) covering map $p:(\tilde{X},\tilde{x_0})\longrightarrow(X,x_0)$ is $(\pi_1(\tilde{X},\tilde{x_0}),p_*(\pi_1(\tilde{X},\tilde{x_0}))$-(semi) covering map.

LEMMA 2.2. *If $p:\tilde{X}\longrightarrow X$ is a $(G,H)$-(semi) covering map and $G\leq G', H'\leq H$, then $p$ is a $(G',H')$-(semi) covering map.*

The following corollary is a consequence of the above lemma.

COROLLARY 2.3. *If $p:\tilde{X}\longrightarrow X$ is a $(G_j,H)$-(semi) covering map for every $j\in J$, then $p$ is a $(\cap_{j\in J}G_j,H)$-(semi) covering map.*

COROLLARY 2.4. *If $p:\tilde{X}\longrightarrow X$ is a $(G,H_i)$-(semi) covering map for every $i\in I$, then $p$ is a $(G,<\cup_{i\in I}H_i>)$-(semi) covering map.*

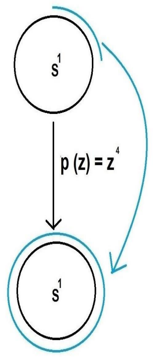



FIGURE 1. $p:S^1\longrightarrow S^1$ defined by $p(z)=z^4$.

In the following theorem, we show that every $(1,\pi_1(X,x_0))$-covering map, is a universal covering map

THEOREM 2.5. *If $p:\tilde{X}\longrightarrow X$ is a $(1,\pi_1(X,x_0))$-covering map, then $p$ is a universal covering map.*

PROOF. Let $\alpha\in\pi_1(\tilde{X},\tilde{x_0})$, so by definition of coveing map $[p\circ\alpha]=p_*([\alpha])\in\pi_1(X,x_0)$. So $[p\circ\alpha]=1$, thua $\pi_1(\tilde{X},\tilde{x_0})=1$. Therefore by definitionof universal covering map, $p$ is a universal covering map. □

The following corollary is a consequence of the above theorem.

COROLLARY 2.6. *Every (semi) covering map $p:(\tilde{X},\tilde{x_0})\longrightarrow(X,x_0)$ is $(\pi_1(\tilde{X},\tilde{x_0}),1)$-(semi) covering map.*

COROLLARY 2.7. *If $p$ is a $(G,H)$-(semi) covering map, then $H\leq p_*(\pi_1(\tilde{X},\tilde{x_0}))$.*

PROOF. Let $p : \tilde{X} \longrightarrow X$ is a $(G, H)$-(semi) covering map and $\beta \in H \leq \pi_1(X, x_0)$, so $\beta$ has a lifting $\alpha$ in $\pi_1(\tilde{X}, \tilde{x_0})$ so $\beta = [p \circ \alpha] \in \pi_1(\tilde{X}, \tilde{x_0})$. So $H \leq p_*(\pi_1(\tilde{X}, \tilde{x_0}))$. □

In the following example, we intruduced a $(\mathbb{Z}, 4\mathbb{Z})$-(semi) covering map such that it is not a $(2\mathbb{Z}, 4\mathbb{Z})$-(semi) covering map, where $\mathbb{Z}$ is an integer number.

EXAMPLE 2.8. Consider the famous covering map $p : S^1 \longrightarrow S^1$ defined by $p(z) = z^4$ (See Figure 1), it is a $(\mathbb{Z}, 4\mathbb{Z})$-(semi) covering map where $\mathbb{Z}$ is an integer number but it is not a $(2\mathbb{Z}, 4\mathbb{Z})$-(semi) covering map.

In the definition $(G, H)$-(semi) covering map, we suppose that $G \leq \pi_1(\tilde{X}, \tilde{x_0})$. Note that $G \leq \pi_1(\tilde{X}, \tilde{x_0})$ is not a subgroup of $\pi_1(\tilde{X}, \tilde{x})$ for any point $\tilde{x} \neq \tilde{x_0}$. To present a similar fact, we can consider subgroups corresponding to $G$ in $\pi_1(\tilde{X}, \tilde{x})$ by the isomorphism $\psi_\alpha : \pi_1(\tilde{X}, \tilde{x_0}) \to \pi_1(\tilde{X}, \tilde{x})$ for every path $\alpha$ from $\tilde{x_0}$ to $\tilde{x}$. We denote $[\alpha]^{-1}G[\alpha]$ by $\alpha^{-1}G\alpha$.

LEMMA 2.9. *If $p : \tilde{X} \longrightarrow X$ is a $(G, H)$-(semi) covering map and $\alpha$ is a path in $\tilde{X}$ with starting at $\tilde{x_0}$ and $\alpha(1) = x$, then $p$ is an $(\alpha^{-1}G\alpha, (p \circ \alpha)^{-1}H(p \circ \alpha))$-(semi) covering map.*

The following corollary is a consequence of the above lemma.

COROLLARY 2.10. *If $p : \tilde{X} \longrightarrow X$ is a $(G, H)$-(semi) covering map and $\alpha$ is a path in $\tilde{X}$ with starting at $\tilde{x_0}$ and $\alpha(1) = x$ and $\alpha(1) = x$ such that $p \circ \alpha$ is a loop. $H$ is a normal subgroup of $\pi_1(\tilde{X}, \tilde{x})$, then $p$ is an $(\alpha^{-1}G\alpha, H)$-(semi) covering map.*

PROOF. Let $p : \tilde{X} \longrightarrow X$ is a $(G, H)$-(semi) covering map and $\alpha$ is a path in $\tilde{X}$ with starting at $\tilde{x_0}$ and $\alpha(1) = x$ such that $p \circ \alpha$ is a loop. Since $[p \circ \alpha]$ is a loop, $[p \circ \alpha] \in \pi_1(\tilde{X}, \tilde{x})$ and$H$ is a normal subgroup of $\pi_1(\tilde{X}, \tilde{x})$, so $(\alpha^{-1}G\alpha, (p \circ \alpha)^{-1}H(p \circ \alpha)) = H$. Thus by Lemma 2.9 $p$ is an $(\alpha^{-1}G\alpha, H)$-(semi) covering map. □

In the following corollary, we show that every $(G, H)$-(semi) covering map, is a $(G, (p \circ \alpha)^{-1}H(p \circ \alpha))$-(semi) covering map, where $\alpha$ is a loop in $\tilde{X}$ at $\tilde{x_0}$ and $G$ is a normal subgroup of $\pi_1(\tilde{X}, \tilde{x_0})$.

COROLLARY 2.11. *If $p : \tilde{X} \longrightarrow X$ is a $(G, H)$-(semi) covering map and $\alpha$ is a loop in $\tilde{X}$ at $\tilde{x_0}$ and $G$ is a normal subgroup of $\pi_1(\tilde{X}, \tilde{x_0})$, then $p$ is a $(G, (p \circ \alpha)^{-1}H(p \circ \alpha))$-(semi) covering map.*

PROOF. Let $p : \tilde{X} \longrightarrow X$ is a $(G, H)$-(semi) covering map and $\alpha$ is a loop in $\tilde{X}$ at $\tilde{x_0}$. Since $G$ is a normal subgroup of $\pi_1(\tilde{X}, \tilde{x_0})$,$(\alpha^{-1}G\alpha = G$. So by the Lemma 2.9 $p$ is a $(G, (p \circ \alpha)^{-1}H(p \circ \alpha))$-(semi) covering map. □

COROLLARY 2.12. *If $p : \tilde{X} \longrightarrow X$ is a $(G, H)$-(semi) covering map and $\lambda$ is a path in $X$ with starting at $x$ and $\tilde{\lambda}$ is lifting of $\lambda$ with starting at $\tilde{x_0}$, then $p$ is a $(\tilde{\lambda}^{-1}G\tilde{\lambda}, \lambda^{-1}H\lambda)$-(semi) covering map.*

## References

1. J. Brazas, *Semicoverings: A generalization of covering space theory*, Homology Homotopy Appl. **14** (1) (2012) 33–63.
2. M. Kowkabi, B. Mashayekhy and H. Torabi, *When is a local homeomorphism a semicovering map*?, Acta Math. Vietnam. **42** (4) (2017) 653–663.
3. E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

E-mail: majid.kowkabi@gonabad.ac.ir
E-mail: h.torabi@ferdowsi.um.ac.ir

# Projective Vector Field on Finsler Spaces

Behnaz Lajmiri*
Department of Mathematics and Computer Science Amirkabir University of Technology
(Tehran Polytechnic) 424 Hafez Ave. 15914 Tehran, Iran

Abstract. The collection of all projective vector fields on a Finsler space $(M, F)$ is a finite-dimensional Lie algebra with respect to the usual Lie bracket, called projective algebra and is denoted by $p(M, F)$. It is the Lie algebra of the projective group $P(M, F)$. After a short review of the definitions of Randers metric and projective vector field. we show that for Randers space with isotropic $S$-curvature and $\beta$ is not close, every affine vector field is invariant affine.
**Keywords:** Projective vector, Isotropic $S$-curvature, Finsler.
**AMS Mathematical Subject Classification [2010]:** 53B40, 53C60.

## 1. Introduction

In general relativity, many spacetimes possess certain symmetries that can be characterised by vector fields on the spacetime. Projective vector fields are a class of important vector fields on differential manifolds, included some important concepts such as Killing vector fields, affine vector fields. All those fields describe some symmetries of the space. Indeed, a projective vector field is related to a projective transformation, which preserve the geodesics.

There are lots of Finsler metrics in this class, for example Randers metrics are the most popular Finsler metrics in differential geometry and physics simply obtained by a Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)$ was introduced by $G$. Randers in [5] in the context of general relativity. They arise naturally as the geometry of light rays in stationary space times [4]. One may refer to [3, 6] for an extensive series of results about the Einstein Randers metrics and the Randers metrics. In present paper we investigated what is the result about the Randers metric with projective vector field, and we show that for Randers space with isotropic $S$-curvature and $\beta$ is not close, every affine vector field is invariant affine.

## 2. Main Results

Theorem 2.1. *Let $(M, F)$ be a Randers space with non-isotropic $S$-curvature, $s_{ij} = 0$ and $V$ is affine vector field then the relation projective transformation for it is following*

$$\nabla_0 L_{\hat{V}} \beta = -2\sigma e_{00}.$$

*Speaker

THEOREM 2.2. *Let $(M, F)$ be Randers space with isotropic $S$-curvature, $\beta$ is close, then every affine vector field is invariant affine.*

## 3. Preliminaries and Notations

A *Finsler structure* on a differentiable manifold $M$ is a continuous function $F : TM \to [0, \infty)$, with the following properties; $F$ is differentiable on $TM_0 := TM\backslash\{0\}$ and positively 1-homogeneous on the fibers of $TM$. The vertical Hessian of $F^2$ with the following components is positive-definite on $TM_0$, $(g_{ij}) := \Big(\big[\frac{1}{2}F^2\big]_{y^i y^j}\Big)$.

The Finsler structure $F$ defines a fundamental tensor $g : \pi^*TM \otimes \pi^*TM \to [0, \infty)$, called *Finsler metric* with the components $g(\partial_i|_v, \partial_j|_v) = g_{ij}(x, y)$, where $V = y^i \frac{\partial}{\partial x^i}$ is a section of $\pi^\star TM$, and $v = V|_x = y^i \frac{\partial}{\partial x^i}|_x$. The pair $(M, g_{ij})$ is called a *Finsler manifold.* by one of the present authors in [2]. Let $(M, F)$ be a Riemannian space and $\beta = b_i(x)y^i$ be a 1-form defined on $M$ such that $\|\beta\|_x := \sup \frac{\beta(y)}{\alpha(y)} < 1$. The Finsler metric $F = \alpha + \beta$ is called a Randers metric on a manifold $M$. Denote the geodesic spray coefficients of $\alpha$ and $F$ by the notions $G^i_\alpha$ and $G^i$, respectively and the Levi-Civita connection of $\alpha$ by $\nabla$. Define $\nabla_j b_i$ by $(\nabla_j b_i)\theta^j := db_i - b_j\theta^j_i$, where $\theta^i := dx^i$ and $\theta^j_i := \Gamma^j_{ik}dx^k$ denote the Levi-Civita connection forms and $\nabla$ denotes its associated covariant derivation of $\alpha$. Let us put

$$r_{ij} := \frac{1}{2}(\nabla_j b_i + \nabla_i b_j), s_{ij} := \frac{1}{2}(\nabla_j b_i - \nabla_i b_j),$$

$$s^i_j := a^{ih}s_{hj}, s_j := b_i s^i_j, e_{ij} := r_{ij} + b_i s_j + b_j s_i.$$

Then $G^i$ are given by

$$G^i = G^i_\alpha + (\frac{e_{00}}{2F} - s_0)y^i + \alpha s^i_0,$$

where $e_{00} := e_{ij}y^i y^j, s_0 := s_i y^i, s^i_0 := s^i_j y^j$ and $G^i$ denote the geodesic coefficients of $\alpha$. Notice that the $S$-curvature of a Randers metric $F = \alpha + \beta$ can be obtained as follows

$$S = (n+1)\{\frac{e_{00}}{F} - s_0 - \rho_0\},$$

where $\rho = \ln\sqrt{1 - \|\beta\|}$ and $\rho_0 = \frac{\partial \rho}{\partial x^k}y^k$.

**3.1. Non-Riemannian Quantities and Special Finsler Spaces.** Let us consider the volume form on $\mathbb{R}^n$ and the distortion scalar function on $TM_0$ as follows (See [8]),

$$\tau(x, y) := \ln\Big[\frac{\sqrt{\det\Big(g_{ij}(x, y)\Big)}}{\mathrm{Vol}(\mathrm{B}^n(1))}\mathrm{Vol}\Big\{(y^i) \in \mathbb{R}^n \Big|\ F\Big(y^i\frac{\partial}{\partial x^i}|_x\Big) < 1\Big\}\Big].$$

Consider the *mean Cartan torsion* defined by $\mathbf{I}_y := I_i(x, y)dx^i$ where $I_i(x, y) := \frac{\partial \tau}{\partial y^i}(x, y) = \frac{1}{2}g^{jk}(x, y)\frac{\partial g_{jk}}{\partial y^i}(x, y)$. Set $L_{ijk} := C_{ijk|s}y^s$,$C_{ijk}$ is cartan tensor and $J_i := g^{jk}L_{ijk}$. $\mathbf{L}$ is called *Landsberg tensor*, and $\mathbf{J}$ is called *mean Landsberg tensor*. A Finsler metric is called a *Landsberg metric* (resp. *weakly Landsberg metric*) if $L = 0$ (resp. $J = 0$).

Definition 3.1. The *S-curvature* $\mathbf{S} = \mathbf{S}(x, y)$ is defined by

$$\mathbf{S}(x,y) := \frac{d}{dt}\left[\tau(\sigma(t), \dot{\sigma}(t))\right]|_{t=0}. \tag{1}$$

It is positively $\mathbf{y}$-homogeneous of degree one, $\mathbf{S}(x, \lambda y) = \lambda \mathbf{S}(x, y),\ \lambda > 0$. Let $\sigma(t)$ be a geodesic and define

$$\tau(t) := \tau(\sigma(t), \dot{\sigma}(t)), \qquad S(t) := S(\sigma(t), \dot{\sigma}(t)).$$

By means of (1) we have $S(t) = \tau'(t)$. A Finsler metric $F$ is said to have *isotropic S-curvature* if $S = (n + 1)cF$, where $c = c(x)$ is a scalar function on $M$. Differentiating the $S$-curvature twice, gives rise to the following quantity

$$E_{ij} := \frac{1}{2} S_{y^i y^j}(x, y).$$

For $y \in T_xM\backslash 0,\ \ E_y = E_{ij}(x, y)dx^i \otimes dx^j$ is a symmetric bilinear form on $T_xM$.

**3.2. Projective Vector Fields on Finsler Spaces.** A diffeomorphism between two Finsler manifolds $(M, F)$ and $(M, \bar{F})$ is called a *projective transformation* if it takes every forward (resp. backward) geodesic to a forward (resp. backward) geodesic. A projective transformation is called an *affine transformation* if it leaves invariant the connection coefficients. Every vector field $X$ on $M$ induces naturally an infinitesimal coordinate transformations $(x^i, y^i) \longrightarrow (\bar{x}^i, \bar{y}^i)$ on $TM$, given by $\bar{x}^i = x^i + X^i dt$, and $\bar{y}^i = y^i + y^k \frac{\partial X^i}{\partial x^k} dt$. It leads to the notion of the *complete lift* $\hat{X}$ of a vector field $X$ on $M$ to a vector field $\hat{X} = X^i \frac{\partial}{\partial x^i} + y^k \frac{\partial X i}{\partial x^k} \frac{\partial}{\partial y^i}$ on $TM_0$, see for instance [10]. In Finsler geometry, almost all geometric objects depend on both position and direction. Hence, the Lie derivatives of these objects in direction of a vector field $X$ on $M$ must be considered in relation to the complete lift vector field $\hat{X}$.

Let $X$ be a vector field on the Finsler manifold $(M, F)$. We denote its complete lift to $TM_0$ by $\hat{X}$ where, $\hat{X} = X^i \frac{\delta}{\delta x^i} + \nabla_0 X^i \frac{\partial}{\partial y^i}$. It's a remarkable observation that, $\pounds_{\hat{X}} y^i = 0$, $\pounds_{\hat{X}} dx^i = 0$ and the differential operators $\pounds_{\hat{X}}$, $\frac{\partial}{\partial x^i}$, the exterior differential operator $d$ and $\frac{\partial}{\partial y^i}$ commute, see for instance [1, 10].

A smooth vector field $X$ is called a *projective vector field* or *affine vector field* on $(M, F)$ if the associated local flow is a projective or affine transformation, respectively. There are several approaches for definition of a projective vector field on a Finsler manifold. We frequently use the following Lemma.

Lemma 3.2. [9] *A vector field $X$ on the Finsler manifold $(M, F)$ is a* projective vector field *if and only if there is a function $\Psi = \Psi(x, y)$ on $TM_0$, positively 1-homogeneous on $y$, such that*

$$\pounds_{\hat{X}} G^i = \Psi(x, y) y^i. \tag{2}$$

*$X$ is an* affine vector field *if and only if $\Psi(x, y) = 0$.*

Lemma 3.3. *Let $(M, F)$ be a Finsler manifold. For a projective vector field $X$ on $(M, F)$ we have*

(3) $$\pounds_{\hat{X}} G^i_k = \Psi_k y^i + \Psi \delta^i_k, \qquad where \quad \Psi_k := \Psi_{,k} := \frac{\partial \Psi}{\partial y^k}.$$

(4) $$\pounds_{\hat{X}} G^i_{jk} = \delta^i{}_j \Psi_k + \delta^i_k \Psi_j + y^i \Psi_{k,j}.$$

(5) $$\pounds_{\hat{X}} G^i_{jkl} = \delta^i_j \Psi_{k,l} + \delta^i_k \Psi_{j,l} + \delta^i_l \Psi_{k,j} + y^i \Psi_{k,j,l}.$$

$$\pounds_{\hat{X}} E_{jl} = \frac{1}{2}(n+1)\Psi_{j,l}.$$

$$\pounds_{\hat{X}} I_k = f_{,k}. \ where f = X^i{}_{|i} + I_i X^i_{|m} y^m,$$

$$\pounds_{\hat{X}} J_k = f_{,k|m} y^m + \Psi I_k.$$

$$(n+1)\Psi_k = f_{|k} + f_{,k|m} y^m, where\ \Psi = (\frac{1}{n+1})(f)_{|s} y^s.$$

$$\pounds_{\hat{X}} K^i_{jkl} = \delta^i_j (\Psi_{l|k} - \Psi_{k|l}) + \delta^i_l \Psi_{j|k} - \delta^i_k \Psi_{j|l} + y^i (\Psi_{l|k} - \Psi_{k|l})_{,j}.$$

$$\pounds_{\hat{X}} K_{jl} = \Psi_{l|j} - n\Psi_{j|l} + \Psi_{l,j|0}.$$

Proof. Let $(M, F)$ be a non-Riemannian Finsler manifold. By a vertical derivative of (2) we have the first assertion (3). Again, a vertical derivative of (3) leads to the second assertion (4). Another vertical derivative of (4) yields the third assertion (5). Respectively, we can see another equation. □

Theorem 3.4. [7] *Let $(M, F = \alpha + \beta)$ be an $n$-dimensional Randers space and $V$ be a special projective vector field then $F$ contain isotropic $S$-curvature or $V$ is conformal vector field on $(M, h)$.*

Theorem 3.5. [7] *Let $(M, F = \alpha + \beta)$ be an $n$-dimensional Randers space. If $s^i_j \neq 0$, then $V$ is $F$-projective vector field if and only if it is a $\alpha$-homothety and $L_{\hat{V}} d\beta = \mu d\beta$ and $L_{\hat{V}} s_{ij} = \mu s_{ij}$.*

Theorem 3.6. *Let $(M, F)$ be a randers space with non-isotropic $S$-curvature, $s_{ij} = 0$ and $V$ is affine vector field then the relation projective transformation for it is following*

$$\nabla_0 L_{\hat{V}} \beta = -2\sigma e_{00}.$$

Proof. Let $s_{ij} = 0$. Therefore

$$G^i = G^i_\alpha + \frac{e_{00}}{2F} y^i,$$

From Theorem 3.4 we have $L_{\hat{V}} h^2 = 2\sigma h^2$. Since $V$ is an affine hence

$$L_{\hat{V}} G^i = 0 \Rightarrow L_{\hat{V}}(G^i_\alpha + \frac{e_{00}}{2F} y^i) = 0 \Rightarrow \eta y^i + L_{\hat{V}}(\frac{e_{00}}{2F}) y^i = 0 \Rightarrow L_{\hat{V}}(\frac{e_{00}}{2F}) + \eta = 0$$
$$\Rightarrow L_{\hat{V}}(\frac{e_{00}}{2F}) + \eta = 0,$$

by derivative from last equation we have

$$\frac{(L_{\hat{V}} e_{00})F - e_{00} L_{\hat{V}} F}{2F^2} + \eta = 0 \Rightarrow \frac{L_{\hat{V}} e_{00}(\alpha + \beta) - e_{00} L_{\hat{V}}(\alpha + \beta)}{2F^2} + \eta = 0,$$
$$\Rightarrow \frac{\alpha L_{\hat{V}} e_{00} + \beta L_{\hat{V}} e_{00} - e_{00} L_{\hat{V}} \alpha - e_{00} L_{\hat{V}} \beta}{2F^2} + \eta = 0.$$

By multiply both of side above equation in $2F^2$ we get

$$\alpha L_{\hat{V}}e_{00} + \beta L_{\hat{V}}e_{00} - e_{00}\frac{t_{00}}{2\alpha} - e_{00}L_{\hat{V}}\beta + \eta 2F^2 = 0.$$

By multiply both of side above equation in $2\alpha$ we get

$$2\alpha^2 L_{\hat{V}}e_{00} + 2\alpha\beta L_{\hat{V}}e_{00} - e_{00}t_{00} - 2\alpha e_{00}L_{\hat{V}}\beta + 4\eta\alpha F^2 = 0.$$

Therefore $\alpha(2(\alpha^2+\beta^2)\eta + \beta L_{\hat{V}}e_{00} - 2e_{00}L_{\hat{V}}\beta) + (4\alpha^2\beta\eta + 2\alpha^2 L_{\hat{V}}e_{00} - e_{00}t_{00}) = 0$. Hence

$$\begin{aligned} &Irrat\{2(\alpha+\beta)\eta + \beta L_{\hat{V}}e_{00} - 2e_{00}L_{\hat{V}}\beta = 0\},\\ &Rat\{4\alpha^2\beta\eta + 2\alpha^2 L_{\hat{V}}e_{00} - e_{00}t_{00}) = 0\}. \end{aligned}$$

$$\begin{aligned} L_{\hat{V}}e_{00} = L_{\hat{V}}\nabla_i b_j = L_{\hat{V}}(\partial_i b_j - b_r\Gamma^r_{ij}) &= \partial_i L_{\hat{V}}b_j - \Gamma^r_{ij}L_{\hat{V}}b_r - b_r L_{\hat{V}}\Gamma^r_{ij}\\ &= \nabla_i L_{\hat{V}}b_j - \eta_i b_j - \eta_j b_i. \end{aligned}$$

Then

$$L_{\hat{V}}e_{00} = \nabla_0 L_{\hat{V}}\beta - 2\eta\beta,$$

know we can get the proof of theorem. □

Using Theorem 3.4 and Theorem 3.5 we can see the proof of Theorem 2.2.

## References

1. H. Akbar-Zadeh, *Initiation to Global Finslerian Geometry*, North-Holland Mathematical Library, Vol. 68, Elsevier Science B. V., Amsterdam, 2006.
2. B. Bidabad, *A remark on the Gauss-Bonnet theorem in Finsler geometry*, Proceedings. International Conference of Differential Geometry and Dynamical Systems (DGDS-2008). V-th International Colloquium of Mathematics in Engineering and Numerical Physics [Mathematics Sections] (MENP-5), 42–46, BSG Proc., 16, Geom. Balkan Press, Bucharest, 2009.
3. D. Bao and C. Robles, *On Randers metrics of constant curvature*, Rep. Math. Phys. **51** (2003) 9–42.
4. G. W. Gibbons, C. A. R. Herdeiro, C. M. Warnick and M. C. Werner *Stationary metrics and optical Zermelo Randers Finsler geometry*, Phys. Rev. D **79** (2009) 044022.
5. G. Randers, *On an asymmetric metric in the four-space of general relativity*, Phys. Rev. **59** (1941) 195–199.
6. C. Robles, *Einstein Metrics of Randers Type*, Ph.D. Thesis, University of British Columbia, Canada, 2003.
7. M. Rafie-Rad and B. Rezaei, *On the projective algebra of Randers metrics of constant flag curvature,* SIGMA **7** (2011) 085.
8. Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, Dordrecht, 2001.
9. H. Tian, *Projective vector fields on Finsler manifolds*, Appl. Math. J. Chinese Univ. Ser. B **29** (2) (2014) 217–229.
10. K. Yano, *The Theory of Lie Derivatives and its Applications*, North-Holland Publishing Co., Amsterdam; P. Noordhoff Ltd., Groningen; Interscience Publishers Inc., New York, 1957.

E-mail: behnaz.lajmiri@aut.ac.ir

# Some Anti-de Sitter Space in Different Dimensions and Coordinates

Samira Latifi*
Faculty of Mathematical Science, University of Mohaghegh Ardabili, Ardabil, Iran
and Nemat Abazari
Faculty of Mathematical Science, University of Mohaghegh Ardabili, Ardabil, Iran

Abstract. We want to introduce sphere, hyperboloid, de-sitter and especially anti-de sitter and obtain the coordinates of the anit-de sitter space in different coordinates and we will describe its features according to each coordinate.
**Keywords:** Anti-de sitter space, Differential equations, Hyperboloid, Sausage coordinate, Stereographic coordinate.
**AMS Mathematical Subject Classification [2010]:** 97Gxx, 97G20, 11F23.

## 1. Introduction

Anti-de sitter space is the maximally symmetric solution of Einstein's equations with an attractive cosmological constant included (Anti-de sitter [2]).

The Anti-de sitter space has negative curvature $R < 0$, with a negatic cosmological constant it solves Einstein's equations

$$R_{\alpha\beta} = \lambda_{g_{\alpha\beta}}.$$

The $n$-dimensional Anti-de sitter space which is represented by $Ads_n$ embedded in a $(n+1)$-dimensional flat space $R^{n-1,2}$ which the metric

$$ds^2 = dX_0^2 + dX_1^2 + \cdots + dX_{n-2}^2 - dX_{n-1}^2 - dX_n^2,$$

and $Ads_n$ is defined as hyperboloid as follows,

$$X_0^2 + X_1^2 + \cdots + X_{n-2}^2 - X_{n-1}^2 - X_n^2 = -r^2 \quad (r \in R^+).$$

To write this article, several sources have been studied and helped, the most important of which are sources [3, 4, 6].

## 2. De-Sitter and Anti-De Sitter Space

Anti-de sitter space be considered to belong to a wide class of homogeneous spaces that are defined as quadratic surfaces in flat vector spaces.

The $n$-dimensional sphere $S^n$ defined as

$$X_1^2 + \cdots + X_{n+1}^2 = r^2,$$

that embedded in an Euclidean $n+1$ dimensional space.

*Speaker

With a change of sign in the above phrase, we will have a hyperboloid of two sheets:

$$X_1^2 + \cdots + X_n^2 - U^2 = -1.$$

If we consider a one sheeted hyperboloid as follows:

$$X_1^2 + \cdots + X_n^2 - X_{n+1}^2 = 1,$$

that embedded in Minkowski space we obtain de sitter space $ds[n]$, which is a space with a lorentzian metric of constant curvature.

Now to obtain anti-de sitter space we change the sign in the de sitter space and we have:

$$X_1^2 + \cdots + X_{n-1}^2 - U^2 - V^2 = -1,$$

embedded in a flat $n+1$ dimensional space which its metric is as follows:

$$ds^2 = dX_1^2 + \cdots + dX_{n-1}^2 - dU^2 - dV^2.$$

According to these definitions, we can conclude that two dimensional anti-de sitter space is a one sheeted hyperboloid embedded in a three dimensional Minkowski space and also in two dimensional we can say that de sitter space and anti-de sitter space become one. Then, in general, we can define 4-dimensional anti-de sitter space as

$$Ads_4 : X_1^2 + X_2^2 + X_3^2 - U^2 - V^2 = -r^2 (r \in R^+),$$

here, value the cosmological constant is $\lambda = -3$.

## 3. Anti-de Sitter Space in Sausage Coordinate and Stereographic Coordinate

We let $U = cost$, $V = R \sin t$ $X_1^2 + X_2^2 + X_3^2 - U^2 - V^2 = -1$, and the quadratic is

$$X_1^2 + X_2^2 + X_3^2 - R^2 \cos^2 t - R^2 \sin^2 t = -1.$$

Then

$$X_1^2 + X_2^2 + X_3^2 - R^2 = -1,$$

and the metric becomes

$$\begin{aligned} ds^2 &= dX_1^2 + dX_2^2 + dX_3^2 - dU^2 - dV^2 \\ &= dX_1^2 + dX_2^2 + dX_3^2 - (dR^2 \cos^2 t + dR^2 \sin^2 t) - (R^2 \sin^2 t dt^2 + R^2 \cos^2 t dt^2). \end{aligned}$$

Thus

$$ds^2 = dX_1^2 + dX_2^2 + dX_3^2 - dR^2 - R^2 dt^2,$$

if let $t$ be constant, last sentence disappears which this equations hyperbolic three-space embedded in four dimensional Minkowski space.

Now, let

$$dX_1^2 + dX_2^2 + dX_3^2 - dR^2 = d\sigma^2.$$

Then $ds^2 = d\sigma^2 - R^2 dt^2$, where $R$ is some definite function of the intrinsic coordinates on hyperbolic three-space. It is a static metric because $R$ does not depend on $t$.

We have to introduce intrinsic coordinates on $H^3$ so that we can draw a picture. So we can use stereographic coordinates, for that purpose

$$X_1 = \frac{2\rho}{1-\rho^2}\sin\theta\cos\varphi, \quad X_2 = \frac{2\rho}{1-\rho^2}\sin\theta\cos\varphi, \quad X_3 = \frac{2\rho}{1-\rho^2}\cos\theta,$$

$$V = \frac{1+\rho^2}{1-\rho^2}\sin t, \quad U = \frac{1+\rho^2}{1-\rho^2}\cos t,$$

where $R = \frac{1+\rho^2}{1-\rho^2}$ and for $0 \leq \rho < 1$ the angular coordinates have their usual range.

Now, let $\theta = \frac{\pi}{2} \Rightarrow X_3 = 0$. then we have three dimensional anti-de sitter space sliced with hyperbolic planes.

That is $X_1^2 + X_2^2 - U^2 - V^2 = -1$, and for get metric, we have

$$ds^2 = dX_1^2 + dX_2^2 + dX_3^2 - dR^2 - R^2 dt^2,$$

$$\begin{aligned}
\left\langle \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \theta} \right\rangle &= \left\langle \left( \frac{2\rho}{1-\rho^2}\cos\theta\cdot\cos\varphi, \frac{2\rho}{1-\rho^2}\cos\theta\cdot\sin\varphi, -\frac{2\rho}{1-\rho^2}\sin\theta \right), \right. \\
&\quad \left. \left( \frac{2\rho}{1-\rho^2}\cos\theta\cos\varphi, \frac{2\rho}{1-\rho^2}\cos\theta\cdot\sin\varphi, -\frac{2\rho}{1-\rho^2}\sin\theta \right) \right\rangle \\
&= \frac{4\rho^2}{(1-\rho^2)^2}\cos^2\theta\cos^2\varphi + \frac{4\rho^2}{(1-\rho^2)^2}\cos^2\theta\sin^2\varphi + \frac{4\rho^2}{1-\rho^2}\sin^2\theta \\
&= \frac{4\rho^2}{(1-\rho^2)^2}, \qquad (1)
\end{aligned}$$

$$\begin{aligned}
\left\langle \frac{\partial f}{\partial \varphi}, \frac{\partial f}{\partial \varphi} \right\rangle &= \left\langle \left( -\frac{2\rho}{1-\rho^2}\sin\theta\cdot\sin\varphi, \frac{2\rho}{1-\rho^2}\sin\theta\cdot\cos\varphi, 0 \right), \right. \\
&\quad \left. \left( -\frac{2\rho}{1-\rho^2}\sin\theta\cdot\sin\varphi, \frac{2\rho}{1-\rho^2}\sin\theta\cdot\cos\varphi, 0 \right) \right\rangle \\
&= \frac{4}{(1-\rho^2)^2}\rho^2\sin^2\theta, \qquad (2)
\end{aligned}$$

$$\begin{aligned}
\left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \right\rangle &= \left\langle \left( 0,0,0,-\left(\frac{1+\rho^2}{1-\rho^2}\right)^2 \sin t, \left(\frac{1+\rho^2}{1-\rho^2}\right)^2 \cos t \right), \right. \\
&\quad \left. \left( 0,0,0,-\left(\frac{1+\rho^2}{1-\rho^2}\right)^2 \sin t, \left(\frac{1+\rho^2}{1-\rho^2}\right)^2 \cos t \right) \right\rangle \\
&= \left(\frac{1+\rho^2}{1-\rho^2}\right)^2 \sin^2 t + \left(\frac{1+\rho^2}{1-\rho^2}\right)^2 \cos^2 t = \left(\frac{1+\rho^2}{1-\rho^2}\right)^2, \qquad (3)
\end{aligned}$$

$$\begin{aligned}
\left\langle \frac{\partial f}{\partial \rho}, \frac{\partial f}{\partial \rho} \right\rangle &= \Bigg\langle \Bigg( \frac{2(\rho^2+1)}{(1-\rho^2)^2}\sin\theta\cdot\cos\varphi, \frac{2(\rho^2+1)}{(1-\rho^2)^2}\sin\theta\sin\varphi, \\
&\quad \frac{2(\rho^2+1)}{(1-\rho^2)^2}\cos\theta, \frac{4\rho}{(1-\rho^2)^2}\cos t, \frac{4\rho}{(1-\rho^2)^2}\sin t \Bigg), \\
&\quad \frac{2(\rho^2+1)}{(1-\rho^2)^2}\cos\theta, \frac{4\rho}{(1-\rho^2)^2}\cos t, \frac{4\rho}{(1-\rho^2)^2}\sin t \Bigg) \Bigg\rangle \\
&= \frac{4(1+\rho^2)^2}{(1-\rho^2)^4} - \frac{16\rho^2}{(1-\rho^2)^4} \\
&= \frac{4\rho^4+8\rho^2+4-16\rho^2}{(1-\rho^2)^4} \\
&= \frac{4(1-\rho^2)^2}{(1-\rho^2)^4} \\
&= \frac{4}{(1-\rho^2)^2}. \qquad (4)
\end{aligned}$$

So using relationships (1), (2), (3) and (4) anti-de sitter metric in sausage coordinate becomes:

$$ds^2 = \frac{4\rho^2}{(1-\rho^2)^2}d\theta^2 + \frac{4\rho^2}{(1-\rho^2)^2}\sin^2\theta d\varphi^2 - \left(\frac{1+\rho^2}{1-\rho^2}\right)^2 dt^2 + \frac{4}{(1-\rho^2)^2}d\rho^2,$$

then

$$ds^2 = \frac{4}{(1-\rho^2)^2}(\rho^2 d\theta^2 + \rho^2\sin^2\theta d\varphi^2 + d\rho^2) - \left(\frac{1+\rho^2}{1-\rho^2}\right)^2 dt^2,$$

by considering these coordinates and writing differential equations and Euler equations, we can obtain elastics and geodesics and more.

The classical curve known as the elastica is the solution to a variational problem proposed by Daniel Bernoulli to Leonhard Euler in 1744 that of minimizing the bending energy of a thin inextensible wire [**5, 7**].

A geodesic on the surface is an embedded simple curve on the surface such that for any two points on the curve the portion of the curve connecting them is also the shortest path between them on the surface [**1**].

## References


1. P. N. Azariadis and N. A. Aspragathos, *Geodesic curvature preservation in surface flattening through constrained global optimization*, Comput-Aided Des. **33** (8) (2001) 581–591.
2. I. Bengtsson, *Anti-de sitter space*, Lecture Notes (1998). `http://3dhouse.se/ingemar/relteori/Kurs.pdf`
3. M. P. Do Carmo, *Differential Geometry of Curves and Surface*, Prentice-Hall, Englewood Cliffs, New Jersey, 1976.
4. W. kühnel, *Differential Geometry Curves-Surfaces-Manifolds*, American Mathematical Society, Providence, RI, 2002.
5. A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, Cambridge University Press, Cambridge, England, 1927.
6. B. O'Neil, *Elementary Differential Geometry*, Academic Press, Orlando, Florida, 1966.
7. D. A. Singer, *Lectures on Elastic Curves and Rods*, AIP Conf. Proc. **1002** (1) (2008). `DOI: 10.1063/1.2918095`

E-mail: samira_lti@yahoo.com
E-mail: abazari@uma.ac.ir

# On Some Questions Concerning Rings of Continuous Ordered-Field Valued Functions

Sudip Kumar Acharyya
Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kolkata, India
and Mehdi Parsinia*
Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran

Abstract. In this paper, we investigate answers to some questions in the context of rings of ordered field-valued continuous functions raised by Acharyya et al. in [*A Generic method to construct P-spaces through ordered fields,* Southeast Asian Bull. Math. **28** (2004) 783–790] and [*Structure spaces for intermediate rings of ordered field continuous functions,*Topology Proc. **47** (2015) 163–176].
**Keywords:** Zero-dimensional space, $P$-Space, $P_F$-Space, Almost $P$-space, Almost $P_F$-space.
**AMS Mathematical Subject Classification [2010]:** 54C30, 46E25.

## 1. Introduction

For a given topological space $X$, which is assumed to be completely regular and Hausdorff throughout this paper, and a totally ordered field $F$ equipped with the order topology induced by its order, $C(X,F)$ denotes the collection of all $F$-valued continuous functions on $X$ and $\mathcal{B}(X,F)$ (resp., $C^*(X,F)$) denotes the subcollection of $C(X,F)$ consisting of all bounded elements (resp., all elements such that $\mathrm{cl}_F f(X)$ is compact in $F$). It is easy to observe that $C(X,F)$ together with pointwise-defined operations of addition and multiplication is a commutative lattice ordered unitary ring and $B(X,F)$ and $C^*(X,F)$, with the inherited operations, are subrings and sublattices of $C(X,F)$. Also, $C^*(X,F) \supseteq B(X,F) \supseteq C(X,F)$. In the case $F=\mathbb{R}$, $C(X,F)$ is simply denoted by $C(X)$ and it is manifest that in this case $B(X,F)=C^*(X,F)$ which is denoted $C^*(X)$. However, the same equality does not hold, in general, for the case $F \neq \mathbb{R}$. For example, let $X=F=\mathbb{Q}$ with the usual metric and $i : X \to F$ be the identity mapping which is clearly continuous. Then, $g = f \wedge 1$ is a member of $B(X,F)$, however, $\mathrm{cl}_F f(X) = [0,1] \cap F$ is not compact in $F$ and thus $g \notin C^*(X,F)$. The classical theory of rings of continuous real-valued functions, as is well-known, have been extensively studied from the past to present. The reader is referred to [6] for notations and fundamental terminologies concerning rings of continuous real-valued functions. The author is referred to [10] to see a comprehensive survey of rings of continuous functions with values in topological rings other than $\mathbb{R}$.

*Speaker

Let us remember some notations and terminologies concerning $C(X,F)$ and $C(X)$ which are used throughout the paper. For each $f \in C(X,F)$, the $F$-zeroset of $f$; $\{x \in X : f(x) = 0\}$ is denoted by $Z_{X,F}(f)$, and the $F$-cozeroset of $X$ is the complement of $Z_{X,F}(f)$ with respect $X$ which is denoted by $Coz_{X,F}(f)$. The collection of all $F$-zerosets and $F$-cozero-sets of elements of $C(X,F)$ are denoted by $Z(C(X,F))$ and $Coz(C(X,F))$, respectively; for sake of brevity, $Z_{\mathbb{R}}(f)$ (resp., $Coz_{\mathbb{R}}(f)$) is denoted by $Z(f)$ (resp., $Coz(f)$) for each $f \in C(X)$ and $Z(C(X))$ is denoted by $Z(X)$. A topological space $X$ is called completely $F$-regular if it is Hausdorff and, for each closed $G$ of $X$ not containing an element $x \in X$, there exists $f \in C(X,F)$ such that $f(x) = 0$ and $f(G) = \{1\}$; i.e., separates $G$ and $x$. Whenever $F$ is connected, then the completely $F$-regular spaces are exactly Tychonoff spaces and whenever $F$ is disconnected, then $X$ would be zero-dimensional (a Hausdorff space containing a base of clopen sets is called a zero-dimensional space). In fact, zero-dimensional spaces are exactly completely $F$-regular spaces for an arbitrary disconnected field $F$. Note that a topological field is either connected or totally disconnected; i.e., a topological space in which any subset containing more than one point is disconnected.

This paper aims to give answers to some questions in the context of $C(X,F)$ raised in [**2**, **3**]. The notion of $P_F$-spaces has been introduced in [**2**] as zero-dimensional spaces $X$ for which $C(X,F)$ is a (Von-Newman) regular ring. and the authors have asserted that they do not know whether this notion is identical with the notion of $P$-spaces or not. We give a wide class of $P_F$-spaces which are not $P$-spaces and $P$-spaces which are not $P_F$-spaces. These imply that these two notions are independent, in general. Moreover, the notion of almost $P_F$-spaces has been introduced in [**5**] as zero-dimensional spaces $X$ for which every nonempty $F$-zeroset has nonempty interior. We give a wide class of examples of almost $P_F$-spaces which are not almost $P$-spaces. Moreover, the authors of [**3**] have stated that they do not know whether the rings $C(X,F)$ and $C^*(X,F)$ generate the same family of zero-sets? The same question also has been raised in [**1**]. By giving an appropriate example, we show that the two mentioned rings do not necessarily generate the same family of zerosets.

## 2. New Results

In [**2**], the class of $P_F$-spaces is introduced as zero-dimensional spaces $X$ for which every prime ideal in the ring $C(X,F)$ is maximal; i.e., $C(X,F)$ is a regular ring. Various characterizations of $P_F$-spaces are given in Theorem 3.2 of the same paper and in the comments after this theorem, the authors have asserted that they do not know, in general, whether the properties of being a $P$-space and a $P_F$-space with $F \neq R$ are independent. However, in Theorem 3.4 and Theorem 3.5 of the same paper, they have shown that whenever $F$ is a Cauchy complete ordered field with $cf(F) = \omega_0$, then $P$-spaces and $P_F$-spaces coincide. It easily follows from [**10**, Theorem 12.3] that whenever $F$ is a subfield of $\mathbb{R}$, then $P_F$-spaces and $P$-spaces coincide. We will show that these two notions are independent by giving examples of $P_F$-spaces which are not $P$-spaces and vice-versa. We also show that these two notions coincide for some classes of non-complete ordered fields and thus [**2**, Theorem 3.8] is incorrect. Note that a subset $Q$ of a partially ordered set $(P, \leq)$ is said to be cofinal in $P$ if for every $x \in P$, there exists some $y \in Q$ with $x \leq y$. The least cardinality of a cofinal subset of $P$, denoted by $cf(P)$, is called the cofinality character of $P$.

We need the following statement which follows from [7, Proposition 2.2], [9, Theorem 2] and the fact that totally ordered fields contain no isolated points.

PROPOSITION 2.1. *The following statements are equivalent for a totally ordered field $F$.*

a) *$F$ is metrizable.*
b) *$F$ containing a countable set having no upper (or no lower) bounds.*
c) *$F$ is not a $P$-space.*
d) *$F$ is a first countable space.*
e) *$F$ contains a non-discrete countable subspace.*

Equivalence of parts (a) and (b) of Proposition 2.1 implies that an ordered field $F$ is metrizable if and only if $cf(F) = \omega_0$. It should be emphasized that metrizable spaces are not necessarily Archimedean, however, by Proposition 2.1, every Archimedean ordered field is metrizable, see [2, Theorem 3.9]. Hence, whenever $F$ is a subfield of $\mathbb{R}$ or has a countable cofinality character, then the two notions of $P_F$-spaces and $P$-spaces coincide for any zero-dimensional space $X$.

Note that from Proposition 2.1 and [6, 13P], it follows that whenever $X$ is a non-pseudocompact Tychonoff space and $K = \frac{C(X)}{M}$ is the ordered field of residue class field of a maximal ideal $M$ of $C(X)$ (See [6, 5.4(c)]), then $K$ is metrizable if and only if $K$ is isomorphic to $\mathbb{R}$ (i.e., $M$ is a real-maximal ideal of $C(X)$). Thus, whenever $M$ is a hyper-real maximal ideal, then $K$ would be a $P$-space and hence not metrizable.

REMARK 2.2. We can infer from the facts mentioned above that [2, Theorem 3.8] is incorrect. Indeed, $\mathbb{Q}$ is an ordered filed with countable cofinality character which clearly is not Cauchy complete, however, by [10, Theorem 12.3], for any zero-dimensional space $X$, $X$ is a $P$-space if every prime ideal of $C(X, \mathbb{Q})$ is maximal; i.e., $X$ is a $P_{\mathbb{Q}}$-space.

The next statement investigates a large class of $P_F$-spaces which are not $P$-spaces, namely, compact spaces. We remind that a topological space $X$ is said to be strongly zero-dimensional if disjoint zerosets in $X$ are separated by disjoint clopen sets. or equivalently, $\beta X$ is zero-dimensional.

THEOREM 2.3. *Let $X$ be a non-compact strongly zero-dimensional infinite space. Then $\beta X$ is a $P_F$-space which is not a $P$-space for each non-metrizable totally ordered field $F$.*

PROOF. By Proposition 2.1, $F$ is a $P$-space. Thus, for each $f \in C(\beta X, F)$, $f(X)$ should be finite. Let $f(\beta X) = \{a_1, \dots, a_n\}$. It follows that $\beta X = f^{-1}(a_1) \cup \dots \cup f^{-1}(a_n)$ which implies that $f^{-1}(a_i)$ is a clopen subset of $\beta X$ for each $1 \leq i \leq n$. Hence, $Z(f)$ would be empty or a clopen subset of $\beta X$. Therefore, $\beta X$ is a $P_F$-space. □

It follows that $\beta X$ for each infinite discrete space $X$, and $\beta F$ are examples of $P_F$-spaces which are not $P$-spaces whenever $F$ is a non-metrizable ordered field. Moreover, for examples of $P$-spaces which are not $P_F$-spaces, we can easily observe that every non-metrizable ordered field $F$ is a $P$-space which is not a $P_F$-space. Thus, the two notions of $P_F$-spaces and $P$-spaces are independent.

Remark 2.4. By using Theorem 2.3, we could easily observe that some basic properties of $P$-spaces fail to hold for $P_F$-spaces which are not $P$-spaces. For example,

a) every subspace of a $P_F$-space need not be a $P_F$-space; whenever $F$ is a non-metrizable ordered field, $\beta F$ is a $P_F$-space, however, $F$ is not a $P_F$-space,
b) every countable subspace of $P_F$-space need not be $C$-embedded; for each non-metrizable ordered field $F$, $\beta\mathbb{N}$ is a $P_F$-space containis the countable set $\mathbb{N}$ and $\mathbb{N}$ is not $C$-embedded in $\beta\mathbb{N}$,
c) a countable subspace of a $P_F$-space is not necessarily discrete; $\beta\mathbb{Q}$ is a $P_F$-space, for each non-metrizable ordered field $F$ containing the countable set $\mathbb{Q}$ which is not a discrete subspace of $\beta\mathbb{Q}$.

Remember that a Tychonoff space $X$ is called an almost $P$-space, if every nonemepty zero-set of elements of $C(X)$ has a nonempty interior, see [**4**] for more details. In [**5**], the notion of almost $P_F$-spaces has been introduced as a generalization of the notion of almost $P$-spaces via an ordered field $F$ as follows: a zero-dimensional space $X$ is called an almost $P_F$-space if each non-empty zero set in $X$ of $F$-valued continuous functions has non-empty interior. In the same paper, the authors have stated that they do not aware of any totally ordered field $F$ and a suitable zero-dimensional space $X$ such that $X$ is almost $P_F$-space without being an almost $P$-space, however, in Theorem 3.11, they showed that for the class of Cauchy complete ordered fields with countable cofinality character, the two notions of almost $P_F$-space and almost $P$-space coincide. By using Theorem 2.3, we now investigate a large class of almost $P_F$-spaces which are not almost $P$-space.

Example 2.5. Let $X$ be an infinite discrete space and $F$ be a non-metrizable ordred field. As shown in Theorem 2.3, $\beta X$ is a $P_F$-space and hence is an almost $P_F$-space. Moreover, it is easy to see that $\beta X$ is not an almost $P$-space.

Remark 2.6. Let $X$ be a zero-dimensional space and $F$ be a proper subfield of $\mathbb{R}$. It is easy to see that every $Z \in Z(X)$ contains a countable intersection of clopen sets in $X$ and thus, by [**10**, Property 1.1], contains an element of $Z(C(X, F))$. It follows from this fact and [**2**, Theorem 2.1] that $X$ is an almost $P$-space if and only if it is an almost $P_F$-space. Therefore Cauchy completeness of the ordered field $F$, in general is not necessary for the two notions of almost $P$-spaces and almost $P_F$-spaces to become identical, compare with [**5**, Theorem 3.11].

In [**3**], it is stated that, for any completely $F$-regular space $X$, the structure space of any intermediate ring between $C(X, F)$ and $B(X, F)$ has an identical structure space with $C(X, F)$. However, the auothors have asserted that the structure space of intermediate rings between $C(X, F)$ and $C^*(X, F)$ may not be identical. Also, they said that it is not known to them, whether in general $C^*(X, F)$ and $C(X, F)$ produce the same family of zero-sets. The following example shows that $C(X, F)$ and $C^*(X, F)$ do not produce the same family of zero-sets.

Example 2.7. Let $F$ be a non-metrizable ordered field (for example a hyperreal field). Then, as $F$ is a $P$-space, each element of $C^*(F, F)$ has a finite image and thus, $Z(f)$, for each $f \in C^*(F,F)$, is an open subset of $X$. But, the identity mapping $i : F \to F$ is clearly continuous and $Z(i) = \{0\}$ is not an open subset of $F$, since ordered fields have no isolated points. Hence, $Z(i) \notin Z(C^*(F,F))$ which implies that $Z(C^*(F,F))$ is not identical with $Z(C(F,F))$.


## Acknowledgment

The second author would like to express his deep gratitude to Dr. Malihe Charkhab for all of her valuable efforts and kind cooperations and this paper is dedicated to her.


## References


1. S. K. Acharyya, K. C. Chattopadhyaya and P. P. Ghosh, *Constructing the Banaschewski compactification without the Dedekind completeness axiom*, Int. J. Math. Sci. **69** (2004) 3799–3816.
2. S. K. Acharyya, K. C. Chattopadhyay and P. P. Ghosh, *A Generic method to construct P-spaces through ordered fields*, Southeast Asian Bull. Math. **28** (2004) 783–790.
3. S. K. Acharyya and P. Rooj, *Structure spaces for intermediate rings of ordered field continuous functions*, Top. Proc. **47** (2015) 163–176.
4. F. Azarpanah, *On almost P-spaces,* Far East J. Math. Sci. Special Vol. I (2000) 121–132.
5. S. Bag, S. K. Acharyya and D. Mandal, *$z^\circ$-ideals in intermediate rings of ordered field valued continuous functions,* Questions Answers Gen. Top. **38** (1) (2020) 43–52.
6. L. Gillman and M. Jerison, *Rings of Continuous Functions*, Springer, New York, 1960.
7. C. Liu and Y. Tanaka, *Metrizability of ordered additive groups,* Tsukuba J. Math. **35** (2) (2011) 169–183.
8. Y. Tanaka, *Topology on ordered fields,* Comment. Math. Univ. Carolin. **53** (1) (2012) 139–147.
9. Y. Tanaka, *Ordered fields and metrizability*, Bull. Tokyo Gakugei Univ. Nat. Sci. **61** (2009) 1–9.
10. E. M. Vechtomov, *Rings of continuous functions with values in a topological division ring,* J. Math. Sci. **78** (6) (1996) 702–753.



E-mail: sdpacharyya@gmail.com
E-mail: parsiniamehdi@gmail.com

# Building Different Types of Curves in a Specific Formula

Ali Asghar Rezaei*
Faculty of Mathematical Sciences, University of Kashan, Kashan, Iran

Abstract. In this paper, using two differential functions, we present a parametric formula for space curves so that the curvature and tursion of the curve can be expressed in terms of these two functions. We then obtain some conditions on the functions to characterize some families of curves, including planar curves, helices, and Bertrand curves.
**Keywords:** Space curve, Helix, Planar curves, Bertrand curves, Curvature.
**AMS Mathematical Subject Classification [2010]:** 53A04.

## 1. Introduction

According to the fundamental theorem of curves, the geometric shape of any regular curve with positive curvature is determined, up to position, by its curvature and torsion. More precisely, let $I$ be an interval on the real line, $\kappa > 0$ a $C^1$ function on $I$, and $\tau$ a continuous function on $I$. Then there exists a $C^3$ regular curve $\alpha : I \to \mathbb{R}^3$ such that the curvature and torsion of $\alpha$ are equal to $\kappa$ and $\tau$ respectively. Theoretically, this is a deep result. But in practice sometimes finding such a curve requires solving nonlinear differential equations that may not have a preliminary solution. However, finding a way to generate a particular family of curves is theoretically and practically useful.

Some types of curves such as helices and Bertrand curves have been widely studied for long times [1, 2], and some new kinds such as slant helices and Mannheim curves have been studied in recent decades [3, 4].

Here is a brief overview of these curves (we recall that for the curve $\alpha(s)$, the Frenet-Serret apparatus will be denoted by $\{T(s), N(s), B(s), \kappa(s), \tau(s)\}$ as usual):

A *helix* is a curve $\alpha$ such that for some fixed unit vector $U$, $\langle T(s), U\rangle$ is constant. An important characterization for helices due to Lancret [5] asserts that a unit speed curve $\alpha$ with $\kappa \neq 0$ is a helix if and only if $\tau/\kappa$ is constant. Note that $\kappa$ and $\tau$ need not be constants. The case for which $\kappa$ and $\tau$ are constants the helix is called *circular helix*.

A *slant helix* is a curve $\alpha$ such that for some fixed unit vector $U$, $\langle N(s), U\rangle$ is constant. In [3] the authors showed that $\alpha$ is a slant helix if and only if the function

$$\sigma(s) = \left(\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'\right)(s), \tag{1}$$

is constant. Obviously, every helix is a slant helix, but the converse is not true.

*Speaker

A curve $\alpha$ is called a *Bertrand curve*[1] if there exist a curve $\beta \neq \alpha$ such that for each $s_0$, the normal line to $\alpha$ at $s_0$ is the same as the normal line to $\beta$ at $s_0$($\alpha$ and $\beta$ need not be unit speed). The well-known characterization for Bertrand curves asserts that a unit speed curve $\alpha$ with $\kappa\tau \neq 0$ is a Bertrand curve if and only if there are constants $a \neq 0$ and $b$ with

$$1 = a\kappa + b\tau. \tag{2}$$

Finally, we define our last type [4]: A curve $\alpha$ is called a *Mannheim curve* if there exists a curve $\beta$ such that at the corresponding points of the curves, the principal normal lines of $\alpha$ coincide with the binormal lines of $\beta$. It is just known that a curve $\alpha$ is a Mannheim curve if and only if

$$\kappa = \lambda(\kappa^2 + \tau^2),$$

for some constant $\lambda \neq 0$.

The organization of the paper is as follow: First we use two functions to obtain a parametric formula for curves such that the curvature and torsion can be expressed in term of that functions. We then derive some results concerning to the functions and give some necessary and sufficient conditions under which the mentioned formula generates a type of curves.

## 2. Main Results

To make the desired curves, we use the following lemma. Throughout this section, $I$ is an interval about zero.

LEMMA 2.1. *Let $f > 0$ and $g$ are real valued differential functions on $I$, and $\alpha : I \to \mathbb{R}^3$ be defined as*

$$\alpha(t) = \left( \int_0^t f(u)\sin u du, \int_0^t f(u)\cos u du, \int_0^t f(u)g(u) \right). \tag{3}$$

*Then the curvature and tursion of $\alpha$ are*

$$\kappa = \frac{1}{f}\sqrt{\frac{1+g^2+g'^2}{(1+g^2)^3}}, \qquad \tau = -\frac{1}{f}\frac{g+g''}{(1+g^2+g'^2)}. \tag{4}$$

PROOF. According to the fundamental theorem of calculus, we have

$$\alpha'(t) = (f(t)\sin t, f(t)\cos t, f(t)g(t)) = f(t)\beta(t),$$

where $\beta(t) = (\sin t, \cos t, g(t))$. So

$$\alpha''(t) = f'(t)\beta(t) + f(t)\beta'(t),$$
$$\alpha'''(t) = f''(t)\beta(t) + 2f'(t)\beta'(t) + f(t)\beta''(t).$$

Replacing $\beta'(t) = (\cos t, -\sin t, g'(t))$, $\beta''(t) = (-\sin t, -\cos t, g''(t)$, and using the formulas

$$\kappa = \frac{|\alpha' \times \alpha''|}{|\alpha|^3}, \qquad \tau = \frac{(\alpha' \times \alpha'').\alpha'''}{|\alpha' \times \alpha''|^2},$$

we get Eq. (4). □

The first conclusion is about planar curves.

PROPOSITION 2.2. *Every planar curve with positive curvature, is congruent to a curve such as (3) where $f$ is a positive $C^2$ function and $g(t) = a\sin t + b\cos t$, for come $a, b \in \mathbb{R}$.*

PROOF. The curve $\alpha$ is planar if and only if $\tau \equiv 0$, and from Eq. (4) this happens if and only if $g + g'' = 0$. Solving this differential equation we get $g(t) = a\sin t + b\cos t$ for some $a, b \in \mathbb{R}$.

Now let $\beta$ be an arbitrary planar curve with curvature $\kappa > 0$. In (3) take $g(t) := a\sin t + b\cos t$ and

$$f := \frac{1}{\kappa}\sqrt{\frac{1 + g^2 + g'^2}{(1 + g^2)^3}}.$$

Then $\kappa_\alpha = \kappa = \kappa_\beta$ and $\tau_\alpha = 0 = \tau_\beta$. So $\alpha$ and $\beta$ are congruent according to the fundamental theorem of corves. □

COROLLARY 2.3. *Every circle of radius $R$ is congruent to a curve such as (3), where $g(t) = a\sin t + b\cos t$ and $f := R\sqrt{\frac{1+a^2+b^2}{(1+g^2)^3}}$.*

PROOF. The circle is a planar curve with curvature $\kappa = 1/R$. So the previous proposition holds (note: $1 + g^2 + g'^2 = 1 + a^2 + b^2$). □

PROPOSITION 2.4. *Let $f, g : I \to \mathbb{R}$ are $C^2$ functions which $f$ is positive and $g$ satisfies the equation*

$$g + g'' + c\left(\frac{a + g^2 + g'^2}{1 + g^2}\right)^{3/2} = 0, \tag{5}$$

*for some $c \in \mathbb{R}$. Then the curve $\alpha$ in (3) is a helix.*

PROOF. We know that $\alpha$ is a helix if and only if $\tau/\kappa = c$ for some $c \in \mathbb{R}$. Replacing the $\kappa$ and $\tau$ in Eq. (4) we get to the desire Eq. (5). □

The next three propositions describe how other families of curves are constructed in the format (3).

PROPOSITION 2.5. *Let $g : I \to \mathbb{R}$ be a $C^4$ function and $f : I \to \mathbb{R}$ be defined as*

$$f := a\sqrt{\frac{1 + g^2 + g'^2}{(1 + g^2)^3}} - \frac{b(g + g'')}{(1 + g^2 + g'^2)}, \tag{6}$$

*for some real numbers $a \neq 0$ and $b$. Then the curve $\alpha$ in (3) is a Bertrand curve.*

PROOF. The curve $\alpha$ is a Bertrand curve if and only if $a\kappa + b\tau = 1$ for some real numbers $a \neq 0, b$. Replacing the $\kappa$ and $\tau$ in Eq. (4) we have

$$1 := \frac{a}{f}\sqrt{\frac{1 + g^2 + g'^2}{(1 + g^2)^3}} - \frac{b}{f}\frac{g + g''}{(1 + g^2 + g'^2)},$$

and one can write this equation as (6). □

Proposition 2.6. *Let $g : I \to \mathbb{R}$ be a $C^4$ function and $f : I \to \mathbb{R}$ be defined as*

$$f := \lambda \left( \sqrt{\frac{1+g^2+g'^2}{(1+g^2)^3}} - \frac{(g+g'')^2(1+g^2)^{3/2})}{(1+g^2+g'^2)^{3/2}} \right), \tag{7}$$

*for some real numbers $\lambda \neq 0$. Then the curve $\alpha$ in (3) is a Mannheim curve.*

Proof. The necessary and sufficient condition for $\alpha$ to be a Mannheim curve is that the equation $\kappa = \lambda(\kappa^2 + \tau^2)$ holds for some real numbers $\lambda \neq 0$. Replacing the $\kappa$ and $\tau$ in Eq. (4) we will have the desire Eq. (7). □

Proposition 2.7. *Let $g : I \to \mathbb{R}$ be a non-constant $C^4$ function such that $g + g'' \neq 0$ and $f : I \to \mathbb{R}$ be defined as*

$$f := \frac{\left((1+g^2+g'^2)^3 + (g+g'')^2(1+g^2)^3\right)^{3/2}}{\lambda(1+g^2)^{3/2}(1+g^2+g'^2)^4 \left(\frac{(g+g'')(1+g^2)^{3/2}}{(1+g^2+g'^2)^{3/2}}\right)'},$$

*for some real numbers $\lambda \neq 0$. Then the curve $\alpha$ in (3) is a slant helix.*

Proof. First note that $g' \neq 0$, since $g$ is not constant. We also have $g+g'' \neq 0$, so the denominator never vanishes and $f$ is well defined. Now $\alpha$ is a slant helix if and only if in (1) we have $\sigma = \lambda$ some real numbers $\lambda \neq 0$. Replacing the $\kappa$ and $\tau$ in Eq. (4) we will have the desire Eq. (2). □

Here we give an example to illustrate the above propositions.

Example 2.8. Let $g(t) = t$ for $t \in \mathbb{R}$. To generate a Bertrand curve we mast take $f$ as

$$f(t) := a\sqrt{\frac{1+g^2+g'^2}{(1+g^2)^3}} - \frac{b(g+g'')}{(1+g^2+g'^2)} = a\sqrt{\frac{2+t^2}{(1+t^2)^3}} - \frac{bt}{(2+t^2)}.$$

Similarly to generate a Mannheim curve the function $f$ will be as

$$f(t) := \lambda \left( \sqrt{\frac{2+t^2}{(1+t^2)^3}} - \frac{t^2(1+t^2)^{3/2})}{(2+t^2)^{3/2}} \right),$$

and by taking

$$f(t) := \frac{\left((2+t^2)^3 + t^2(1+t^2)^3\right)^{3/2}}{\lambda(1+t^2)^{3/2}(2+t^2)^4 \left(\frac{t(1+t^2)^{3/2}}{(2+t^2)^{3/2}}\right)'},$$

we will have a slant helix.


## Acknowledgement

The author is grateful to the University of Kashan for supporting this work by Grant No. 985987.

## References

1. J. Bertrand, *Memoire sur la theorie des courbes a double courbure*, J. Math. Pures Appl. **82** (1850) 332–350.
2. U. Ciftci, *A generalization of Lancert's theorem*, J. Geom. Phys. **59** (2009) 1597–1603.
3. S. Izumiya and N. Tkeuchi, *New special curves and developable surfaces*, Turk. J. Math. **28** (2004) 153–163.
4. H. L. Liu and F. Wang, *Mannheim partner curves in* 3*-space*, J. Geom. **88** (2008) 120–126.
5. M. A. Lancret, *Mémoire sur les courbes à double courbure*, Mémoires présentés a l'Institut **1** (1806) 416–454.

E-mail: a_rezaei@kashanu.ac.ir

# Ricci Flow and Estimations for Derivatives of Cartan Curvature in Finsler Geometry

Mohammad Yar Ahmadi*
Department of Mathematics, Faculty of Mathematical Sciences and Computer, Shahid Chamran University of Ahvaz, Ahvaz, Iran

ABSTRACT. Here, we first derive evolution equation for the hh-curvature tensor of Cartan connection. Then we establish estimates for the covariant derivatives of the Cartan curvature tensor. It is proved the long time existence theorem for the Finsler Ricci flow as long as its hh-curvature remains bounded.
**Keywords:** Finsler geometry, Ricci flow, Cartan curvature.
**AMS Mathematical Subject Classification [2010]:** 53C60, 53C44.

## 1. Introduction

Hamilton in 1982 introduced the concept of Ricci flow which was subject of numerous progress in Riemannian geometry. In fact the length of a path on Riemannian geometry as well as on Finslerian geometry is computed, in terms of the metric tensor $g_{ij}$, by the integral $\int_\gamma \sqrt{g_{ij}dx^i dx^j}$. The Ricci and scalar curvatures in both geometries are defined by first and second derivatives of the metric tensor $g_{ij}$. Hamilton considered the relationship between time variation of metric tensor and Ricci tensor $\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$, for which the solutions are known as Ricci flow in Riemannian geometry. More intuitively, the metric is required to change with time so that distances decrease in directions of positive curvature. The Ricci flow equation is essentially a parabolic differential equation and behaves much like the heat equation studied by physicists, that is, if one heats one end of a cold metallic bar, then the heat will progressively flow throughout the bar until the other end attains a same temperature. Likewise, Poincarè make a conjecture and claims that, one may hope in certain 3-dimensional manifolds, under the Ricci flow, positive curvature would tend to spread out until, in the limit, the manifold would attain constant curvature. This conjecture is proved by Prelmann using Hamilton's Ricci flow.

The concept of Ricci flow on Finsler manifolds in analogy with the Ricci flow in Riemannian geometry is first defined by D. Bao [1]. The convergence of evolving Finslerian metrics first in a general flow and so under Finsler Ricci flow is obtained. In fact, it has been shown that a family of Finslerian metrics $g(t)$ which are solutions to the Finsler Ricci flow converges in $C^\infty$ to a smooth limit Finslerian metric as $t$ approaches the finite time $T$, see [3]. Moreover, the existence and uniqueness of solution to the Finsler Ricci flow equation is also studied by Bidabad and Sedaghat, see [4]. Also, it is considered another significant Ricci flow for

*Speaker

Finsler n-manifolds and is obtained evolution of Cartan hh-curvature, as well as Ricci tensor and scalar curvature, see [2].
In the present work, we establish estimates for the covariant derivatives of the Cartan curvature tensor. It is proved the long time existence theorem for the Finsler Ricci flow as long as its hh-curvature remains bounded.

## 2. Estimations for Derivatives of Cartan Curvature Tensor

We denote by $\nabla^m A$ and $D^m A$ the $m^{th}$ order iterated horizontal Cartan covariant derivative of the tensor $A$. Let $A$ and $B$ be two tensor fields defined on $\pi^*TM$. We denote by $A*B$ any linear combination of these tensors obtained by the tensor product $A\otimes B$ and any of the following operations;

I. summation over pairs of matching upper and lower indices,
II. contraction on upper indices with respect to the metric $g$,
III. contraction on lower indices with respect to the inverse metric of $g$.

Here, we establish estimates for covariant derivatives of the Cartan curvature tensor. Throughout this section, we suppose that $M$ is a compact manifold and $g(t)$, $t\in[0,\tau]$, is a solution to the Finslerian Ricci flow.

Lemma 2.1. *Suppose that $R(X,Y,W,V)=R(W,V,X,Y)$. Then*

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^m R =&\Delta^h\nabla^m R+\sum_{l=0}^{m}\nabla^l R*\nabla^{m-l}R+\sum_{l=0}^{m}\nabla^{m-l}R*\nabla^l P\\ &+\sum_{l=0}^{m}\nabla^{l+1}R*\nabla^{m-l}P+\sum_{l=0}^{m}\nabla^{m-l}R*\nabla^{l+1}P+1*\nabla^{m+1}R,\end{aligned}$$

*for $m=0,1,2,\ldots$.*

Now for a $(p,q)-$tensor field $\Omega$ define

$$|\Omega|_g^2:=\Omega^{j_1\ldots j_q}_{i_1\ldots i_p}\Omega^{i_1\ldots i_p}_{j_1\ldots j_q}=g^{j_1l_1}\cdots g^{j_ql_q}g_{i_1k_1}\cdots g_{i_pk_p}\Omega^{k_1\ldots k_p}_{l_1\ldots l_q}\Omega^{i_1\ldots i_p}_{j_1\ldots j_q}.$$

Proposition 2.2. *Let $(M,g(t))$, $t\in[0,\tau]$, be a solution to the compact Finsler Ricci flow, and $\sup_{SM}|R_{g(t)}|\leq\tau^{-1}$. Moreover, suppose that for each non-negative integer $m$, there exists a positive constant $C_m$ such that $\sup_{SM}|\nabla^m P_{g(t)}|^2\leq C_m$ for all $t\in[0,\tau]$. Then for any integer $m\geq1$, there exists a positive constant $B_m$, such that*

$$\sup_{SM}|\nabla^m R_{g(t)}|\leq B_m\tau^{-1}t^{-m},$$

*for all $t\in(0,\tau]$.*

Corollary 2.3. *Under the conditions of Proposition 2.2, there exists a positive constant $B$, such that*

$$\sup_{SM}|\nabla^m R_{g(t)}|\leq B\tau^{-1}\tau^{-m},$$

*for all $t\in[\frac{\tau}{2},\tau]$.*

Now, we suppose that $H$ is a smooth tensor field on $\pi^*TM$ which satisfies an evolution equation of the form

$$\frac{\partial}{\partial t}H = \Delta^h H + R * H. \tag{1}$$

In order to estimate the tensor $\nabla^m H$, we need the following lemma.

Lemma 2.4. *We have*

$$\frac{\partial}{\partial t}\nabla^m H = \Delta^h \nabla^m H + \sum_{l=0}^{m} \nabla^l R * \nabla^{m-l} H + \sum_{l=0}^{m} \nabla^l P * \nabla^{m-l} H, \qquad \forall m \geq 1.$$

Proposition 2.5. *Let $H$ be a smooth tensor field satisfying in evolution equation (1) and for all $t \in [0,\tau]$,*

$$\sup_{SM} |H| \leq \lambda, \quad \sup_{SM} |R_{g(t)}| \leq \tau^{-1},$$

*where $\lambda$ is a positive constant. Moreover, suppose that for each non-negative integer $m$, there exists $C_m$ such that*

$$\sup_{SM} |\nabla^m P_{g(t)}|^2 \leq C_m,$$

*for all $t \in [0,\tau]$. Given any integer $m \geq 1$, we can find a positive constant $B$ such that*

$$\sup_{SM} |\nabla^m H|^2 \leq B\lambda^2 t^{-m},$$

*for all $t \in (0,\tau]$.*

Corollary 2.6. *Under the conditions of Proposition 2.5, for any integer $m \geq 1$, we can find a positive constant $B$ such that*

$$\sup_{SM} |\nabla^m H|^2 \leq B\lambda^2 \tau^{-m},$$

*for all $t \in [\frac{\tau}{2},\tau]$.*

Here, assume that that $g(t)$ is a maximal solution to the Finslerian Ricci flow defined on a finite time interval $[0,T)$. We need the following Lemma.

Lemma 2.7. [4] *Let $M$ be a differentiable manifold. Given any initial Finsler structure $F_0$ with metric tensor $g_0$, there exists a real number $T$ and a smooth one-parameter family of Finsler structures $F(t)$, $t \in [0,T)$, with metric tensors $g(t)$, such that $F(t)$ is a solution to the Finsler Ricci flow and $F(0) = F_0$.*

Theorem 2.8. *Let $(M,g(t))$, $t \in [0,T)$ be a maximal solution to the compact Finsler Ricci flow and $T < \infty$. Moreover, suppose that $|\nabla^m P_{g(t)}| < \infty$, for $m \geq 0$ and $R(X,Y,V,W) = R(V,W,X,Y)$. Then*

$$\lim_{t \longrightarrow T} \sup \sup_{SM} |R_{g(t)}| = \infty.$$

Proof. Suppose this is false. Then the Cartan curvature tensor of $g(t)$ is uniformly bounded for all $t \in [0,T)$. Using Corollary 2.3, we obtain

$$\lim_{t \longrightarrow T} \sup \sup_{SM} |\nabla^m R_{g(t)}| < \infty,$$

for $m = 1, 2, \ldots$. For Simplicity, we write $\frac{\partial}{\partial t} g(t) = \omega(t)$, where $\omega(t) = -2Ric_{g(t)}$. Then

$$\limsup_{t \longrightarrow T} \sup_{SM} |\nabla^m \omega(t)|_{g(t)} < \infty,$$

for $m = 0, 1, 2, \ldots$. Therefore the Finslerian metrics $g(t)$ converge in $C^\infty$ to a limit Finslerian metric $\bar{g}$. Thus the Finsler structures $F(t)$ converge in $C^\infty$ to a limit Finsler structure $\bar{F}$ with metric tensor $\bar{g}$. By means of Lemma 2.7 we can extend the solution beyond $T$. This contradicts the maximality of $T$. □

## References


1. D. Bao, *On two curvature-driven problems in Riemann-Finsler geometry*, Adv. stud. pure Math. **48** (2007) 19–71.
2. B. Bidabad, A. Shahi and M. Yar Ahmadi, *Deformation of Cartan curvature on Finsler manifolds*, Bull. Korean Math. Soc. **54** (6) (2017) 2119–2139.
3. B. Bidabad and M. Yar Ahmadi, *Convergence of Finslerian metrics under Ricci flow*, Sci. China. Math **59** (4) (2016) 741–750.
4. B. Bidabad and M. K. Sedaghat, *Hamilton's ricci flow on finsler spaces*, (2015). arXiv:1508.02893



E-mail: m.yarahmadi@scu.ac.ir; m.yarahmadi@aut.ac.ir;
E-mail: m.yarahmadi@ipm.ac.ir

# Generalized Ricci Solitons on Four-Dimensional Non-Reductive Homogeneous Spaces of Signature $(2,2)$

Parvaneh Atashpeykar
Department of Mathematics, Basic Sciences Faculty, University of Bonab, Bonab 5551761167, Iran
and Amirhesam Zaeim*
Department of Mathematics, Payame noor University , P. O. Box 19395-3697, Tehran, Iran

Abstract. We classify generalized Ricci solitons on four-dimensional non-reductive homogeneous spaces of neutral signature.
**Keywords:** Non-reductive homogeneous space, Pseudo-Riemannian metric, Neutral signature, Generalized Ricci soliton.
**AMS Mathematical Subject Classification [2010]:** 53C30, 53C44.

## 1. Introduction

The notion of generalized Ricci solitons was introduced by P. Nurowski and M. Randall [5] in 2016. A generalized Ricci soliton is a pseudo-Riemannian manifold $(M,g)$ admitting a smooth vector field $X$, such that

$$\mathcal{L}_X g + 2\alpha X^\flat \odot X^\flat - 2\beta Ric = 2\lambda g, \tag{1}$$

for some real constants $\alpha, \beta, \lambda$, where $\mathcal{L}_X$ denotes the Lie derivative in the direction of $X$, $X^\flat$ denotes a 1-form such that $X^\flat(Y) = g(X,Y)$ and $Ric$ is the Ricci tensor. For particular values of the constants $\alpha, \beta, \lambda$, several important equations occur as special cases of Eq. (1). In particular, one has:

Table 1. Examples of generalized Ricci solitons.

| Equation | $\alpha$ | $\beta$ | $\lambda$ |
|---|---|---|---|
| Killing vector field equation | 0 | 0 | 0 |
| Homothetic vector field equation | 0 | 0 | $*$ |
| Ricci soliton equation | 0 | 1 | $*$ |
| Cases of Einstein-Weyl equation | 0 | $-\frac{1}{n-2}$ | $*$ |
| Metric projective structure with a skew-symmetric Ricci tensor in the projective class | 1 | $-\frac{1}{n-1}$ | 0 |
| Vacuum near-horizon geometry equation | 1 | $\frac{1}{2}$ | $*$ |

Equation (1) corresponds to an overdetermined system of partial differential equations of finite type. The study of this system was undertaken in the fundamental paper [5].

*Speaker

A homogeneous pseudo-Riemannian manifold $(M, g)$ is reductive if it can be realized as a coset space $M = \frac{G}{H}$, such that the Lie algebra $\mathfrak{g}$ can be decomposed into a direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is an Ad(H)-invariant subspace of $\mathfrak{g}$. When H is connected, this condition is equivalent to the algebraic condition $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$.

While all homogeneous Riemannian manifolds are reductive(and the same is true for two- and three-dimensional Lorentzian manifolds), in dimension four there exist homogeneous pseudo-Riemannian manifolds that do not admit any reductive decomposition. These spaces, both Lorentzian and of neutral signature, have been classified in [4]. The aim of this paper is to provide a systematic study of the geometry of four-dimensional non-reductive homogeneous pseudo-Riemannian manifolds of neutral signature, with particular regard to the existence of homogeneous generalized Ricci solitons. All calculations have also been checked using $Maple16^{©}$.

## 2. Four-Dimensional Non-Reductive Homogeneous Spaces of Signature $(2, 2)$

The classification of non-reductive homogeneous pseudo-Riemannian four-manifolds $(M = \frac{G}{H}, g)$ was given in [1] in terms of the corresponding non-reductive Lie algebras. Their invariant pseudo-Riemannian metrics $g$, together with their connection and curvature, were explicitly described in [2, 3], which we may refer for more details.

We report below the complete list of non-reductive homogeneous four-manifolds of signature $(2, 2)$, together with the description of their invariant metrics and Ricci tensor, as calculated in [1, 2, 3].

(B1) $\mathfrak{g} = \mathfrak{b}_1$ is the 5-dimensional Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathbb{R}^2$, admitting a basis $\{e_1, \ldots, e_5\}$, where the non-zero Lie brackets are

$$[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = e_4,$$
$$[e_1, e_5] = -e_5, \quad [e_2, e_5] = e_4, \quad [e_3, e_4] = e_5,$$

and the isotropy is $\mathfrak{h} = Span\{h_1 = e_3\}$. Thus, taking

$$\mathfrak{m} = Span\{u_1 = e_1, u_2 = e_2, u_3 = e_4, u_4 = e_5\}.$$

We find the invariant metrics and corresponding Ricci tensors of the form

$$g = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & c & a \\ a & c & d & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \qquad Ric = \begin{pmatrix} 0 & 0 & \frac{3d}{2a} & 0 \\ 0 & \frac{3(6bd-5c^2)}{2a^2} & \frac{3cd}{2a^2} & \frac{3d}{2a} \\ \frac{3d}{2a} & \frac{3cd}{2a^2} & \frac{3d^2}{2a^2} & 0 \\ 0 & \frac{3d}{2a} & 0 & 0 \end{pmatrix},$$

where $a, b, c, d$ are arbitrary real constants. The invariant metric $g$ is nondegenerate whenever $a \neq 0$. Moreover, the above equation easily yields that the Ricci tensor $Ric$ is nondegenerate if and only if $d \neq 0$.

(B2) $\mathfrak{g} = \mathfrak{b}_2$ is the 6-dimensional Schroedinger Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathfrak{n}(3)$, but with isotropy is $\mathfrak{h} = Span\{h_1 = e_3 - e_6, h_2 = e_5\}$. Then, we take

$$\mathfrak{m} = Span\{u_1 = e_1, u_2 = e_2, u_3 = e_3 + e_6, u_4 = e_4\},$$

and we find

$$g = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & a & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & -\frac{a}{2} \end{pmatrix}, \qquad Ric = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -\frac{8b}{a} & -3 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{2} \end{pmatrix},$$

where $a, b$ are arbitrary real constants. The invariant metric $g$ is nondegenerate whenever $a \neq 0$. The Ricci tensor $Ric$ is always nondegenerate.

(B3) $\mathfrak{g} = \mathfrak{b}_3$ is the 6-dimensional Lie algebra $(\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^2) \times \mathbb{R}$. It admitts a basis $\{u_1, \ldots, u_4, h_1 = u_5, h_2 = u_6\}$, such that $\mathfrak{h} = Span\{h_1, h_2\}$, $\mathfrak{m} = Span\{u_1, \ldots, u_4\}$, and the non-zero Lie brackets are

$[h_1, u_2] = u_1, \quad [h_1, u_3] = -u_4, \quad [h_2, u_2] = -2h_2, \quad [h_2, u_3] = -u_2,$
$[h_2, u_4] = u_1, \quad [u_1, u_2] = -u_1, \quad [u_1, u_3] = u_4, \quad [u_2, u_3] = -2u_3,$
$[u_2, u_4] = -u_4.$

Thus, the invariant metrics are of the form

$$g = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & 0 \end{pmatrix},$$

where $a, b$ are arbitrary real constants. The invariant metric $g$ is nondegenerate whenever $a \neq 0$. Moreover the Ricci tensor identically vanishes, that is, $g$ is Ricci-flat.

## 3. Generalized Ricci Solitons

We shall now look for solutions of Eq. (1) for four-dimensional non-reductive, homogeneous, pseudo-Riemannian manifolds of neutral signature corresponding to Lie algebras $\mathfrak{b}_i$, $(1 \leq i \leq 3)$.

(B1) Let $(M = \frac{G}{H}, g)$ be four-dimensional non-reductive homogeneous pseudo-Riemannian manifold corresponding to the Lie algebra $\mathfrak{b}_1$ and $X = X_i u_i \in \mathfrak{m}$, then with respect to the global bases $\{u_i\}$, we have

$$\mathcal{L}_X g = \begin{pmatrix} 2aX_3 & 2bX_2 + cX_3 & -aX_1 + 2cX_2 + dX_3 & aX_2 \\ 2bX_2 + cX_3 & -4bX_1 + 2cX_4 & -3cX_1 + dX_4 & -aX_1 - cX_2 \\ -aX_1 + 2cX_2 + dX_3 & -3cX_1 + dX_4 & -2dX_1 & -dX_2 \\ aX_2 & -aX_1 - cX_2 & -dX_2 & 0 \end{pmatrix}.$$

Thus, Eq. (1) becomes

$$\alpha a^2 X_2^2 = 0, \; aX_2 + 2\alpha a^2 X_2 X_3 = 0, \; aX_3 + \alpha a^2 X_3^2 = 0,$$
$$-dX_2 + 2\alpha a X_2(aX_1 + cX_2 + dX_3) = 0,$$
$$2bX_2 + cX_3 + 2\alpha a X_3 + (bX_2 + cX_3 + aX_4) = 0,$$
$$-2dX_1 + 2\alpha(aX_1 + cX_2 + dX_3)^2 - 3\beta(\frac{d}{a})^2 = 2\lambda d,$$
$$-aX_1 - cX_2 + 2\alpha a X_2(bX_2 + cX_3 + aX_4) - \frac{3\beta d}{a} = 2\lambda a, \tag{2}$$

$$-4bX_1 + 2cX_4 + 2\alpha(bX_2 + cX_3 + aX_4)^2 - \frac{3\beta(6bd - 5c^2)}{a^2} = 2\lambda b,$$
$$-3cX_1 + dX_4 + 2\alpha(bX_2 + cX_3 + aX_4)(aX_1 + cX_2 + dX_3) - \frac{3\beta cd}{a^2} = 2\lambda c,$$
$$-aX_1 + 2cX_2 + dX_3 + 2\alpha aX_3(aX_1 + cX_2 + dX_3) - \frac{3\beta d}{a} = 2\lambda a.$$

We then solve (2), obtaining the following classification result.

THEOREM 3.1. *Let $(M = \frac{G}{H}, g)$ be a four-dimensional non-reductive homogeneous pseudo-Riemannian manifold corresponding to the Lie algebra $\mathfrak{b}_1$, then $(M, g)$ is a non-trivial (i.e. not Ricci soliton) generalized Ricci solitons if and only if one of the following cases occurs:*

1) $b = c = 0 \neq d, \lambda = -\frac{d(6\alpha\beta+1)}{4\alpha a^2}$ *and* $X = \frac{d}{2\alpha a^2}u_1$ *for all* $\alpha \neq 0$ *and* $\beta \in \mathbb{R}$.
2) $bd - c^2 = 0, \lambda = -\frac{d(6\alpha\beta+1)}{4\alpha a^2}$ *and* $X = \frac{1}{2\alpha a^2}(du_1 + cu_4)$ *for all* $\alpha \neq 0$ *and* $\beta \in \mathbb{R}$.
3) $c = 0, \beta = -\frac{1}{10\alpha}, \lambda = -\frac{d}{10\alpha a^2}$ *and* $X = \frac{d}{2\alpha a^2}u_1$ *for all* $\alpha \neq 0 \in \mathbb{R}$.
4) $\beta = -\frac{1}{10\alpha}, \lambda = \frac{\beta d}{a^2}$ *and* $X = -\frac{5\beta}{a^2}(du_1 + cu_2)$ *for all* $\alpha \neq 0 \in \mathbb{R}$.
5) $bd - c^2 = 0, \lambda = -\frac{d(6\alpha\beta+1)}{4\alpha a^2}$ *and* $X = \frac{1}{2\alpha a^2}(du_1 - 2au_3 + cu_4)$ *for all* $\alpha \neq 0$ *and* $\beta \in \mathbb{R}$.
6) $\beta = -\frac{1}{10\alpha}, \lambda = -\frac{d}{10\alpha a^2}$ *and* $X = \frac{1}{2\alpha a^2}(du_1 - 2au_3 + cu_4)$ *for all* $\alpha \neq 0 \in \mathbb{R}$.

(B2) Let $(M = \frac{G}{H}, g)$ be a four-dimensional non-reductive homogeneous pseudo-Riemannian manifold corresponding to the Lie algebra $\mathfrak{b}_2$ and $X = X_iu_i \in \mathfrak{m}$, then with respect to the global bases $\{u_i\}$, we have

$$\mathcal{L}_X g = \begin{pmatrix} 0 & 2bX_2 & aV_2 & -\frac{1}{2}aX_4 \\ 2bX_2 & -4bX_1 & -aX_1 & 0 \\ aX_2 & -aX_1 & 0 & 0 \\ -\frac{1}{2}aX_4 & 0 & 0 & aX_1 \end{pmatrix}.$$

Thus, Eq. (1) becomes

$$\begin{aligned} &\alpha a^2 X_2^2 = 0,\ \alpha a^2 X_2 X_4 = 0,\ \alpha a X_4(bX_2 + aX_3) = 0, \\ &aX_2 + 2\alpha a^2 X_1 X_2 = 0,\ aX_4 + 2\alpha a^2 X_1 X_4 = 0, \\ &bX_2 + 2\alpha a X_1(aX_3 + bX_2) = 0,\ \alpha a^2 X_1^2 + 3\beta = \lambda a, \\ &-aX_1 + 2\alpha a X_2(bX_2 + aX_3) + 6\beta = 2\lambda a, \\ &2aX_1 + \alpha a^2 X_4^2 - 6\beta = 2\lambda a, \\ &-4bX_1 + 2\alpha(aX_3 + bX_2)^2 + \frac{17\beta b}{a} = 2\lambda b. \end{aligned} \tag{3}$$

We then solve (3), obtaining the following classification result.

THEOREM 3.2. *Let $(M = \frac{G}{H}, g)$ be a four-dimensional non-reductive homogeneous pseudo-Riemannian manifold corresponding to the Lie algebra $\mathfrak{b}_2$, then $(M, g)$ is a non-trivial generalized Ricci solitons if and only if one of the following cases occurs:*

1) $b \neq 0, \lambda = \frac{3\beta}{a}$ *and* $X = \pm\sqrt{\frac{-5\beta d}{\alpha a^3}}u_3$ *for all* $\alpha \neq 0$ *and* $\beta \in \mathbb{R}$ *satisfying* $\frac{\beta d}{\alpha a} < 0$.

2) $b = 0, \lambda = -\frac{12\alpha\beta+1}{4\alpha a}$ *and* $X = -\frac{1}{2\alpha a}(u_1 \pm \sqrt{2}u_4)$ *for all* $\alpha \neq 0$ *and* $\beta \in \mathbb{R}$.
3) $\beta = -\frac{3}{20\alpha}, \lambda = -\frac{1}{5\alpha a}$ *and* $X = -\frac{1}{2\alpha a}(u_1 \pm \sqrt{2}u_4)$ *for all* $\alpha \neq 0$ *and* $\beta \in \mathbb{R}$.

## References

1. G. Calvaruso and A. Fino, *Ricci solitons and geometry of four-dimensional non-reductive homogeneous spaces,* Canad. J. Math. **64** (4) (2012) 778–804.
2. G. Calvaruso and A. Zaeim, *Geometric structures over non-reductive homogeneous* 4*-spaces,* Adv. Geom. **14** (2) (2014) 191–214.
3. G. Calvaruso and A. Zaeim, *Four-dimensional homogeneous Lorentzian manifolds,* Monatsh. Math. **174** (2014) 377–402.
4. M. E. Fels and A. G. Renner, *Non-reductive homogeneous pseudo-Riemannian manifolds of dimension four,* Canad. J. Math. **58** (2) (2006) 282–311.
5. P. Nurowski and M. Randall, *Generalized Ricci solitons,* J. Geom. Anal. **26** (2016) 1280–1345.

E-mail: p.atashpeykar@ubonab.ac.ir
E-mail: zaeim@pnu.ac.ir

# Bundle-Like Metric on Foliated Manifold with Semi-Symmetric Metric Connection

Elham Zangiabadi*
Faculty of Mathematical Sciences, University of Vali-e-Asr, Rafsanjan, Iran
Nasrin Mohamadi
Department of Mining Engineering, Higher Education Complex of Zarand, Zarand, Iran
and Zohre Nazari
Faculty of Mathematical Sciences, University of Vali-e-Asr, Rafsanjan, Iran

Abstract. Let $(M, g, \mathcal{F})$ be a semi-Riemannian foliated manifold with structural distribution $\mathcal{D}$ on $\mathcal{F}$. We define a semi-symmetric metric connection on $\mathcal{D}$ and $\mathcal{D}^{\perp}$, where $TM = \mathcal{D} \oplus \mathcal{D}^{\perp}$. In particular it is presented a characterization of bundle-like metric of $\mathcal{F}$ by means of semi-symmetric metric connection.
**Keywords:** Foliation, Bundle-like metric, Semi-symmetric metric connection.
**AMS Mathematical Subject Classification [2010]:** 53C12, 53B05.

## 1. Introduction

In 1924, Friedman and Schouten [2] introduced the notion of semi-symmetric linear connection on a differentiable manifold. A linear connection $\nabla$ on a semi-Riemannian manifold $(M, g)$ is said to be semi-symmetric if the torsion tensor $T$ of the connection $\nabla$ satisfies

$$T(X, Y) = \omega(Y)X - \omega(X)Y,$$

for any vector fields $X, Y$ on $M$ and $\omega$ is a 1-form given by $\omega(X) = g(X, W)$, where $W$ is the vector field associated with the 1-form $\omega$.

If $\nabla g = 0$, then the connection $\nabla$ is said to be a metric connection; otherwise, it is non-metric [3]. The study of a semi-symmetric metric connection was further developed by Yano [5]. Let $M$ be an $(n+p)-$dimensional manifold endowed with an $n-$foliation $\mathcal{F}$. Denote by $\mathcal{D}$ the tangent distribution to $\mathcal{F}$ and suppose that there exists a complementary distribution $\mathcal{D}^{\perp}$ to $\mathcal{D}$ in the tangent bundle $TM$ of $M$, that is, we have

$$TM = \mathcal{D} \oplus \mathcal{D}^{\perp}. \qquad (1)$$

We call $\mathcal{D}$ and $\mathcal{D}^{\perp}$ the structural distribution and transversal distribution on the foliated manifold $(M, g)$ and Denote by $\mathcal{P}$ and $\mathcal{Q}$ the morphisms of $TM$ on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively.

*Speaker

Denoting respectively by $\tilde{D}$ and $\tilde{D}^{\perp}$ the Levi-Civita connections induced on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ from $\tilde{\nabla}$. First, according to (1) we write

$$a)\ \tilde{\nabla}_X \mathcal{P}Y = \tilde{D}_X \mathcal{P}Y + \tilde{H}(X, \mathcal{P}Y),$$
$$b)\ \tilde{\nabla}_X \mathcal{Q}Y = \tilde{D}^{\perp}_X \mathcal{Q}Y + \tilde{H}^{\perp}(X, \mathcal{Q}Y),$$

where

$$a)\ \tilde{D}_X \mathcal{P}Y = \mathcal{P}\tilde{\nabla}_X \mathcal{P}Y,$$
$$b)\ \tilde{D}^{\perp}_X \mathcal{Q}Y = \mathcal{Q}\tilde{\nabla}_X \mathcal{Q}Y,$$

and

$$a)\ \tilde{H}(X, \mathcal{P}Y) = \mathcal{Q}\tilde{\nabla}_X \mathcal{P}Y, \tag{2}$$
$$b)\ \tilde{H}^{\perp}(X, \mathcal{Q}Y) = \mathcal{P}\tilde{\nabla}_X \mathcal{Q}Y,$$

for all $X, Y \in \Gamma(TM)$. Where $\tilde{H}$ and $\tilde{H}^{\perp}$ are respectively the second fundamental forms of $\mathcal{D}$ and $\mathcal{D}^{\perp}$ with respect to $\tilde{\nabla}$ [1].

## 2. Bundle-Like Metric Semi-Symmetric Metric Connection

We now suppose that the semi-Riemannian manifold $(M, g)$ admits a semi-symmetric metric connection given by

$$\nabla_X Y = \tilde{\nabla}_X Y + \omega(Y)X - g(X, Y)W,$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $(M, g)$, $\omega$ is a 1-form and $W$ is the vector field defined by

$$g(W, X) = \omega(X),$$

for any vector field $X$ of $M$ (See [4, 5]).

The semi-symmetric metric connection $\nabla$ on $(M, g)$ induces two semi-symmetric metric connections $D$ and $D^{\perp}$ on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively. By (1)

$$a)\ \nabla_X \mathcal{P}Y = D_X \mathcal{P}Y + H(X, \mathcal{P}Y), \tag{3}$$
$$b)\ \nabla_X \mathcal{Q}Y = D^{\perp}_X \mathcal{Q}Y + H^{\perp}(X, \mathcal{Q}Y),$$

where

$$a)\ D_X \mathcal{P}Y = \mathcal{P}\nabla_X \mathcal{P}Y,$$
$$b)\ D^{\perp}_X \mathcal{Q}Y = \mathcal{Q}\nabla_X \mathcal{Q}Y,$$

and

$$a)\ H(X, \mathcal{P}Y) = \mathcal{Q}\nabla_X \mathcal{P}Y, \tag{4}$$
$$b)\ H^{\perp}(X, \mathcal{Q}Y) = \mathcal{P}\nabla_X \mathcal{Q}Y.$$

We call $H$ (resp. $H^{\perp}$) the second fundamental forms of $\mathcal{D}$ (resp. $\mathcal{D}^{\perp}$) with respect to $\nabla$.

Since $\nabla$ is metric by using (3a) and (3b) we obtain that

$$g(H(X, \mathcal{P}Y), \mathcal{Q}Z) = -g(H^{\perp}(X, \mathcal{Q}Z), \mathcal{P}Y).$$

By definition of the semi-symmetric metric connection $\nabla$ and by (2a), (2b), (4a) and (4b) we deduce that

$$H(X, \mathcal{P}Y) = \tilde{H}(X, \mathcal{P}Y) + \omega(\mathcal{P}Y)\mathcal{Q}X - g(X, \mathcal{P}Y)\mathcal{Q}W,$$

and

$$H^{\perp}(X,\mathcal{Q}Y)=\tilde{H}^{\perp}(X,\mathcal{Q}Y)+\omega(\mathcal{Q}Y)\mathcal{P}X-g(X,\mathcal{Q}Y)\mathcal{P}W.$$

Therefore

$$H(\mathcal{P}X,\mathcal{P}Y)=\tilde{H}(\mathcal{P}X,\mathcal{P}Y)-g(\mathcal{P}X,\mathcal{P}Y)\mathcal{Q}W,$$

and

$$H^{\perp}(\mathcal{Q}X,\mathcal{Q}Y)=\tilde{H}^{\perp}(\mathcal{Q}X,\mathcal{Q}Y)-g(\mathcal{Q}X,\mathcal{Q}Y)\mathcal{P}W.$$

The symmetric second fundamental form $H_s^{\perp}$ of $\mathcal{D}^{\perp}$ is

(5) $$H_s^{\perp}(\mathcal{Q}Y,\mathcal{Q}Z)=\frac{1}{2}(H^{\perp}(\mathcal{Q}Y,\mathcal{Q}Z)+H^{\perp}(\mathcal{Q}Z,\mathcal{Q}Y)).$$

We say that $\mathcal{F}$ is a foliation with bundle-like metric $g$ if each geodesic in $(M,g)$ which tangent to the transversal distribution $\mathcal{D}^{\perp}$ at one point remain tangent for its entire length. Then a necessary and sufficient condition for $g$ to be bundle-like for $\mathcal{F}$ is that

$$g(\tilde{\nabla}_{\mathcal{Q}Y}\mathcal{P}X,\mathcal{Q}Z)+g(\tilde{\nabla}_{\mathcal{Q}Z}\mathcal{P}X,\mathcal{Q}Y)=0.$$

Several characterizations of bundle-like metrics are presented in the next theorem.

Theorem 2.1. *Let $(M,g,\mathcal{F})$ be a folioated semi-Riemannian manifold where $\mathcal{F}$ is a non-degenerate foliation. Then the following assertions are equivalent:*

i) *$g$ is a bundle-like metric.*
ii) $g(\nabla_{\mathcal{Q}Y}\mathcal{P}X,\mathcal{Q}Z)+g(\nabla_{\mathcal{Q}Z}\mathcal{P}X,\mathcal{Q}Y)=-2\omega(\mathcal{P}X)g(\mathcal{Q}Y,\mathcal{Q}Z)$.
iii) $\mathcal{L}_{\mathcal{P}X}g(\mathcal{Q}Y,\mathcal{Q}Z)=-2\omega(\mathcal{P}X)g(\mathcal{Q}Y,\mathcal{Q}Z)$.
iv) $g(\mathcal{P}X,\nabla_{\mathcal{Q}Y}\mathcal{Q}Z+\nabla_{\mathcal{Q}Z}\mathcal{Q}Y)=2\omega(\mathcal{P}X)g(\mathcal{Q}Y,\mathcal{Q}Z)$.
v) $H_s^{\perp}(\mathcal{Q}Y,\mathcal{Q}Z)=g(\mathcal{Q}Y,\mathcal{Q}Z)\mathcal{P}W$.

Proof. Let $g$ be a bundle-like metric. Therefore,

$$g(\tilde{\nabla}_{\mathcal{Q}Y}\mathcal{P}X,\mathcal{Q}Z)+g(\tilde{\nabla}_{\mathcal{Q}Z}\mathcal{P}X,\mathcal{Q}Y)=0,$$

and by definition of $\nabla$ we have

$$g(\nabla_{\mathcal{Q}Y}\mathcal{P}Z+\omega(\mathcal{P}X)\mathcal{Q}Y,\mathcal{Q}Z)+g(\nabla_{\mathcal{Q}Z}\mathcal{P}X+\omega(\mathcal{P}X)\mathcal{Q}Z,\mathcal{Q}Y)=0.$$

Thus $g(\nabla_{\mathcal{Q}Y}\mathcal{P}X,\mathcal{Q}Z)+g(\nabla_{\mathcal{Q}Z}\mathcal{P}X,\mathcal{Q}Y)=-2\omega(\mathcal{P}X)g(\mathcal{Q}Y,\mathcal{Q}Z)$ and we obtain the equivalence of (i) and (ii).
We know that

$$\mathcal{L}_{\mathcal{P}X}g(\mathcal{Q}Y,\mathcal{Q}Z)=g(\nabla_{\mathcal{Q}Y}\mathcal{P}X,\mathcal{Q}Z)+g(\nabla_{\mathcal{Q}Z}\mathcal{P}X,\mathcal{Q}Y),$$

and therefore we deduce that (ii) and (iii) are equivalent. Since $\nabla$ is metric we have

$$g(\mathcal{P}X,\nabla_{\mathcal{Q}Y}\mathcal{Q}Z)=-g(\nabla_{\mathcal{Q}Y}\mathcal{P}X,\mathcal{Q}Z),$$

and

$$g(\mathcal{P}X,\nabla_{\mathcal{Q}Z}\mathcal{Q}Y)=-g(\nabla_{\mathcal{Q}Z}\mathcal{P}X,\mathcal{Q}Y),$$

thus we obtain that

$$g(\mathcal{P}X,\nabla_{\mathcal{Q}Y}\mathcal{Q}Z+\nabla_{\mathcal{Q}Z}\mathcal{Q}Y)=-\{g(\nabla_{\mathcal{Q}Y}\mathcal{P}X,\mathcal{Q}Z)+g(\nabla_{\mathcal{Q}Z}\mathcal{P}X,\mathcal{Q}Y)\},$$

and it follows that (ii) and (iv) are equivalent.
By (5), $g(\mathcal{P}X, \nabla_{\mathcal{Q}Y}\mathcal{Q}Z + \nabla_{\mathcal{Q}Z}\mathcal{Q}Y) = g(\mathcal{P}X, 2H_s^{\perp}(\mathcal{Q}Y, \mathcal{Q}Z))$, and since

$$2\omega(\mathcal{P}X)g(\mathcal{Q}Y, \mathcal{Q}Z) = g(\mathcal{P}X, 2g(\mathcal{Q}Y, \mathcal{Q}Z)W),$$

(iv) and (v) are equivalent. □

## References


1. A. Bejancu and H. R. Farran, *Foliation and Geometric Structures*, Springer, Netherlands, 2005.
2. A. Friedmann and J. A. Schouten, *Über die geometrie der halbsymmetrischen Übertragungen*, Math. Z. **21** (1924) 211–223.
3. B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, London, 1983.
4. A. Sharfuddin and S. I. Hussain, *Semi-symmetric metric connexions in almost contact manifolds*, Tensor (N.S.) **30** (2) (1976) 133–139.
5. K. Yano, *On semi-symmetric metric connection*, Rev. Roumaine Math. Pures Appl. **15** (1970) 1570–1586.



E-mail: e.zangiabadi@vru.ac.ir
E-mail: mohamadinasrin@yahoo.com
E-mail: z.nazari@vru.ac.ir

# On the Compactness of Minimal Prime Spectrum of $C_c(X)$

Rostam Mohamadian
Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran
and Maryam Zeinali*
Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran

Abstract. The ring $C_c(X)$ as a subring of $C(X)$ consists of all functions with countable image. We show that $C_c(X)$ has countable annihilator condition and property(A). Let $Min(C_c(X))$ denote the minimal prime spectrum of $C_c(X)$. $Min(C_c(X))$ as a subspace of $Spec(C_c(X))$ is not generally compact. Also, in the class of basically disconnected spaces $Min(C_c(X))$ and $Min(C(X))$ are homeomorphic. We consider some relations between the topological properities of the spaces $X$ and $Min(C_c(X))$, for which $Min(C_c(X))$ becomes a compact space. Finally, while introducing $z_c^\circ$-ideals and $c-cc-$spaces, we study the compactness of $Min(C_c(X))$.

**Keywords:** Zero-dimensional space, Basically disconnected space, $z_c^\circ$-ideals, Compact space, Minimal prime spectrum.
**AMS Mathematical Subject Classification [2010]:** 54C05, 54C30, 54C40.

## 1. Introduction

Let $C(X)$ denote the ring of all real valued continuous functions on a topological space $X$. The ring $C_c(X)$ as a subalgebra of $C(X)$, consisting of all functions with countable image are studied in [3]. For the ring $R$, the space $Spec(R)$ is a space of prime ideals of $R$ with Zariski topology and the space $Min(R)$ as a subspace of $Spec(R)$, is the space of minimal prime ideals of $R$, see [5, 7].

We recall that a zero-dimensional space is a Hausdorff space with a base consisting of clopen sets. Furthermore, a Tychonoff space $X$ is called strongly zero-dimensional if each pair of disjoint zero-sets are contained in disjoint clopen sets. Moreover, a Thychonoff space $X$ is strongly zero-dimensional iff $\beta X$ is zero-dimensional. Banaschewski has shown that for every zero-dimensional space$X$, there is a unique zero-dimensional compactification, denoted by $\beta_0 X$ in which each continuous function from $X$ into a compact and zero-dimensional space $T$, has a continuous extension from $\beta_0 X$ into $T$. It is shown that $\beta X = \beta_0 X$ iff $X$ is a strongly zero-dimensional space, see [2]. It is proved that $Spec(R)$ is a compact and $T_0$-space whereas $Min(R)$ is a Hausdorff and zero-dimensional space but not necessarily compact, see [5]. Furthermore, the space $Min(R)$ is dense in $Spec(R)$. A reduced ring $R$ satisfies the annihilator condition (or a.c.) if for each $a, b \in R$, there exists $c \in R$ such that $Ann(c) = Ann(a) \cap Ann(b)$. $C_c(X)$ has a.c., let $f, g \in C_c(X)$, we put $h = f^2 + g^2$. Obviously, $h \in C_c(X)$ and $Ann_c(h) = Ann_c(f) \cap Ann_c(g)$. Furthermore, a reduced ring $R$ satisfies countable annihilator condition (or c.a.c.) if for every sequence $\{r_n\}_{n\in\mathbb{N}}$ in $R$, there

*Speaker

exists $s \in R$ in which $Ann(s) = \bigcap_{n\in\mathbb{N}} Ann(r_n)$. $C(X)$ has c.a.c. [5]. Moreover, let $Z = Zd(R)$ be the set of all zero divisors of ring $R$. Also, $R$ has property(A) if for every finitely generated ideal $I$ in $R$ such that $I \subseteq Z$, we have $Ann(I) \neq 0$. By [5], we recall that for a reduced ring $R$ with a.c., $Min(R)$ is compact and extremally disconnected iff for each $S \subseteq R$, there exists $y \in R$ such that $Ann(S) = Ann(y)$.

Furthermore, if $R$ satisfies c.a.c., then $Min(R)$ is countably compact. Also, if $R$ has c.a.c. and $Min(R)$ is locally compact, then $Min(R)$ is basically disconnected. For each $f \in C_c(X)$, we set:

$$V_c(f) = \{P \in Min(C_c(X)) : f \in P\},$$
$$D_c(f) = \{P \in Min(C_c(X)) : f \notin P\}.$$

The family $\mathcal{B} = \{V_c(f) : f \in C_c(X)\}$ is a base for closed sets, also the family $\mathcal{B}' = \{D_c(f) : f \in C_c(X)\}$ is a base for open sets for the Zariski topology on $Min(C_c(X))$. Furthermore, $D_c(f)$ and $V_c(f)$ are disjoint clopen sets. The space $Min(C_c(X))$ that is equipped with this topology is the hull-kernel or Zariski space. For each $f \in C_c(X)$, the zero-set (cozero-set) of $f$ is denoted by $Z_c(f)$ $(Coz_c(f))$. The set of all zero-sets in $X$ is denoted by $Z_c(X)$. Also, $Z_c(X)$ is closed under countable intersection property. Furthermore, $Z_c(X) = Z(X)$ iff $X$ is strongly zero-dimensional. see [2, Proposition 2.4.].

For each $f \in C_c(X)$, the annihilator of $f$ is denoted by $Ann_c(f)$. An ideal $I$ in $C_c(X)$ is a $z_c$-ideal if for each $f \in I$, $g \in C_c(X)$, $Z_c(f) = Z_c(g)$ we have $g \in I$. Similar to the concept of $\mathcal{M}_p$ in $C(X)$, the fixed maximal ideals in $C_c(X)$ is denoted by $\mathcal{M}_{cp}$ $(p \in X)$. Also, if $X$ is a zero-dimensional space, the set of all maximal ideals in $C_c(X)$ is denoted by $\mathcal{M}_c^p$ $(p \in \beta_0 X)$, Moreover, $\mathcal{M}_c^p = \mathcal{M}_{cp}$ if $p \in X$. Also, similar to the concept of the ideals $O^p$, $p \in \beta X$, for the zero-dimensional space $X$, we have the ideals $O_c^p$ in $C_c(X)$. Furthermore, $O_c^p = O_{cp}$ if $p \in X$. For the zero-dimensional space $X$, $O_c^p$ is a $z_c$-ideal. For more results about the ideals $\mathcal{M}_c^p$, $O_c^p$, see [2].

A space $X$ is called $CP$-space if $C_c(X)$ is regular. A zero-dimensional space $X$ is an $F_c$-space if and only if $O_c^p$ is a prime ideal in $C_c(X)$ for each $p \in \beta_0 X$. Every $F$-space is an $F_c$-space. The converse is not necessarily true unless $X$ is strongly zero. For more results of $CP$-spaces and $F_c$-spaces, see [2, 3].

We recall that $X$ is a basically (extremally) disconnected space if every cozero-set (open set) has an open closure. Also, $X$ is $c$-basically disconnected if for each $f \in C_c(X)$, $intz(f)$ is closed. Every basically disconnected space is zero-dimensional, see [4, 14O(3)]. It is shown that every basically disconnected space is an $F_c$-space. The space $Min(C(X))$ is not generally compact, basically disconnected and extremally disconnected. It is proved that $Min(C(X))$ is compact if and only if the classical ring of quotients of $C(X)$ is a regular ring, see [5]. Furthermore, if $X$ is a basically disconnected space, then $Min(C(X))$ and $\beta X$ are homeomorphic, so $Min(C(X))$ is compact. Moreover, $Min(C(X))$ is basically disconnected if it is locally compact, see [7].

Similar to the concept of $z^\circ$-ideals in $C(X)$, see [1], we introduce $z_c^\circ$-ideals in $C_c(X)$ and consider the relations between $z_c^\circ$-ideals and the compactness of $Min(C_c(X))$. Moreover, we study the conditions when the minimal prime ideals in $C_c(X)$ and $z_c^\circ$-ideals coincide. Finally, while introducing countably cozero complemented or $c-cc-$spaces , We study its relation with the compactness of $Min(C_c(X))$. We recall that $cc-$spaces are the spaces for which $Min(C(X))$ is compact for the topological space $X$, see [6].

## 2. Main Results

PROPOSITION 2.1. *The following statements are hold:*

a) $C_c(X)$ *has c.a.c.*
b) $C_c(X)$ *has property(A).*

COROLLARY 2.2. *Let $q_c(X)$ be the classical ring of quotients of $C_c(X)$. The following statements are equivalent:*

a) $Min(C_c(X))$ *is a compact space.*
b) $q_c(X)$ *is a Von-Neumann regular ring.*

COROLLARY 2.3. *The following statements are hold:*

a) $Min(C_c(X))$ *is countably compact.*
b) *If $X$ is a $CP$-space, then $Min(C_c(X))$ is compact and basically disconnected.*
c) *If $X$ is a discrete space, then $Min(C_c(X))$ is compact and extremally disconnected.*
d) *If every prime ideal of $C(X)$ contracts to a minimal prime ideal of $C_c(X)$, then $Min(C_c(X))$ is compact.*

EXAMPLE 2.4. $Min(C_c(\mathbb{N}))$ is compact and extremally disconnected.

THEOREM 2.5. *Let $X$ be a zero-dimensional space and $\varphi_c$ be the mapping from $Min(C_c(X))$ into $\beta_0 X$ by $\varphi_c(P) = p$, then we have the following statements:*

a) *$\varphi_c$ is a continuous mapping of $Min(C_c(X))$ onto $\beta_0 X$.*
b) *$\varphi_c$ is a mapping in which for each proper closed subset $F \subseteq Min(C_c(X))$, we have $\varphi_c(F) \neq \beta_0 X$.*
c) *$\varphi_c$ is a one-to-one mapping if and only if $O_c^p$ is a prime ideal for each $p \in \beta_0 X$.*
d) *Let $X$ be a pseudocompact space, then we have $\varphi_c$ is a homeomorphism if and only if $X$ is c-basically disconnected.*
e) *If $X$ is a pseudocompact and $F_c$-space, then $Min(C_c(X))$ is compact if and only if $X$ is c-basically disconnected.*

COROLLARY 2.6. *Let $X$ be a basically disconnected space, then $Min(C_c(X))$ and $Min(C(X))$ are homeomorphic and compact spaces.*

EXAMPLE 2.7. 1) Let $X = \beta\mathbb{N} \setminus \mathbb{N}$. Since $X$ is a strongly zero-dimensional and $F$-space which is not basically disconnected, [4, 6w], then $Min(C_c(X))$ is not compact.
2) Let $X = \mathbb{N}$ be the space of positive integers. Since $X$ is a strongly zero-demensional and $F$-space, then $\beta X = \beta_0 X$ are $F$-spaces. Also, these spaces are extremally disconnected [4, 6M.1]. Thus, $Min(C_c(X))$, $Min(C_c(\beta X))$ are compact.

Definition 2.8. A proper ideal $I$ in $C_c(X)$ is a $z_c^\circ$-ideal if for each $f \in I$, we have $P_f \subseteq I$ in which $P_f = \bigcap\{P \in Min(C_c(X)) : f \in P\}$, $P_f$ is a basic $z_c^\circ$-ideal.

Proposition 2.9. *For each $f \in C_c(X)$, we have $P_f = \{g \in C_c(X) : Ann_c(f) \subseteq Ann_c(g)\}$ in which $P_f$ is a basic $z_c^\circ$-ideal in $C_c(X)$.*

Corollary 2.10. *The following statements are hold:*

a) *Every minimal prime ideal in $C_c(X)$ is a $z_c^\circ$-ideal.*
b) *If $I$ is a $z_c^\circ$-ideal in $C_c(X)$ and $P$ is a prime ideal in $C_c(X)$ in which $P \in Min(I)$, then $P$ is a $z_c^\circ$-ideal.*

Lemma 2.11. *Let $X$ be a $CP-$space and $f, g \in C_c(X)$. The following statements are equivalent:*

a) $Z_c(f) = Z_c(g)$,
b) $D_c(f) = D_c(g)$,
c) $P_f = P_g$.

Theorem 2.12. *Every $z_c^\circ$-ideal in $C_c(X)$ is a contraction of a $z^\circ$-ideal in $C(X)$.*

Proposition 2.13. *Let $X$ be a strongly zero-dimensional space, then every $z_c^\circ$-ideal in $C_c(X)$ is a contraction of a unique $z^\circ$-ideal in $C(X)$.*

Proposition 2.14. *Let $X$ be a zero-dimensional and $F_c$-space. The following statements are equivalent:*

a) *$Min(C_c(X))$ is a compact space.*
b) *$X$ is basically disconnected.*
c) *$Min(C_c(X))$ and $\beta_0 X$ are homeomorphic.*
d) *$q_c(X)$, the classical ring of quotionts of $C_c(X)$, is regular.*
e) *Every $z_c^\circ$-ideal in $C_c(X)$ is a minimal prime ideal.*
f) *Let $I$ be a $z_c^\circ$-ideal in $C_c(X)$, then there exists $p \in \beta_0 X$ such that $I = O_c^p$. Furthermore, if $X$ is a $F$-space, there exists $p' \in \beta X$ such that $O_c^p = O^{p'} \cap C_c(X)$.*

Definition 2.15. (1) A Space $X$ is called countably cozero complemented or $c-cc$-space if for each $f \in C_c(X)$, there exists $g \in C_c(X)$ such that $Coz_c(f) \cap Coz_c(g) = \phi$, $\overline{Coz_c(f) \cup Coz_c(g)} = X$.
(2) A space $X$ is said to be countably perfectly normal or $c$-perfectly normal if for disjoint closed sets $A$ and $B$ in $X$, there exists $f \in C_c(X)$ such that $A = f^{-1}(\{0\})$, $B = f^{-1}(\{1\})$. Also, the support of $f$ is denoted by $spt_c(f)$ in which $f \in C_c(X)$, i.e., $spt_c(f) = \overline{Coz_c(f)}$.

Proposition 2.16. *Let $X$ be a c-perfectly normal space, then each open set in $X$ is a cozero set.*

Corollary 2.17. *Let $X$ be a c-perfectly normal space and $G \subseteq X$ be an open set in $X$, then $\overline{G} = spt_c(f)$ in which $f \in C_c(X)$. Similar to* [5, Theorem 5.6] *we have the next theorem.*

Theorem 2.18. *The following statements are hold:*

a) *If for each $f \in C_c(X)$, $spt_c(f)$ is a zero set, then $Min(C_c(X))$ is compact and basically disconnected.*

b) *If $X$ is a c-perfectly normal space, then $Min(C_c(X))$ is compact and extremally disconnected.*

PROPOSITION 2.19. *$Min(C_c(X))$ is a compact space if and only if for each $f \in C_c(X)$, there exists $g \in C_c(X)$ such that $Z_c(f) \cup Z_c(g) = X$, $int[Z_c(f) \cap Z_c(g)] = \phi$.*

THEOREM 2.20. *The following statements for the space $X$ are equivalent.*

a) *$Min(C_c(X))$ is a compact space.*
b) *For each $f \in C_c(X)$, there exists $g \in C_c(X)$ such that $Ann_c(Ann_c(f)) = Ann_c(g)$.*
c) *$V_c(f) = V_c(Ann_c(g))$ in which $f, g \in C_c(X)$.*
d) *For each $f \in C_c(X)$, there exists $g \in C_c(X)$ such that $spt_c(f) \cup spt_c(g) = X$, $int[spt_c(f) \cap spt_c(g)] = \phi$.*
e) *$X$ is a $c - cc-$space.*

By the definition of $cc-$spaces and $c - cc-$spaces we conclude that these spaces are not equivalent unless $X$ is strongly zero-dimensional.

EXAMPLE 2.21. (1) Let $S$ be an uncountable space in which all points are isolated except for the distinguished point $s$ with the defined topology, see [4, 4N]. The space $S$ is basically disconnected. So, $S$ is both $cc-$space and $c - cc-$space, equivalently, $MinC_c(S)$ and $Min(C(S))$ are compact.
(2) Let $D$ be an infinite discrete space and $X = \beta D \setminus D$. So, $X$ is not basically disconnected. Thus, $Min(C(X))$ and $Min(C_c(X))$ are not compact. Consequently, $X$ is neither $cc-$space nor $c - cc-$space.

## Acknowledgement

We would like to thank the referee for his/her suggestions, for improving the quality of our exposition.

## References

1. F. Azarpanah, O. A. S. Karamzadeh and A. Rezai Aliabad, *On $z^\circ$-ideals in $C(X)$*, Fund. Math. **160** (1) (1999) 15–25.
2. F. Azarpanah, O. A. S. Karamzadeh, Z. Keshtkar and A. R. Olfati, *On maximal ideals of $C_c(X)$ and the uniformity of its localizations*, Rocky Mountain J. Math. **48** (2) (2018) 1–42.
3. M. Ghadermazi, O. A. S Karamzadeh and M. Namdari, *On the functionally countable subalgebra of $C(X)$*, Rend. Sem. Mat.Univ. Padova. **129** (2) (2013) 47–69.
4. L. Gilman and M. Jerison, *Rings of Continuous Functions*, Graduate texts in Maths., Vol. 43, Springer-Verlag, Berlin-Heidelberg-New York, 1976.
5. M. Henriksen and M. Jerison, *The space of minimal prime ideals of a commutative ring*, Trans. Amer. Math. Soc. **115** (1965) 110–130.
6. M. Henriksen and R. G. Woods, *Cozero complemented spaces, when the space of minimal prime ideals of $C(X)$ is compact*, Topology Appl. **141** (1-3) (2004) 147–170.
7. E. Matlis, *The minimal prime spectrum of a reduced ring*, Illinois J. Math. **27** (3) (1983) 353–391.

E-mail: mohamadian-r@scu.ac.ir
E-mail: Mzeinaly@yahoo.com

# Projective Vector Fields on the Cotangent Bundle of a Manifold

Mosayeb Zohrehvand*
Faculty of Mathematical Sciences and Statistic, Malayer University, Malayer, Iran

Abstract. Let $\nabla$ be a symmetric connection on an $n$-dimensional manifold $M_n$ and $T^*M_n$ its cotangent bundle. In this paper, firstly, we determine the fiber-preserving projective vector fields on $T^*M_n$ with respect to the Riemannian connection of the modified Riemannian extension $\tilde{g}_{\nabla,C}$, where $C$ is a symmetric $(0,2)$- tensor field on $M_n$. Then we prove that, if $(T^*M_n, \tilde{g}_{\nabla,C})$ admits a non-affine fiber-preserving projective vector field, then $M_n$ is locally flat, where $\nabla$ is the Levi-Civita connection of a Riemannian metric $g$ on $M_n$.
**Keywords:** Modified Riemannian extension, Fiber-preserving vector fields, Projective vector fields.
**AMS Mathematical Subject Classification [2010]:** 53C07, 53C22, 53B20.

## 1. Introduction

Let $M_n$ be a connected $n$-dimension manifold and $T^*M_n$ its cotangent bundle. We assume that the all geometric objects, which will be considered in this paper, are differentiable of class $C^\infty$. Also the set of all tensor fields of type $(r,s)$ on $M_n$ and $T^*M_n$ are denoted by $\mathrm{Im}^r_s(M_n)$ and $\mathrm{Im}^r_s(T^*M_n)$, respectively.

Let $\nabla$ be an affine connection on $M_n$. If a transformation on $M_n$ preserves the geodesics as point sets, then it is called projective transformation. Also, a transformation on $M_n$ which preserves the connection is called affine transformation. Therefore, an affine transformation is a projective transformation which preserves the geodesics with the affine parameter.

A vector field $V$ on $M_n$ with the local one-parameter group $\{\phi_t\}$ is called an infinitesimal projective (affine) transformation, if for every $t$, $\phi_t$ be a projective (affine) transformation on $M_n$.

It is well known that, a vector field $V$ is an infinitesimal projective transformation if and only if for every $X, Y \in \mathrm{Im}^1_0(M_n)$, we have

$$(L_V\nabla)(X,Y) = \Omega(X)Y + \Omega(Y)X,$$

where $\Omega$ is an one form on $M_n$ and $L_V$ is the Lie derivation with respect to $V$. In this case $\Omega$ is called the associated one form of $V$. One can see that $V$ is an infinitesimal affine transformation if and only if $\Omega = 0$ [10].

Now let $\tilde{\phi}$ be a transformation on $T^*M_n$. If $\tilde{\phi}$ preserves the fibers, then it is called the fiber-preserving transformation. Let $\tilde{V}$ be a vector field on $T^*M_n$ and $\{\tilde{\phi}_t\}$ the local one-parameter group generated by $\tilde{V}$. If $\tilde{\phi}_t$, for every $t$, be a fiber-preserving transformation, then $\tilde{V}$ is called an infinitesimal fiber-preserving transformation. Infinitesimal fiber-preserving transformations form a rich class of

*Speaker

infinitesimal transformations on $T^*M_n$ which include infinitesimal complete lift, horizontal lift and vertical lift transformations as special subclasses. For more details see [9].

Let $\nabla$ be a torsion free linear connection on $M_n$. Patterson and Walker defined a pseudo-Riemannian metric $\tilde{g}_\nabla$ on $T^*M_n$, the cotangent bundle of $M_n$, as follow

$$\begin{aligned}
&\tilde{g}_\nabla({}^H X, {}^H Y) = \tilde{g}_\nabla({}^V \omega, {}^V \theta) = 0,\\
&\tilde{g}_\nabla({}^H X, {}^V \omega) = \tilde{g}_\nabla({}^V \omega, {}^H X) = \omega(X),
\end{aligned}$$

where ${}^H X, {}^H Y$ and ${}^V \omega, {}^V \theta$ are horizontal and vertical lift of $X, Y \in \mathrm{Im}^1_0(M_n)$ and $\omega, \theta \in \mathrm{Im}^0_1(M_n)$, respectively [8]. The metric $\tilde{g}_\nabla$ is called the Riemannian extension of symmetric affine connection $\nabla$ and investigated by many authors [1, 2]. These metrics are interesting, because they are the simplest examples of non-Lorentzian Walker metrics. Walker metrics play a distinguished role in geometry and physics [6, 7].

In [3] a modification of Riemannian extension is defined that denoted by $\tilde{g}_{\nabla,C}$ where $C \in \mathrm{Im}^0_2(M_n)$ is a symmetric tensor field. In fact

$$\begin{aligned}
&\tilde{g}_{\nabla,C}({}^H X, {}^H Y) = C(X, Y),\\
&\tilde{g}_{\nabla,C}({}^H X, {}^V \omega) = \tilde{g}_{\nabla,C}({}^V \omega, {}^H X) = \omega(X),\\
&\tilde{g}_{\nabla,C}({}^V \omega, {}^V \theta) = 0,
\end{aligned}$$

$\tilde{g}_{\nabla,C}$ is a pseudo-Riemannian metric on $T^*M_n$ of signature $(n, n)$ and is called modified Riemannian extension.

The aim of this paper is to study of the infinitesimal fiber-preserving projective (IFP) transformations on $T^*M_n$ with respect to the Levi-Civita connection of the modified Riemannian extension $\tilde{g}_{\nabla,C}$ , where $C \in \mathrm{Im}^0_2(M_n)$ is a symmetric tensor field on $M_n$.

## 2. Preliminaries

Here, we give some of the necessary definitions and theorems on $M_n$ and $T^*M_n$, that are needed later. In this paper, indices $a, b, c, i, j, k, \ldots$ have range in $\{1, \ldots, n\}$.

Let $M_n$ be a manifold and covered by local coordinate systems $(U, x^i)$, where $x^i$ are the coordinate functions on the coordinate neighborhood $U$. The cotangent bundle of $M_n$ is defined by $T^*M_n := \bigcup_{x \in M} T^*_x(M_n)$, where $T^*_x(M_n)$ is the cotangent space of $M_n$ at a point $x \in M_n$. The induced local coordinate system on $T^*M_n$, from $(U, x^i)$, is denoted by $(\pi^{-1}(U), x^i, p_i)$, where $\pi : T^*M_n \to M_n$ is the natural projection and $p_i$ are the components of covector $p$ in each cotangent space $T^*_x(M_n)$, with respect to coframe $\{dx^i\}$.

Let $M_n$ be an $n$-dimensional manifold and $\nabla$ be a symmetric connection on $M_n$. The coefficients of $\nabla$ with respect to frame field $\{\partial_i := \frac{\partial}{\partial x^i}\}$ are denoted by $\Gamma^h_{ji}$, i.e. $\nabla_{\partial_j}\partial_i = \Gamma^h_{ji}\partial_h$.

Now, using the symmetric Connection $\nabla$, we can define the local frame field $\{E_i, E_{\bar{i}}\}$ on each induced coordinate neighborhood $\pi^{-1}(U)$ of $T^*M_n$, as follows

$$E_i := \partial_i + p_a \Gamma^a_{hi} \partial_{\bar{h}}, \quad E_{\bar{i}} := \partial_{\bar{i}},$$

where $\partial_{\bar{i}} := \frac{\partial}{\partial p_i}$. This frame field is called the adapted frame on $T^*M_n$ and can be useful for the tensor calculations on $T^*M_n$. The dual frame of $\{E_i, E_{\bar{i}}\}$ is $\{dx^h, \delta p_h\}$, where $\delta p_h := dp_h - p_b\Gamma^b_{hi}dx^i$.

Let $X$ be a vector field and $\omega$ be a covector field on $M_n$ that expressed by $X = X^i\partial_i$ and $\omega = \omega_i dx^i$ on a local coordinate system $(U, x^i)$, respectively. We can define vector fields horizontal lift ${}^H X$ and complete lift ${}^C X$ of $X$ and vertical lift ${}^V\omega$ of $\omega$ on $T^*M_n$ as follows

$$^H X := X^i E_i, \quad {}^C X := X^i E_i - p_a\nabla_i X^a E_{\bar{i}}, \quad {}^V\omega = \omega_i E_{\bar{i}},$$

where $\nabla_i := \nabla_{\partial_i}$.

An important class of vector fields on $T^*M_n$ is the fiber-preserving vector fields, which is determined in the following lemma.

Lemma 2.1. [9] *Let $\tilde{V} = \tilde{V}^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$ be a vector field on $T^*M_n$. Then $\tilde{V}$ is an infinitesimal fiber-preserving transformation if and only if $\tilde{V}^h$ are functions on $M_n$.*

Thus, the class of fiber-preserving vector fields is include horizontal lift, vertical lift and complete lift vector fields, and any fiber-preserving vector field $\tilde{V} = V^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$ on $T^*M_n$ induces a vector field $V := V^h\partial_h$ on $M_n$.

Now let $\nabla$ be a symmetric affine connection on $M_n$ and $\tilde{g}_{\nabla,C}$ be the modified Riemannian extension on $T^*M_n$. The coefficients of the Levi-Civita connection $\tilde{\nabla}$, of modified Riemannian extension $\tilde{g}_{\nabla,C}$, with respect to the adapted frame field $\{E_i, E_{\bar{i}}\}$ are computed in [4].

Lemma 2.2. [4] *Let $\tilde{\nabla}$ be the Riemannian connection of modified Riemannian extension $\tilde{g}_{\nabla,C}$, where $C \in \mathrm{Im}^0_2(M_n)$ is a symmetric tensor field on $M_n$, then we have*

$$\begin{aligned} &\tilde{\nabla}_{E_j} E_i = \Gamma^h_{ji} E_h + \left\{p_a R^a_{hji} + \tfrac{1}{2}(\nabla_i c_{hj} + \nabla_j c_{hi} - \nabla_h c_{ij})\right\} E_{\bar{h}},\\ &\tilde{\nabla}_{E_j} E_{\bar{i}} = -\Gamma^i_{jh} E_{\bar{h}}, \qquad \tilde{\nabla}_{E_{\bar{j}}} E_i = 0, \qquad \tilde{\nabla}_{E_{\bar{j}}} E_{\bar{i}} = 0,\end{aligned}$$

*where $\Gamma^h_{ji}$ and $R^h_{aji}$ are the coefficients of the symmetric affine connection $\nabla$ and the Riemannian curvature of $\nabla$, respectively and $\nabla_i := \nabla_{\partial_i}$.*

## 3. Main Results

Theorem 3.1. *Let $(M_n, \nabla)$ be a manifold with a symmetric(torsion free) affine connection $\nabla$ and $T^*M_n$ its cotangent bundle with the Riemannian connection of the modified Riemanian extension metric $\tilde{g}_{\nabla,C} = \tilde{g}_\nabla + \pi^*C$ where $C \in \mathrm{Im}^0_2(M_n)$ is a symmetric tensor field. Then $\tilde{V}$ is an infinitesimal fiber-preserving projective(IFP) transformation on $T^*M_n$, with the associated one form $\tilde{\Omega}$, if and only if there exist $\psi \in \mathrm{Im}^0_0(M_n)$, $V = (V^h) \in \mathrm{Im}^1_0(M_n)$, $B = (B_h) \in \mathrm{Im}^0_1(M_n)$ and $A = (A^i_h) \in \mathrm{Im}^1_1(M_n)$, satisfying*

1) $(\tilde{V}^h, \tilde{V}^{\bar{h}}) = (V^h, B_h + p_a A^a_h)$,
2) $(\tilde{\Omega}_i, \tilde{\Omega}_{\bar{i}}) = (\Psi_i, 0)$,
3) $\Psi_i = \partial_i\psi,\ \nabla_j\Psi_i = 0$,
4) $V^a\nabla_a R^h_{bji} + R^h_{bai}\nabla_j V^a + R^h_{bja}\nabla_i V^a + R^a_{bji}A^h_a - R^h_{aji}A^a_b = 0$,
5) $\nabla_i A^j_h = \Psi_i\delta^j_h - V^a R^j_{iah}$,

6) $L_V\Gamma^h_{ji} = \nabla_j\nabla_i V^h + V^a R^h_{aji} = \Psi_i\delta^h_j + \Psi_j\delta^h_i$,
7) $\nabla_j\nabla_i B_a + B_a R^a_{hji} = A^a_h M_{ija} - V^a\nabla_a M_{ijh} - M_{iah}\nabla_j V^a - M_{ajh}\nabla_i V^a$,

*where* $\tilde{V} = (\tilde{V}^h, \tilde{V}^{\bar{h}}) = \tilde{V}^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$, $\tilde{\Omega} = (\tilde{\Omega}_i, \tilde{\Omega}_{\bar{i}}) = \tilde{\Omega}_i dx^i + \tilde{\Omega}_{\bar{i}}\delta p_i$, $\nabla_i := \nabla_{\partial_i}$ *and* $M_{ijh} := \frac{1}{2}(\nabla_i c_{hj} + \nabla_j c_{hi} - \nabla_h c_{ij})$.

Proof. Firstly, we prove the necessary conditions. Let $\tilde{V} = V^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$ be an infinitesimal fiber-preserving projective transformation and $\tilde{\Omega} = \tilde{\Omega}_h dx^h + \tilde{\Omega}_{\bar{h}}\delta y^h$ its the associated one form on $T^*M_n$, thus for any $\tilde{X}, \tilde{Y} \in \mathrm{Im}^1_0(T^*M_n)$, we have

$$(L_{\tilde{V}}\tilde{\nabla})(\tilde{X}, \tilde{Y}) = \tilde{\Omega}(\tilde{X})\tilde{Y} + \tilde{\Omega}(\tilde{Y})\tilde{X}.$$

From

$$(L_{\tilde{V}}\tilde{\nabla})(E_{\bar{j}}, E_{\bar{i}}) = \tilde{\Omega}_{\bar{j}} E_{\bar{i}} + \tilde{\Omega}_{\bar{i}} E_{\bar{j}},$$

we have

$$\partial_{\bar{j}}\partial_{\bar{i}}\tilde{V}^{\bar{h}} = \tilde{\Omega}_{\bar{j}}\delta^h_i + \tilde{\Omega}_{\bar{i}}\delta^h_j. \tag{1}$$

Form (1) we obtain that, there exist $\Phi = (\Phi^i) \in \mathrm{Im}^1_0(M_n)$, $B = (B_h) \in \mathrm{Im}^0_1(M_n)$ and $A = (A^i_h) \in \mathrm{Im}^1_1(M_n)$ which are satisfied

$$\tilde{\Omega}_{\bar{i}} = \Phi^i, \tag{2}$$

$$\tilde{V}^{\bar{h}} = B_h + p_a C^a_h + p_h p_a \Phi^a. \tag{3}$$

From

$$(L_{\tilde{V}}\tilde{\nabla})(E_{\bar{j}}, E_i) = \tilde{\Omega}_{\bar{j}} E_i + \tilde{\Omega}_i E_{\bar{j}},$$

and (2) and (3) we have

$$\Big\{(\nabla_i A^j_h + V^a R^j_{iah}) + p_b\Big((\nabla_i \Phi^j \delta^b_h + \nabla_i \Phi^b \delta^j_h)\Big)\Big\} E_{\bar{h}} = \Phi^j \delta^h_i E_h + \tilde{\Omega}_i \delta^j_h E_{\bar{h}}. \tag{4}$$

Comparing the both sides of the Eq. (4), we see that

$$\Phi_i = 0, \tag{5}$$

$$\tilde{\Omega}_i = \Psi_i = \partial_i\psi, \tag{6}$$

$$\nabla_i A^j_h = V^a R^j_{aih} + \Psi_i \delta^j_h, \tag{7}$$

where $\psi := \frac{1}{n}A^a_a$.
Lastly from

$$(L_{\tilde{V}}\tilde{\nabla})(E_j, E_i) = \tilde{\Omega}_i E_j + \tilde{\Omega}_j E_i,$$

and (5), (6), (7) we obtain

$$\begin{aligned}\Psi_i E_j + \Psi_j E_i = \{\nabla_j\nabla_i V^h + V^a R^h_{aji}\} E_h + \Big\{&\nabla_j\nabla_i B_h + B_a R^a_{hij} + V^a\nabla_a M_{ijh} \\ &+ \nabla_i V^a M_{ajh} + \nabla_j V^a M_{iah} - A^a_h M_{ijh} + p_b\big(V^a\nabla_a R^b_{hji} + R^b_{hai}\nabla_j V^a \\ &+ R^b_{hja}\nabla_i V^a + R^a_{hji} A^b_h - R^b_{aji} A^a_h + \nabla_j \Psi_i \delta^b_h\big)\Big\} E_{\bar{h}},\end{aligned}$$

where $M_{ijh} := \frac{1}{2}\{\nabla_i c_{hj} + \nabla_j c_{hi} - \nabla_h c_{ij}\}$.
From which we have

$$L_V \Gamma^h_{ji} = \nabla_j \nabla_i V^h + V^a R^h_{aji} = \Psi_i \delta^h_j + \Psi_j \delta^h_i, \tag{8}$$
$$\nabla_j \nabla_i B_h + B_a R^a_{hij} = A^a_h M_{ijh} - V^a \nabla_a M_{ijh} - \nabla_i V^a M_{ajh} - \nabla_j V^a M_{iah},$$
$$V^a \nabla_a R^b_{hji} + R^b_{hai} \nabla_j V^a + R^b_{hja} \nabla_i V^a + R^a_{hji} A^b_h - R^b_{aji} A^a_h = 0,$$
$$\nabla_j \Psi_i = 0. \tag{9}$$

This completes the necessary conditions. The proof of the sufficient conditions are easy. □

Now let $\nabla$ be the Levi-Civita connection of a Riemannian metric $g$ on $M_n$. In this case we have the following theorem.

Theorem 3.2. *Let $(M_n, g)$ be a complete $n$-dimensional Riemannian manifold and $T^*M_n$ its cotangent bundle with the Riemannian connection of the modified Riemannian extension metric $\tilde{g}_{\nabla,C} = \tilde{g}_\nabla + \pi^* C$ where $C \in \mathrm{Im}^0_2(M_n)$ is a symmetric tensor field and $\nabla$ is the Levi-Civita connection of $g$. If $(T^*M_n, \tilde{g}_{\nabla,C})$ admits a non-affine infinitesimal fiber-preserving projective transformation then $M_n$ is locally flat.*

Proof. Let $\tilde{V}$ be a non-affine infinitesimal fiber-preserving projective transformation on $(T^*M_n, \tilde{g}_{\nabla,C})$. It is easy to see that $\Psi := (\Psi_i)$ is a nonzero one form on $M_n$ and $\|\Psi\|$ is a constant function.
We put $X := (\nabla_a V^h - A^h_a)\Psi^a$, where $\Psi^a := g^{ai}\Psi_i$. Using of (7), (8) and (9) one can see that

$$\begin{aligned} L_X g_{ji} = \nabla_j X_i + \nabla_i X_j &= (\nabla_j \nabla_a V_i - \nabla_j A_{ia})\Psi^a + (\nabla_i \nabla_a V_j - \nabla_i A_{ja})\Psi^a \\ &= 2(\Psi_a \Psi^a) g_{ji} = 2\|\Psi\| g_{ji}. \end{aligned}$$

This means that $X$ is an infinitesimal non-isometric homothetic transformation on $M_n$. In [5] it is proved that if a complete Riemannian manifold $(M_n, g)$ admits an infinitesimal non-isometric homothetic transformation then $(M_n, g)$ is locally flat. Therefore $M_n$ is locally flat. □

## References


1. Z. Afifi, *Riemann extensions of affine connected spaces,* Q. J. Math. **5** (1) (1954) 312–320.
2. S. Aslanci, S. Kazimova and A. A. Salimov, *Some remarks concerning Riemannian extensions,* Ukrainian Math. J. **62** (5) (2010) 661–675.
3. E. Calvino-Louzao, E. Garca-Ro, P. Gilkey and A. Vazquez-Lorenzo, *The geometry of modified Riemannian extensions,* Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **465** (2107) (2009) 2023–2040.
4. A. Gezer, L. Bilen and A. Cakmak, *Properties of modified Riemannian extensions,* Zh. Mat. Fiz. Anal. Geom. **11** (2015) 159–173.
5. S. Kobayashi, *A theorem on the affine transformation group of a Riemannian manifold,* Nagoya Math. J. **9** (1955) 39–41.
6. P. Law and Y. Matsushita, *A spinor approach to walker geometry,* Commun. Math. Phys. **282** (2008) 577–623.
7. M.A. Magid, *Indefinite Einstein hypersurfaces with nilpotent shape operators,* Hokkaido Math. J. **13** (1984) 241–250.
8. E. M. Patterson and A. G. Walker, *Riemann extensions,* Quart. J. Math. **3** (1) (1952) 19–28.

9. K. Yamauchi, *On infinitesimal conformal transformations of the tangent bundles over Riemannian manifolds,* Ann. Rep. Asahikawa. Med. Coll. **16** (1995) 1–6.
10. K. Yano, *The Theory of Lie Derivatives and Its Applications*, Bibliotheca mathematica, North Holland Pub. Co., 1957.

E-mail: m.zohrehvand@malayeru.ac.ir; m.zohrevand61@gmail.com

# Contributed Talks

## Graphs and Combinatorics

# On Covering Set of Dominated Coloring in Some Graph Operations

Fatemeh Choopani*
Department of Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran
and Doost Ali Mojdeh
Department of Mathematics, University of Mazandaran, Babolsar, Iran

Abstract. The dominated coloring of a graph G is a proper coloring of $G$ such that each color class is dominated by at least one vertex. The dominated chromatic number of a graph $G$ is the smallest number of colors needed to color the vertices of $G$ by this way, denoted by $\chi_{dom}$. In this paper, we define the covering set related to $\chi_{dom}$ as a new concept. For a minimum dominated coloring of $G$, a set of vertices $S$ is called a covering set of dominated coloring if each color class is dominated by a vertex of $S$. We call the minimum cardinality of a covering set of dominated coloring of $G$, covering number and we denote by $\theta_{\chi_{dom}}$. We obtain some bounds on $\theta_{\chi_{dom}}$ and finally we study about covering number of subdivision, middle and total graph of paths and cycles.
**Keywords:** Dominated coloring, Dominated chromatic number, Covering set, Covering number.
**AMS Mathematical Subject Classification [2010]:** 05C69, 05C15.

## 1. Introduction

Let $G = (V, E)$ be a simple, undirected, and finite graph of order $n$. A subset $S \subseteq V$ is a dominating set of $G$ if every vertex in $V - S$ has a neighbor in $S$ and is a total dominating set, if every vertex in $V$ has a neighbor in $S$. The domination number (total domination number) of $G$, denoted by $\gamma(G)$ ($\gamma_t(G)$), is the minimum cardinality of a dominating set (total dominating set).

A support vertex is defined as a vertex adjacent to a leaf and a leaf or a pendant vertex is a vertex of degree 1 in a tree. A support vertex with one pendant vertex (one leaf) is called a weak support vertex, while a strong support vertex is a support vertex with at least two pendant vertices (two leaves). The $k$-th power of $G$, $G^k$, is a graph whose vertex set is that of $G$ and two vertices in it are adjacent if their distance in $G$ is at most $k$. The graph $G^2$ is also referred to as the square of $G$. The subdivision graph $S(G)$ is a graph obtained from $G$ by subdividing of each edge exactly once.

The Middle graph $M(G)$ of a graph $G$ is defined as a graph with vertex set $V \cup E$ and two vertices x and y of $M(G)$ are adjacent in $M(G)$ if either $x$ and $y$ are adjacent edges in $G$ or $x$ is a vertex in $G$, $y$ is an edge of $G$ and they are incident in $G$. The total graph $T(G)$ of a graph $G$ is a graph with the vertex set $V \cup E$ in which two vertices $x$ and $y$ of $T(G)$ are adjacent in $T(G)$ if either they are adjacent

*Speaker

vertices or adjacent edges in $G$ or $x$ is a vertex in $G$, $y$ is an edge of $G$ and they are incident in $G$. In this paper, we take the vertices set of middle and total graphs as a sequence of vertices in the form of $\{v_1, e_1, v_2, \dots, v_i, e_i, v_{i+1}, \dots v_n\}$ that $e_i$ is between $v_i$ and $v_{i+1}$.

A proper coloring of $G$ is an assignment of colors to the vertices of $G$ such that no two adjacent vertices are assigned the same color. A proper coloring of $G$ with $k$ colors is also called a $k$-proper coloring of $G$. Merouane et al. [5], defined the dominated coloring of a graph as follows.

A $k$-dominated coloring of $G$ is a proper $k$-coloring of $G$ with color classes $C_1, C_2, \dots, C_k$ such that for each $i$ $(1 \leq i \leq k)$, there exists a vertex $u \in V$ and $C_i \subseteq N(u)$ (i.e. vertices in $C_i$ are dominated by vertex $u$); such vertex $u$ is called a dominating vertex. The minimum number of colors among all dominated colorings of $G$ is called its dominated chromatic number, denoted by $\chi_{dom}(G)$. Obviously, a graph has a dominated coloring if it has no isolated vertices. Therefore, hereafter we assume that graphs in the paper have no isolated vertex. The $k$-dominated coloring has also been studied by Choopani et al. in [1].

Now we define a variant of dominated coloring, namely covering set of dominated coloring that is defined as follows.

Definition 1.1. Let $C_1, C_2, \dots, C_{\chi_{dom}}$ be the color classes of a minimum dominated coloring of $G$. A set $S \subseteq V$ is called a *covering set of dominated coloring* of graph $G$ if every $C_i$ is dominated by a vertex in $S$. The minimum cardinality of such set $S$ is called the covering number of dominated chromatic of $G$, denoted by $\theta_{\chi_{dom}(G)}$, and the set $S$ is called a $\theta_{\chi_{dom}}(G)$-set.

It is clear that $\theta_{\chi_{dom}}(G) \leq \chi_{dom}(G)$. In the next section, we investigate the properties of $\theta_{\chi_{dom}}(G)$.

## 2. Main Results

### 2.1. Some Existence Results.

Proposition 2.1. *For any graph $G$, $\gamma(G) \leq \theta_{\chi_{dom}}(G)$.*

In the following theorem, we state that the difference $\theta_{\chi_{dom}}(G) - \gamma(G)$ can be arbitrarily large.

Theorem 2.2. *If $k$ is a non-negative integer, then there exists graph $G$ for which, the covering number of dominated chromatic $\theta_{\chi_{dom}}(G) = a$ and domination number $\gamma(G) = b = a - k$.*

Proposition 2.3. *If $a$ and $b$ are two integers with $a \geq b \geq 2$, then there exists a graph $G$ with dominated chromatic number $\chi_{dom}(G) = a$ and covering number of dominated chromatic $\theta_{\chi_{dom}}(G) = b$.*

**2.2. Some Bounds on the Covering Number of Dominated Coloring of Graphs.** In this section, we find some bounds for $\chi_{dom}$ of a graph $G$. Let $G$ be a graph with $\chi_{dom}$-dominated colored, and $C_1, C_2, \dots, C_{\chi_{dom}}$ as the corresponding color classes. Assume that each color class $C_i$ is dominated by the vertex $u_i$.
Since every color class $C_i$ with at least two members includes only independent vertices, the class $C_i$ must be dominated by a vertex out of $C_i$. Thus we have the following proposition.

Proposition 2.4. *If for a graph $G$, $u_i \in C_i$, $(1 \leq i \leq \chi_{dom})$ and $u_i$ dominates the color class $C_i$, then $C_i = \{u_i\}$.*

Let $v$ be a vertex in $G$ with $deg(v) = |V(G)| - 1$. Then $\gamma(G) = 1$ and $\{v\}$ is a $\gamma(G)$-set. This shows that every color class of a $\chi_{dom}(G)$ dominated coloring, is dominated by the vertex $v$. Thus we may have:

Proposition 2.5. *$\theta_{\chi_{dom}}(G) = 1$ if and only if $\Delta(G) = |V(G)| - 1$.*

Lemma 2.6. *For $n \geq 4$,*

$$\chi_{dom}(P_n) = \chi_{dom}(C_n) = \begin{cases} \frac{n}{2} + 1, & \text{if } n \equiv 2 \pmod 4, \\ \lceil \frac{n}{2} \rceil, & \text{otherwise.} \end{cases}$$

Theorem 2.7. *For $n \geq 4$,*

$$\theta_{\chi_{dom}}(P_n) = \theta_{\chi_{dom}}(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil, & \text{if } n \equiv 0,\ 1 \pmod 4, \\ \lceil \frac{n}{2} \rceil - 1, & \text{otherwise.} \end{cases}$$

Proposition 2.8. *If $\theta_{\chi_{dom}}(G) = 2$, then $2 \leq diam(G) \leq 5$.*

Proposition 2.9. *Let $G$ be a connected graph for which every vertex is support vertex or a pendant vertex. If $s \geq 2$ is the cardinality of support vertices, then $\chi_{dom}(G) = \theta_{\chi_{dom}}(G) = s$.*

In what follows we obtain a sharp bound for $\chi_{dom}(T)$ and $\theta_{\chi_{dom}(T)}$. At first, we pose the following theorem from [2].

Theorem 2.10. [2, Theorem 4.1] *Let $T$ be a tree with $n \geq 3$ vertices, $l$ leaves, and $s$ support vertices. Then, $\frac{n+2-l}{2} \leq \gamma_t(T) \leq \frac{n+s}{2}$.*

Proposition 2.11. [5] *$\chi_{dom}(G) \geq \gamma_t(G)$. Also if $G$ is a triangle-free graph, then $\chi_{dom}(G) = \gamma_t(G)$.*

Now from Proposition 2.11 we have.

Corollary 2.12. *Let $T$ be tree. Then $\frac{n+2-l}{2} \leq \chi_{dom}(T) \leq \frac{n+s}{2}$.*

Let $G = (V, E)$ be a graph. For every support vertex $u \in V$, delete all the leaves from $N(u)$ except one. The remaining graph is called the *pruned sub graph* (or *pruned sub tree*, if $G$ is a tree) of $G$ and is denoted by $G_p$ [3, 4]. The next result shows that, for any graph $G$, the number of end vertices for a support vertex dose not affect to the $\chi_{dom}$-coloring and $\theta_{\chi_{dom}}$-covering of $G$, in the other words $\chi_{dom}(G) = \chi_{dom}(G_p)$ and $\theta_{\chi_{dom}}(G) = \theta_{\chi_{dom}}(G_p)$.

Proposition 2.13. *Let $G$ be a graph with support vertices $v_1, v_2, \ldots, v_k$ and $L_{v_i}$ be the set of end vertices corresponding to $v_i$. Let $H = G_p$ be a pruned sub graph of $G$. Then $\chi_{dom}(H) = \chi_{dom}(G)$ and $\theta_{\chi_{dom}}(H) = \theta_{\chi_{dom}}(G)$.*

Theorem 2.14. *Let $T$ be tree of order $n$ with $s$ weak support vertices and $s$ leaves. Then $s \leq \theta_{\chi_{dom}(T)} \leq \lceil \frac{n}{2} \rceil$. The upper bound is sharp. For lower bound, the equality holds if and only if*

(1) *every vertex is a support vertex or a leaf. Otherwise,*
(2) *the neighbor of an end support vertex is a leaf or another support vertex,*

(3) *a vertex that is not support vertex and leaf, has at most distance* 1 *with at least one support vertex.*

COROLLARY 2.15. *Let $T$ be tree of order $n$ with $s$ weak support vertices and $l$ leaves. Then $s \leq \theta_{\chi_{dom}(T)} \leq \lceil \frac{n}{2} \rceil$. The bounds are sharp.*

**2.3. Covering Number of Dominated Chromatic of $S(G)$ and $M(G)$.** In this section, we study the dominated chromatic number and covering number of dominated chromatic of graphs $S(P_n)$, $S(C_n)$, $M(P_n)$, $M(C_n)$ of $n$-path $P_n$, the $n$-cycle $C_n$.

THEOREM 2.16.

i) *For $n \geq 2$,*
   1) $\chi_{dom}(S(P_n)) = n$.
   2) $\theta_{\chi_{dom}}(S(P_n)) = \begin{cases} n-1, & \text{if } n \text{ is even},\\ n, & \text{if } n \text{ is odd}.\end{cases}$

ii) *For $n \geq 3$,*
   1) $\chi_{dom}(S(C_n)) = \begin{cases} n+1, & \text{if } n \text{ is odd},\\ n, & \text{if } n \text{ is even}.\end{cases}$
   2) $\theta_{\chi_{dom}}(S(C_n)) = \begin{cases} n-1, & n \text{ is odd},\\ n, & n \text{ is even}.\end{cases}$

THEOREM 2.17. *For $n \geq 2$, $\chi_{dom}(M(P_n)) = n$ and*

$$\theta_{\chi_{dom}}(M(P_n)) = \begin{cases} \frac{2n}{3}, & n \equiv 0 \pmod 3,\\ \lceil \frac{2n}{3} \rceil, & n \equiv 1 \pmod 3,\\ \lfloor \frac{2n}{3} \rfloor, & n \equiv 2 \pmod 3.\end{cases}$$

THEOREM 2.18. *For $n \geq 3$, $\chi_{dom}(M(C_n)) = n$ and*

$$\lfloor 2n/3 \rfloor \leq \theta_{\chi_{dom}}(M(C_n)) \leq \lceil 3n/4 \rceil.$$

**2.4. Covering Number of Dominated Chromatic of $T(G)$.**

THEOREM 2.19. *Let $P_n$ and $C_n$ be paths and cycles with $n$ vertices. Then*

i) *For $n \geq 2$,* $\chi_{dom}(T(P_n)) = \begin{cases} n, & n \equiv 0,1 \pmod 3,\\ n+1, & n \equiv 2 \pmod 3.\end{cases}$

ii) *For $n \geq 3$,* $\chi_{dom}(T(C_n)) = \begin{cases} n, & n \equiv 0 \pmod 3,\\ n+1, & n \equiv 1,2 \pmod 3.\end{cases}$

THEOREM 2.20. *For $n \geq 2$,*

$$\theta_{\chi_{dom}}(T(P_n)) = \begin{cases} \frac{2n}{3} - 1, & n \equiv 0 \pmod 3,\\ \lfloor \frac{2n}{3} \rfloor, & n \equiv 1 \pmod 3,\\ \lfloor \frac{2n}{3} \rfloor - 2, & n \equiv 2 \pmod 3.\end{cases}$$

THEOREM 2.21. *For $n \geq 3$,*

$$\theta_{\chi_{dom}}(T(C_n)) = \begin{cases} \frac{2n}{3}, & n \equiv 0 \pmod 3, \\ \lceil \frac{2n}{3} \rceil, & n \equiv 1 \pmod 3, \\ \lfloor \frac{2n}{3} \rfloor - 1, & n \equiv 2 \pmod 3. \end{cases}$$

## References

1. F. Choopani, A. Jafarzadeh, A. Erfanian and D. A. Mojdeh, *On dominated coloring of graphs and some Nardhaus-Gaddum-type relations*, Turk. J. Math. **42** (2018) 2148–2156.
2. M. H. Henning and A. Yeo, *Total Domination in Graphs*, Springer-Verlag, New York, 2013.
3. D. A. Mojdeh and M. Alishahi, *Outer independent global dominating set of trees and unicyclic graphs*, Electron. J. Graph Theory Appl. **7** (1) (2019) 121–145.
4. D. A. Mojdeh, M. Alishahi and M. Chellali, *Trees with the same global domination number as their square*, Australas. J. Combin. **66** (2) (2016) 288–309.
5. H.B. Merouane, M. Haddad, M. Chellali and H. Kheddouci, *Dominated colorings of graphs*, Graphs Combin. **31** (2015) 713–727.

E-mail: damojdeh@umz.ac.ir
E-mail: fateme.choopani@mail.um.ac.ir

# Maximum Fractional Forcing Number of the Products of Cycles

Javad Ebrahimi Boroujeni
Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran, and, Institute for Studies in Theoretical Physics and Mathematics (IPM), Tehran, Iran
Sajed Karimy*
Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran
and Shayan Ranjbarzadeh
Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran

Abstract. In this work, we find upper and lower bounds on the maximum fractional forcing number of the Cartesian product of even cycles of the same lengths. Our results can extend the result of [2] about the maximum forcing number of $C_{2n}\square C_{2n}$ to that of the product of an arbitrary number of even cycles of the same lengths.
**Keywords:** Fractional perfect matching, Forcing number, Fractional forcing number, Cartesian product of graphs, Perfect matching.
**AMS Mathematical Subject Classification [2010]:** 05C70, 05C72, 05C92.

## 1. Introduction

The notion of the forcing number of a perfect matching, also known as the innate degree of freedom of Kekule structures in chemistry, is an important parameter of graphs due to its exciting theoretical properties as well as application aspects such as computational chemistry.

There are numerous publications about related parameters such as the exact or the approximate value of the maximum or minimum forcing number of all possible perfect matchings of members of certain families of graphs. For instance, in [2], the problem of finding the maximum forcing number among all the perfect matchings in the Cartesian product of two cycles has been investigated.

In [1], Ebrahimi et al. defined the fractional version of the forcing number and proved several analytic properties of this parameter. Built on their results, we obtain upper and lower bounds on the maximum fractional forcing number of the Cartesian products of even cycles. Our result, in particular, provides an upper bound on the maximum forcing number of such graphs which itself can be regarded as a generalization of the result of [2, Corollary 4.6].

*Speaker

## 2. Preliminaries

Let $G$ be a graph, where the set of *vertices* and *edges* are denoted by $V(G)$ and $E(G)$, respectively. Each edge is an unordered pair of vertices. If $\{v_i, v_j\} \in E(G)$, we write $v_i \overset{G}{\sim} v_j$ and we say $v_i$ is *neighbor* of $v_j$.

Definition 2.1. The function $\gamma : E(G) \to \mathbb{R}^{\geq 0}$ is called a *fractional matching* (FM, for short) if for every vertex $v \in V(G)$, $\sum_{e:v\in e} \gamma(e) \leq 1$. $\gamma$ is called a *fractional perfect matching* (FP, for short) if for every vertex $v \in V(G)$, $\sum_{e:v\in e} \gamma(e) = 1$. Note that every integral FP $\gamma$ is a perfect matching. By Supp($\gamma$), we mean the set of all the edges $e$ with $\gamma(e) \neq 0$.

Definition 2.2. Let $G$ be a graph and $\alpha, \alpha' : E(G) \to \mathbb{R}^{\geq 0}$ be two functions. Define the partial order "$\preceq$" on the set $(\mathbb{R}^{\geq 0})^E$ as follows.

$$\alpha \preceq \alpha' \iff \forall e \in E(G) : \alpha(e) \leq \alpha'(e).$$

Let $\alpha$ be an FM and $\gamma$ be an FP in a graph $G$. We say $\alpha$ is *extendable* to $\gamma$ if $\alpha \preceq \gamma$. $\alpha$ is a *forcing function* for $\gamma$ if $\alpha$ is uniquely extendable to $\gamma$, and we write $\alpha \uparrow \gamma$. $\alpha$ is a *minimal forcing function* if $\alpha \uparrow \gamma$ and, whenever $\alpha' \preceq \alpha$ and $\alpha' \uparrow \gamma$ then $\alpha = \alpha'$. In this case we write $\alpha \Uparrow \gamma$.

Definition 2.3. Let $G$ be a graph and $\gamma$ be any FP of $G$. We define the quantities *fractional forcing number of $\gamma$ in G*, *minimum fractional forcing number of G*, and *maximum fractional forcing number of G*, respectively, as follows,

$$\begin{aligned} f_f(G,\gamma) &:= \min_{\alpha:\alpha\Uparrow\gamma} \sum_{e\in E} \alpha(e), \\ f_f(G) &:= \min\{f_f(G,\gamma) : \gamma \text{ is an FP of } G\}, \\ F_f(G) &:= \max\{f_f(G,\gamma) : \gamma \text{ is an FP of } G\}. \end{aligned}$$

Definition 2.4. The Cartesian product of two graphs $G_1$ and $G_2$, denoted by $G_1 \Box G_2$, is a graph with vertices $V = V(G_1) \times V(G_2)$ such that

$$(v_1, v_2) \sim (u_1, u_2) \iff [(v_1 = u_1) \wedge (v_2 \overset{G_2}{\sim} u_2)] \vee [(v_2 = u_2) \wedge (v_1 \overset{G_1}{\sim} u_1)].$$

We denote the product of $k$ copies of $G$ with $G^k$, e.g., $G^2 = G \Box G$.

Let $C_n$ denote the cycle graph with $n$ vertices. Observe that the graph $C_n^k$ is a graph with the vertex set $V = \mathbb{Z}_n^k$ as vertices, where $\mathbb{Z}_n$ is the additive group of integers modulo $n$. We assume $n \geq 3$, to keep the graph simple. Indeed, every edge is the set $\{v, v + e_i\}$ or $\{v, v - e_i\}$ for some $v \in V$ and $i \in \mathbb{Z}_k$, where $e_i$ is an $n$-tuple with a 1 in the $i$th coordinate and 0's elsewhere. One can see that $C_n^k$ is *edge transitive* and *vertex transitive*, where the edge (vertex) transitive graph is a graph for which its automorphism group acts transitively on the set of edges (vertices). Furthermore, $C_n^k$ is bipartite if and only if $n$ is even.

## 3. Main Results

Our main result is to find lower and upper bounds on maximum fractional forcing number of $C_{2n}^k$. Since $C_n^k$ has the perfect matching only for even $n$, we only consider the graphs $C_{2n}^k$. Notice that $C_{2n}^k$ is also a bipartite graph.

Let $G$ be a bipartite graph. Due to Ebrahimi and Ghanbari [1, Theorem 22], $f_f(G) = f(G)$, where $f(G)$ denotes the minimum forcing number of $G$, i.e., the minimum over integral minimal forcing functions.

THEOREM 3.1. *For every $n \geq 2$ and $k \geq 1$, $\frac{k-1}{2k}(2n)^k + \frac{1}{k} \leq F_f(C_{2n}^k) \leq \frac{k-1}{2k}(2n)^k + \frac{1}{k}n^{k-1}$.*

We recall the following lemmas.

LEMMA 3.2. [1, Lemma 12] *Let $G$ be a graph, $\gamma$ be an FP and $\alpha \Uparrow \gamma$, then for every edge $e \in E(G)$, $\alpha(e) \in \{0, \gamma(e)\}$.*

LEMMA 3.3. [1, Corollary 34] *Let $G$ be a vertex and edge transitive graph. Then, the FP that assigns the value $\frac{1}{deg(v)}$ to all the edges, has the maximum fractional forcing number.*

Furthermore, we use a special case of [1, Theorem 15].

LEMMA 3.4. *Let $G$ be a bipartite graph, $\gamma$ be an FP that $Supp(\gamma) = E(G)$ and $S \subseteq E(G)$. There exist an FM $\alpha$ such that $\alpha \uparrow \gamma$ and $Supp(\alpha) = S$ if and only if for every cycle $C$ of $G$ with 2-coloring of the edges of $C$, each color class intersects $S$. Equivalently, for every cycle $C$ of $G$, there are 2 edges in $C \cap S$ with even distance in $C$, i.e. number of edges between them in $C$ is even.*

Note that for a vertex and edge transitive graph $G$, a vertex $v \in V(G)$, and the foresaid FP $\gamma$ in Lemma 3.3, by finding a set $S$ having the mentioned condition in Lemma 3.4, it follows that,

$$\begin{aligned} F_f &\leq \sum_{e \in S} \frac{1}{deg(v)} \\ &= \frac{|S|}{deg(v)}. \end{aligned} \tag{1}$$

Now we are ready to sketch the proof of Theorem 3.1.

PROOF OF THEOREM 3.1. For the lower bound, first observe that $G$ has the following properties.

1) $|V(G)| = (2n)^k$,
2) $\forall v \in V(G): \ deg(v) = 2k$,
3) $|E(G)| = k(2n)^k$.

So if $|S| \leq (k-1)(2n)^k$, the graph remaining by removing edges in $S$ from $E(G)$ has a cycle. Because it still has $(2n)^k$ edges, i.e., the same number of vertices, and a jungle with $l$ vertices has at most $l - 1$ edges. Consequently, we must have $|S| \geq (k-1)(2n)^k + 2$ to have intersection in two edges with the foresaid cycle. By considering weight of each edge, Lemma 3.4, and Lemma 3.2, it implies that $\frac{k-1}{2k}(2n)^k + \frac{1}{k} \leq F_f(C_{2n}^k)$.

For the upper bound, by Eq. (1), we need to find $S_{2n,k}$ for the FP $\gamma$, that assigns the value $\frac{1}{deg(v)}$ to all the edges. We inductively define $S_{2n,k}$ as follows.

$$S_{2n,k} := \begin{cases} \{\{0,1\},\{1,2\}\} & k=1 \\ \bigcup_{i=1}^{4} A_{2n,k,i} & k>1 \end{cases},$$

$$A_{2n,k,i} := \begin{cases} \Big\{\{(v,a),(u,a)\} : \ [\{v,u\} \in E(C_{2n}^{k-1})] \wedge [a \equiv 1 \mod 2]\Big\} & i=1 \\ \Big\{\{(v,a),(u,a)\} : \ [\{v,u\} \in S_{2n,k-1})] \wedge [a \equiv 0 \mod 2]\Big\} & i=2 \\ \Big\{\{v, v+e_k\} : \ [v_k \in \{1\} \cup \{2,4,\dots,2n-2\}] \wedge [\sum_{j=1}^{k-1} v_j \equiv 0 \mod 2]\Big\} & i=3 \\ \Big\{\{v, v+e_k\} : \ [v_k \in \{0\} \cup \{3,5,\dots,2n-1\}] \wedge [\sum_{j=1}^{k-1} v_j \equiv 1 \mod 2]\Big\} & i=4 \end{cases},$$

where $v$ and $u$ are two $(k-1)$-tuples in cases that $i=1$ and $i=2$, and are two $k$-tuples when $i=3$ and $i=4$, and the $j$th coordinate is denoted by $v_j$ and $u_j$. First, we compute the size of $S_{2n,k}$.

$$\begin{aligned} |S_{2n,k}| &= \sum_{i=1}^{4} |A_{2n,k,i}| \\ &= n \times |E(C_{2n}^{k-1})| + n \times |S_{2n,k-1}| + 2 \times |A_{2n,k,3}| \\ &= n(k-1)(2n)^{k-1} + n \times |S_{2n,k-1}| + 2 \times (n \times ((2n)^{k-2}n)) \\ &= \frac{k}{2}(2n)^k + n \times |S_{2n,k-1}|. \end{aligned}$$

By induction on $k$, we prove that $|S_{2n,k}| = (k-1)(2n)^k + 2n^{k-1}$. For the base case, where $k=1$, the statement is correct clearly, i.e., $S_{2n,1} = 2$. For the induction step, note that

$$\begin{aligned} |S_{2n,k}| &= \frac{k}{2}(2n)^k + n \times |S_{2n,k-1}| \\ &= \frac{k}{2}(2n)^k + n((k-2)(2n)^{k-1} + 2n^{k-2}) \\ &= (k-1)(2n)^k + 2n^{k-1}. \end{aligned}$$

Now by substituting the size of $S_{2n,k}$ in (1), we get our upper bound.

$$\begin{aligned} F_f &\leq \frac{1}{deg(v)}|S_{2n,k}| \\ &= \frac{k-1}{2k}(2n)^k + \frac{1}{k}n^{k-1}. \end{aligned}$$

Let us prove that $S_{2n,k}$ intersects with each class of any 2-coloring cycle in $G$. We induct on $k$.

Base case: Note that the only cycle of this graph is itself and $S_{2n,1}$ is two sequential edges. Therefore $S_{2n,1}$ intersects with each color class.

Before the inductive step, we claim following lemma.

Lemma 3.5. *If for a vertex $v$, $v_k = 1$, then, all but one of the edges containing $v$ are in $S_{2n,k}$.*

Note that $2n-2$ edges containing $v$ are in $A_{2n,k,1}$ and the other is in $A_{2n,k,3} \cup A_{2n,k,4}$.

Induction step: Consider a cycle $C : v_1, \dots, v_l$ in $C_{2n}^k$. Let $v_{i,j}$ be the $j$th coordinate of $v_i$.

(Case I) $v_{1,k} = \cdots = v_{l,k}$: First, if $v_{1,k} \equiv 1 \pmod 2$, then, all the edges of $C$ are in $A_{2n,k,1}$. Also if $v_{1,k} \equiv 0 \pmod 2$, by induction hypothesis, $C$ intersects $A_{2n,k,2}$ in two edges with even distance.

(Case II) For some $i$, $v_{i,k} = 2$ or $v_{i,k} = 0$: We may assume that $v_{i,k} = 2$. The other case is similar. also assume that $v_1, \ldots, v_s$ be the longest path with the property that for any $j \in [s]$, $v_{i,k} = 2$. We argued the case $s = l$ in I and also $s$ can not be $l - 1$. Therefore, $s \leq l - 2$. Let $S(v_i)$ be $\sum_{j=1}^{k-1} v_{i,j} \pmod 2$. Note that $|S(v_1) - S(v_s)| = w$, where $w$ is parity of the length of the path. If $S(v_1) = S(v_s) = 0$, then, $\{v_1, v_l\}, \{v_s, v_{s+1}\} \in A_{2n,k,3}$ and parity of their distance in $C$, i.e. w, is an even number and the assertion is valid. If $S(v_1) = S(v_s) = 1$, then, $\{v_1, v_l\}, \{v_s, v_{s+1}\} \notin S_{2n,k}$. In addition, $v_{s+1,k} \equiv v_{l,k} \equiv 1$. By lemma 3.5, the next edges, i.e., $\{v_{s+1}, v_{s+2}\}$ and $\{v_l, v_{l-1}\}$, are in $S_{2n,k}$ and since, parity of their distance in $C$, i.e., $w + 2$, is an even number, the assertion is valid. Note that there is at least two edges between $v_{s+1}$ and $v_l$ in $C$. Since, if $v_1 = v_s$, then $|v_{l,k} - v_{s+1,k}| \geq 2$ and if $v_1 \neq v_s$, then $S(v_l) = S(v_{s+1}) = 1$ and consequently their Hamming distance in both cases is more than two. Finally, in the case which $w = 1$, without loss of generality, assume that $S(v_1) = 0$. Then, as before, $\{v_1, v_l\} \in A_{2n,k,3}$ and $\{v_{s+1}, v_{s+2}\} \in S_{2n,k}$, and since, parity of their distance, i.e., $w + 1$, is an even number, the assertion is valid.

(Case III) Complementary of cases I and II : By not concerning case one, there is a vertex $u$ in $C$ that its $k$th coordinate, is odd. Let $v_1, \ldots, v_s$ be a path in $C$ where $v_{1,k} = v_{s,k} = u_k$ and for any $j$ that $1 < j < s$, $v_{j,k} \neq u_k$. Since for any $j$, $v_{j,k} \neq 2$ and $v_{j,k} \neq 0$, we have $v_{1,k} \neq 1$. Also it implies that $v_1 \neq v_s$ and consequently, such path exists. In addition, parity of the length of this path is still $w := |S(v_1) - S(v_s)|$, since $v_{1,k} = v_{s,k}$ and the number of the edges $\{v, v + e_k\}$ and $\{v, v - e_k\}$ are the same and consequently, sum of them is an even number. By similarity of the cases, we assume that $v_1 + e_k = v_2$ and $v_s + e_k = v_{s-1}$. If $S(v_1) = S(v_s) = 1$, then, $\{v_1, v_2\}$ and $\{v_{s-1}, v_s\}$ are in $A_{2n,k,4}$, and since, parity of their distance in $C$, i.e., $w$, is an even number, the assertion is valid. If $S(v_1) = S(v_s) = 0$, then, $\{v_1, v_2\}$ and $\{v_{s-1}, v_s\}$ are not in $S_{2n,k}$, and by 3.5, $\{v_l, v_1\}$ and $\{v_s, v_{s+1}\}$ are in $S_{2n,k}$. Since, parity of their distance in $C$, i.e., $w$, the assertion is valid. Finally, In the case which $w = 1$, without loss of generality, assume that $S(v_1) = 1$. Then, as before, $\{v_1, v_2\} \in A_{2n,k,4}$ and $\{v_s, v_{s+1}\} \in S_{2n,k}$, and since, their distance, i.e., $w - 1$, is an even number, the assertion is valid. □

## References


1. J. B. Ebrahimi and B. Ghanbari, *Fractional forcing number of graphs*, (2020). arXiv:2011.03087
2. S. Kleinerman, *Bounds on the forcing numbers of bipartite graphs*, Discrete Math. **306** (1) (2006) 66–73.


E-mail: javad.ebrahimi@sharif.edu
E-mail: s.kar78@student.sharif.edu
E-mail: shayanRanjbarzadeh.77@student.sharif.edu

# A Lower Bound on Graph Energy in Terms of Minimum and Maximum Degrees

Somayeh Khalashi Ghezelahmad*
Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran

Abstract. The energy of a graph $G$, denoted by $\mathcal{E}(G)$, is defined as the sum of absolute values of all eigenvalues of $G$. In (*MATCH Commun. Math. Comput. Chem.* **83** (2020) 631–633) it was conjectured that for every graph with maximum degree $\Delta(G)$ and minimum degree $\delta(G)$ whose adjacency matrix is non-singular, $\mathcal{E}(G) \geq \Delta(G) + \delta(G)$ and the equality holds if and only if $G$ is a complete graph. Here, we prove the validity of this conjecture for regular graphs, triangle-free graphs and quadrangle-free graphs.

**Keywords:** Energy of a graph, Regular graph, Triangle-free graph, Quadrangle-free graph.
**AMS Mathematical Subject Classification [2010]:** 05C50.

## 1. Introduction

Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. By *order* and *size* of $G$, we mean the number of vertices and the number of edges of $G$, respectively. The maximum degree of $G$ is denoted by $\Delta(G)$ (or by $\Delta$ if $G$ is clear from the context). The minimum degree of $G$ is denoted by $\delta(G)$ (or simply by $\delta$). For any vertex $v \in V(G)$, the *open neighborhood*of $v$ in $G$ is $N(v) = \{u \in V(G) : uv \in E(G)\}$. Also the degree of $v \in V(G)$ is $d_G(v) = |N(v)|$ or simply $d(v)$. A graph is *triangle-free* and *quadrangle-free* if it has no subgraph isomorphic to $C_3$ and $C_4$, respectively. A $\{1,2\}$*-factor* is a spanning subgraph of $G$ which is a disjoint union of a matching and a 2-regular subgraph of $G$.

Let $G$ be a graph and $V(G) = \{v_1, \ldots, v_n\}$. The *adjacency matrix* of $G$, $A(G) = [a_{ij}]$, is an $n \times n$ matrix, where $a_{ij} = 1$ if $v_iv_j \in E(G)$, and $a_{ij} = 0$, otherwise. Thus $A(G)$ is a symmetric matrix and all eigenvalues of $A(G)$ are real. By eigenvalues of a graph $G$, we mean the eigenvalues of $A(G)$. The largest eigenvalue of $G$ is called the *spectral radius* of $G$. For a graph $G$, let $\det A(G) \neq 0$. Then there exists $\sigma \in S_n$ such that $a_{1\sigma(1)} = \cdots = a_{n\sigma(n)} = 1$. This transversal is corresponding to a $\{1,2\}$-factor in $G$. The *energy* of a graph $G$, $\mathcal{E}(G)$, is defined as the sum of absolute values of eigenvalues of $G$. The concept of graph energy was first introduced by Gutman in 1978, see [6]. For more properties of the energy of graphs we refer to [7]. Some lower bounds for the energy of graphs have been obtained by several authors. For quadrangle-free graphs, Zhou studied the problem of bounding the graph energy in terms of the minimum degree together with other parameters [9]. In [8], it is proved that for a connected graph $G$, $\mathcal{E}(G) \geq 2\delta(G)$ and the equality holds if and only if $G$ is a complete multipartite graph with the

*Speaker

equal size of parts. In [1], this lower bound improved by showing that if $G$ is a connected graph with average degree $\overline{d}$, then $\mathcal{E}(G) \geq 2\overline{d}$ and the equality holds if and only if $G$ is a complete multipartite graph with the equal size of parts. Also in [1] the authors proposed the following conjecture.

Conjecture. *For every graph $G$ whose adjacency matrix is non-singular, $\mathcal{E}(G) \geq \Delta(G) + \delta(G)$ and the equality holds if and only if $G$ is a complete graph.*

In this paper, it is proved that this conjecture holds for triangle-free, quadrangle-free and regular graphs.

Lemma 1.1. [2] *Let $G$ be a graph of order $n$. If $G$ has a $\{1,2\}$-factor, then $\mathcal{E}(G) \geq n$. In particular, if $A(G)$ is non-singular, then $\mathcal{E}(G) \geq n$.*

Lemma 1.2. [3] *Let $G$ be a graph and $H_1, \ldots, H_k$ be its $k$ vertex-disjoint induced subgraphs. Then $\mathcal{E}(G) \geq \sum_{i=1}^{k} \mathcal{E}(H_i)$.*

Lemma 1.3. [2] *If $n$ is an odd positive integer, then $\mathcal{E}(G) \geq n + 1$.*

## 2. The Validity of the Conjecture for Triangle-Free, Quadrangle-Free and Regular Graphs

In this section, it is shown that the conjecture holds for three classes of graphs, triangle-free, quadrangle-free and regular graphs.

Theorem 2.1. *Let $G$ be a triangle-free graph which has a $\{1,2\}$-factor. Then for any two adjacent vertices $u$ and $v$, $\mathcal{E}(G) \geq d(u) + d(v)$.*

Proof. Let $u$ and $v$ be two adjacent vertices of $G$. Since $G$ is triangle-free, $N(u) \cap N(v) = \emptyset$. This implies that $d(u) + d(v) \leq n$, where $n = |V(G)|$. Now, since $G$ has a $\{1,2\}$-factor, by Lemma 1.1, $\mathcal{E}(G) \geq n \geq d(u) + d(v)$. □

Corollary 2.2. *The conjecture holds for triangle-free graphs. In particular, every bipartite graph satisfies the conjecture.*

Theorem 2.3. *Let $G$ be a quadrangle-free graph which has a $\{1,2\}$-factor. Then $\mathcal{E}(G) \geq \Delta(G) + \delta(G)$.*

Proof. The result holds for $K_2$. So, let $G$ be a graph of order $n \geq 3$ and $u$ be a vertex of $G$ with $d(u) = \Delta$. First suppose that $d(u) < n - 1$. Consider a vertex $v$ non-adjacent to $u$. Since $G$ is quadrangle-free, $|N(u) \cap N(v)| \leq 1$. Thus $\Delta + \delta \leq d(u) + d(v) \leq n - 1$. Now, applying Lemma 1.1 yields the result. Next, assume that $d(u) = n - 1$. Since $G$ is quadrangle-free, the degree of each vertex of $N(u)$ is at most 2. If there exists a vertex $w$ with degree is 1, then using Lemma 1.1, we obtain $\mathcal{E}(G) \geq n \geq \Delta(G) + \delta(G)$. Otherwise, for each $w \in N(u)$, $d(w) = 2$. Therefore, $G$ is a union of some edge-disjoint triangles having a vertex in common. Hence, $G$ has a $\{1,2\}$-factor, say $F$, consisting of a triangle and some $P_2$-components. By considering the components of $F$ as vertex-disjoint induced subgraphs and applying Lemmas 1.2 and 1.3, we have $\mathcal{E}(G) \geq n + 1 \geq \Delta + \delta$. □

Now, we prove the validity of the conjecture for the class of graphs whose maximum eigenvalues are integer.

Theorem 2.4. *The conjecture holds for a graph whose spectral radius is integer.*

PROOF. Let $G$ be a graph of order $n$ with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Note that since $A(G)$ is non-singular, for $i = 1, \dots, n$, $\lambda_i \neq 0$. As for every real number $x > 0$, $x - \ln x \geq 1$, we have

$$\mathcal{E}(G) = \lambda_1 + \sum_{i=2}^{n} |\lambda_i| \geq \lambda_1 + (n-1) + \sum_{i=2}^{n} \ln|\lambda_i| = \lambda_1 + (n-1) + \ln\prod_{i=2}^{n} |\lambda_i|.$$

By [5, Theorem 3.8], we know that $\lambda_1 \geq \delta$. Now, since $A(G)$ is non-singular and $\lambda_1$ is integer, $\prod_{i=2}^{n} |\lambda_i| = \frac{|\det A(G)|}{\lambda_1}$ is a non-zero rational number which is an algebraic integer. Hence, $\ln\prod_{i=2}^{n} |\lambda_i| \geq 0$. This implies that $\mathcal{E}(G) \geq \delta + \Delta$. Also the equality holds if and only if $\Delta = n-1$, $\delta = \lambda_1$ and $\prod_{i=2}^{n} |\lambda_i| = 1$. In the equality case, since $\delta = \lambda_1$, by [5, Theorem 3.8], we find that the graph is regular and since $\Delta = n-1$, the graph is complete. □

Since the spectral radius of a regular graph is integer, see [4, Theorem 6.8], as a consequence of Theorem 2.4, we give the following corollary.

COROLLARY 2.5. *The conjecture holds for regular graphs.*

## References


1. S. Akbari and M. A. Hosseinzadeh, *A short proof for graph energy is at least twice of minimum degree*, MATCH Commun. Math. Comput. Chem. **83** (2020) 631–633.
2. S. Akbari, M. Ghahremani, I. Gutman and F. Koorepazan Moftakhar, *Orderenergetic graphs*, MATCH Commun. Math. Comput. Chem. **84** (2020) 325–334.
3. S. Akbari, E. Ghorbani and M. R. Oboudi, *Edge addition, singular values, and energy of graphs and matrices*, Linear Algebra Appl. **430** (2009) 2192–2199.
4. R. B. Bapat, *Graphs and Matrices*, Springer-Verlag, London, 2010.
5. D. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs*, Academic Press, New York, 1980.
6. I. Gutman, *The energy of a graph*, Ber. Math. Statist. Sekt. Forschungsz. Graz. **103** (1978) 1–22.
7. X. Li, Y. Shi and I. Gutman, *Graph Energy*, Springer, New York, 2012.
8. X. Ma, *A low bound on graph energy in terms of minimum degree*, MATCH Commun. Math. Comput. Chem. **81** (2019) 393–404.
9. B. Zhou, *Lower bounds for energy of quadrangle–free graphs*, MATCH Commun. Math. Comput. Chem. **55** (2006) 91–94.


E-mail: s.ghezelahmad@srbiau.ac.ir

# On Distance-Eigenvalues of Complete Multipartite Graphs

Mohammad Reza Oboudi*
Department of Mathematics, Shiraz University, Shiraz, Iran

Abstract. For every connected graph, the distance matrix is a matrix whose entries are the distance between the vertices of the graph. The distance-eigenvalues of a graph are the eigenvalues of its distance matrix. In this paper we study the largest distance-eigenvalue of complete multipartite graphs and obtain some bounds for this parameter.

We obtain some bounds for the distance spectral radius of complete multipartite graphs. In particular, we obtain that

$$\frac{n+a+b-4+\sqrt{(n+a+b)^2-4ab(t+1)}}{2} \leq \mu(K_{n_1,\ldots,n_t}) \leq \frac{2n-t-2+\sqrt{(2n-2t+1)^2+t^2-1}}{2},$$

where $t \geq 2$ and $n_1, \ldots, n_t$ be some positive integers, and $n = n_1 + \cdots + n_t$, $a = \lceil \frac{n}{t} \rceil$ and $b = \lfloor \frac{n}{t} \rfloor$.
**Keywords:** Distance spectral radius, Complete multipartite graphs.
**AMS Mathematical Subject Classification [2010]:** 05C31, 05C50, 15A18.

## 1. Introduction

In this article the graphs are simple. The *order* of a graph is the number of its vertices. The *disjoint union* of two graphs $H$ and $K$ is denoted by $H \cup K$. As usual, the *degree* of a vertex is the number of edges incident with that. A *clique* of a graph is a set of vertices in which they are adjacent to each other. An *independent* set is a set of vertices so that there is no edge between them. The complete multipartite with parts size $n_1, \ldots, n_t$ is denoted by $K_{n_1,\ldots,n_t}$.

In algebraic graph theory they are many matrices associated to graphs. The most important and well known matrix is the adjacency matrix. For a graph $G$ with vertices $v_1, \ldots, v_n$, the *adjacency matrix* of $G$, $A(G)$, is the matrix where its $(i, j)$-entry is equal to 1 if $v_i$ and $v_j$ are adjacent, and is 0 otherwise. Since this matrix is symmetric, all of its eigenvalues are real. We call these eigenvalues as the eigenvalues of the graph.

The other matrices are the Laplacian, signless Laplacain, Seidel matrix. Here we consider the distance matrix as define as follows. Let $G$ be a connected graph with vertices $v_1, \ldots, v_n$. The distance matrix of $G$, $D(G)$, is the matrix whose its $(i, j)$-entry is equal to the distance between $v_i$ and $v_j$ (the length of a shortest path between them). Since the distance matrix is symmetric, its eigenvalues are real. We call these eigenvalues as distance-eigenvalues of graph. The *distance characteristic polynomial* of a graph is the characteristic polynomial of its distance matrix. Notice that is this polynomial is monic and its degree is the order of the graph. As an example for the complete graph $K_n$, the adjacency matrix and the

*Speaker

distance matrix both are $J-I$, where $J$ is the all one matrix and $I$ is the identity matrix of size $n$. Thus the eigenvalues and distance-eigenvalues of $K_n$ are $n-1$ and $-1,\ldots,-1$ ($n-1$ times).

The largest distance-eigenvalue of a graph $G$, $\mu(G)$, is called the *distance spectral radius* of $G$. The study of distance matrix is of great interest. It is proved that among all connected graphs with order $n$ (and so among all tress with $n$ vertices) the paths have the maximum distance spectral radius [11]. In this paper we study the distance spectral radius of the complete multipartite graphs.

For more details related to the eigenvalues of associted matrices to graphs, in particular, distance characteristic polynomial and distance-eigenvalue of graphs, see [1]- [12] and references therein.

## 2. Main Results

We recall the formula for calculating the the distance characteristic polynomial of complete multipartite graphs.

Theorem 2.1. [4] *Let $t\geq 2$ and $n_1,\ldots,n_t$ be some positive integers. Let $n=n_1+\cdots+n_t$. Then*

$$P_d(K_{n_1,\ldots,n_t},x)=(x+2)^{n-t}\Big(\prod_{i=1}^{t}(x+2-n_i)-\sum_{i=1}^{t}n_i\prod_{j=1,j\neq i}^{t}(x+2-n_j)\Big).$$

At first a lower bound for the distance spectral radius of complete multipartite graphs is obtained.

Theorem 2.2. *Let $t\geq 2$ and $m_1,\ldots,m_t$ be some positive integers. Let $m=\max\{m_1,\ldots,m_t\}$. Then*

$$\mu(K_{m_1,\ldots,m_t})\geq m.$$

*The equality happen only when $t=2$ and $m_1=m_2=1$.*

Theorem 2.3. *Let $t\geq 2$ and $m_1\geq\cdots\geq m_t\geq 1$ and $n_1\geq\cdots\geq n_t\geq 1$. If $m_1+\cdots+m_i\geq n_1+\cdots+n_i$, for $i=1,\ldots,t-1$ and $m_1+\cdots+m_t=n_1+\cdots+n_t$, then*

$$\mu(K_{m_1,\ldots,m_t})\geq\mu(K_{n_1,\ldots,n_t}).$$

Let $t\geq 2$ and $n_1\geq 1,\ldots,n_t\geq 1$. Let $S_{n,t}$ be the split graph $K_{n-t+1,\underbrace{1,\ldots,1}_{t-1}}$ and $T_{n,t}$ be the Turán graph

$$K_{\underbrace{\lceil\frac{n}{t}\rceil,\ldots,\lceil\frac{n}{t}\rceil}_{r},\underbrace{\lfloor\frac{n}{t}\rfloor,\ldots,\lfloor\frac{n}{t}\rfloor}_{s}},$$

where $n=n_1+\cdots+n_t$, $r=n-t\lfloor\frac{n}{t}\rfloor$ and $s=t-r$.

In the next result we determine the extremal value of the distance spectral radius among the family of complete multipartite graphs. This theorem shows that the Turán graphs have the minimum while the split graphs have the maximum distance spectral radius.

THEOREM 2.4. *Let $t \geq 2$ and $n_1, \ldots, n_t$ be some positive integers and $n = n_1 + \cdots + n_t$. Then*

$$\mu(T_{n,t}) \leq \mu(K_{n_1,\ldots,n_t}) \leq \mu(S_{n,t}).$$

*Moreover in the first inequality happen only if $K_{n_1,\ldots,n_t} \cong T_{n,t}$ while the second inequality happen only if $K_{n_1,\ldots,n_t} \cong S_{n,t}$.*

Finally we find some bounds for the distance spectral radius of the complete multipartite graphs in terms of the number of vertices and the number of parts.

THEOREM 2.5. *Let $t \geq 2$ and $n_1, \ldots, n_t$ be some positive integers. Let $n = n_1 + \cdots + n_t$, $a = \lceil \frac{n}{t} \rceil$ and $b = \lfloor \frac{n}{t} \rfloor$. Suppose that $K_{n_1,\ldots,n_t} \ncong S_{n,t}$ and $K_{n_1,\ldots,n_t} \ncong T_{n,t}$. Then*

$$\frac{n+a+b-4+\sqrt{(n+a+b)^2-4ab(t+1)}}{2} < \mu(K_{n_1,\ldots,n_t}) < \frac{2n-t-2+\sqrt{(2n-2t+1)^2+t^2-1}}{2}.$$

**Acknowledgement**

This research was in part supported by Iran National Science Foundation (INSF) under the contract No. 98001945.

**References**

1. M. Aouchiche and P. Hansen, *Distance spectra of graphs: A survey,* Linear Algebra Appl. **458** (2014) 301–386.
2. P. Híc and M. Pokorny, *Remarks on D-integral complete multipartite graphs*, Czech. Math. J. **66** (2016) 457–464.
3. Y. L. Jin and X. D. Zhang, *Complete multipartite graphs are determined by their distance spectra,* Linear Algebra Appl. **448** (2014) 285–291.
4. H. Q. Lin, Y. Hong, J. F. Wang and J. L. Shu, *On the distance spectrum of graphs,* Linear Algebra Appl. **439** (2013) 1662–1669.
5. H. Lin, B. Wu, Y. Chen and J. Shu, *On the distance and distance Laplacian eigenvalues of graphs*, Linear Algebra Appl. **492** (2016) 128–135.
6. A. W. Marshall, I. Olkin and B. C. Arnold, *Inequalities: Theory of Majorization and its Applications*, 2nd ed., Springer, New York, 2011.
7. G. J. Ming and T. Sh. Wang, *On the spectral radius of trees,* Linear Algebra Appl. **329** (2001) 1–8.
8. M. R. Oboudi, *A relation between the signless Laplacian spectral radius of complete multipartite graphs and majorization,* Linear Algebra Appl. **565** (2019) 225–238.
9. M. R. Oboudi, *Bipartite graphs with at most six non-zero eigenvalues,* Ars Math. Contemp. **11** (2016) 315–325.
10. M. R. Oboudi, *Energy and Seidel energy of graphs,* MATCH Commun. Math. Comput. Chem. **75** (2016) 291–303.
11. S. N. Ruzieh and D. L. Powers, *The distance spectrum of the path $P_n$ and the first distance eigenvector of connected graphs,* Linear Multilinear Algebra **28** (1990) 75–81.
12. R. Yang and L. Wang, *Distance integral complete r-partite graphs,* Filomat **29** (2015) 739–749.

E-mail: mr_oboudi@yahoo.com; mr_oboudi@shirazu.ac.ir

# Degree-Associated Reconstruction Number of Balanced Trees

Mehrnoosh Shadravan*
Department of Mathematics, Faculty of Mathematical Sciences, Shahid Beheshti University, Tehran, Iran
and Rajab Ali Borzooei
Department of Mathematics, Faculty of Mathematical Sciences, Shahid Beheshti University, Tehran, Iran

Abstract. A card of a graph $G$ is a subgraph formed by deleting one vertex. The reconstruction conjecture states that each graph with at least three vertices is determined by its multiset of cards. A dacard specifies the degree of the deleted vertex along with the card. The degree-associated reconstruction number $drn(G)$ is the minimum number of dacards that determine $G$. Barrus and West conjectured that $drn(G) \leq 2$ for all but finitely many trees. Each connected subtree formed by deleting of a vertex $v$ in $T$ is called the component of $v$. The components of vertex $v$ are denoted by $comp_1(v), comp_2(v), \ldots, comp_{d(v)}(v)$. A vertex $v$ of a tree $T$ is called balanced, if for each $i$, $|comp_i(v)| \leq \frac{n-1}{2}$. A vertex $v$ of $T$ is called parent if it has at least one leaf in its neighborhood. In this paper, we prove that $drn(T) \leq 2$ for any tree $T$ with a balanced parent vertex.

**Keywords:** Degree-associated reconstruction number, Balanced tree, Eq-balanced tree.
**AMS Mathematical Subject Classification [2010]:** 05C05, 05C99.

## 1. Introduction

Let $G$ be a graph. For any vertex $v$ of $G$, the card $C_v$ is the subgraph of $G$ obtained by deleting $v$. The well-known graph reconstruction conjecture [6, 9] asserts that any graph of order at least three can be reconstructed from its deck of cards. Here the deck of graph $G$ is the multiset of cards. For the surveys of results on this conjecture see [3, 4].

For the reconstruction of many graphs we do not need all the cards. Harrary and Plantholt [5], introduced the reconstruction number of a graph $G$, denoted by $rn(G)$, to be the minimum number of cards from the deck of $G$ that suffix to determine $G$, meaning that no graph has the same multiset in its deck. For a survey of some open questions in reconstruction numbers see [1].

A degree associated card or dacard $dC_v$ of the graph $G$ is the ordered pair $(C_v, d_G(v))$ where $d_G(v)$ is the degree of $v$ in $G$. The dadeck of graph $G$, denoted by $dD(G)$, is the multiset of dacards of $G$. Ramachandran [8] defined the degree-associated reconstruction number $drn(G)$ of a graph $G$ to be the size of the smallest submultiset of $dD(G)$ which is not contained in the dadeck of any other graph.

*Speaker

In other words, $drn(G)$ is the minimum number of dacards from the dadeck of $G$ that suffi to determine $G$. Ramachandran studied it for complete graphs, edgeless graphs, cycles, complete bipartite graphs, and disjoint unions of identical graphs. Barrus studied vertex-transitive graphs. Thet proved that $drn(G) \leq 3$ when $G$ is not complete or edgeless.

Monikandan introduced the degree-associated analogue of $arn(G)$. When $G$ is reconstructible from its dadeck, the adversary degree-associated reconstruction number, denoted $adrn(G)$, is the least $k$ such that every set of $k$ dacards determines $G$. Monikandan and Sundar Raj [7] determined adrn for double-stars, for subdivisions of stars, and other classes of graphs.

The skeleton of a tree $T$ is the subtree obtained by deleting all leaves from $T$. Caterpillars are the trees whose skeletons are paths. Barrus and West conjectured that $drn(G) \leq 2$ for all but finitely many trees. They studied caterpillars in [2] and proved that $drn(G) \leq 2$ for all caterpillars except one specific example (a caterpillar tree with 6 vertices). In this paper, any graph is simple and any subgraph is vertex induced subgraph. The degree of vertex $v$ is denoted by $d(v)$. A vertex $v$ in $T$ is called leaf, if $d(v) = 1$. We call a vertex parent, if it has at least one leaf in its neighborhood. In the Section 2, the concept of components of a vertex is defined, then by using this concept, we define a balanced vertex and a balanced tree. We prove the conjecture of Barrus and West for any tree with a balanced parent vertex.

## 2. Reconstruction of Balanced Trees

Let $T$ be a tree on $n$ vertices. A vertex $v$ of $T$ is called parent, if it has at least one leaf in neighborhood. For a leaf $l$ of $T$, the parent of $l$ is denoted by $p(l)$. For each vertex $v$, the number of adjacent vertices of $v$ that are leaf denoted by $dl(v)$. The tree which is obtained by adding and joining a new vertex $l$ to one of the vertices of $T$, we denote by $T + l$.

Definition 2.1. Suppose $T$ is a tree on $n$ vertices. Each connected subtree obtained by deleting of a vertex $v$ of $T$ is called component of $v$. The number of components of $v$ is equal to the degree of $v$. The components of vertex $v$ are denoted by $comp_1(v), comp_2(v), \ldots, comp_{d(v)}(v)$.

Definition 2.2. Suppose $T$ is a tree on $n$ vertices. A vertex $v$ of $T$ is called balanced if for each $i$, we have $|comp_i(v)| \leq \frac{n-1}{2}$.

Definition 2.3. A tree $T$ is called balanced if there is one balanced vertex in tree $T$.

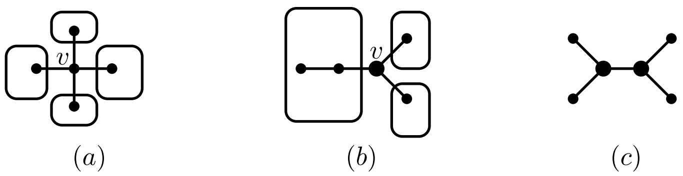



Figure 1. (a) The tree $S_4$ and (b) a tree with a balanced vertex $v$ and (c) a non-balanced tree.

Figure 1 provides some examples of non-balanced and balanced trees.

Our goal is to prove $drn(T) \leq 2$ for any balanced tree with a balanced parent vertex. So, we should find for such a tree two dacards that determine it. After finding two proper dacards, we show that every reconstruction of given dacards is a tree. Notice that for each balanced tree $T$, one of the two chosen dacards is $(C_v, 1)$ where $v$ is a leaf. It is easy to show that the dacard $(C_v, 1)$ which $C_v$ is a tree forces $T$ to be a tree. For the second dacard $(C_u, d(u))$, the vertex $u$ may be a leaf or not. We have a general method to prove $T$ is determined by the two dacards $(C_u, d(u))$ and $(C_v, 1)$. Consider the vertex $p(v)$ in $T$ and denote this vertex by $v*$ in $C_v$. By adding and joining a new vertex $l$ to possible vertices in $C_v$, we show $dC_u \in dD(C_v + l)$ if and only if $l$ is joined to the vertex $v*$ in $C_v$.

LEMMA 2.4. *Let $T$ be a tree. Then there is at most one balanced vertex in $T$.*

THEOREM 2.5. *Let $T$ be a tree with balanced parent vertex $v$ such that for each $i$, we have $|comp_i(v)| < \frac{n-1}{2} - 1$. Then $drn(T) \leq 2$.*

In the Theorem 2.5, we prove tree $T$ is determined by two dacards $dC_v$ and $dC_l$ where $v$ is the balanced vertex of $T$ and $l$ is an adjacent leaf to $v$. The card $C_v$ is a balanced tree, So by using Lemma 4 we can prove this theorem.

A vertex $v$ of $T$ is called *balanced*$_1$, if for some $i$ we have $|comp_i(v)| = \frac{n}{2}$.

Note that $n$ is an even number when $T$ has a *balanced*$_1$ vertex.

LEMMA 2.6. *Let $T$ be a tree with a balanced$_1$ vertex $v$. Then $T$ has exactly two ballanced$_1$ vertices.*

A tree $T$ with exactly two *balanced*$_1$ vertices $u$ and $v$ has a specific structure. Firstly, vertices $u$ and $v$ are adjacent. Furthermore,

$$\sum_{i \neq k'} |comp_i(u)| = \sum_{i \neq k} |comp_i(v)|,$$

and

$$|comp_{k'}(u)| = |comp_k(v)|,$$

where $comp_k(v)$ and $comp_{k'}(u)$ are the largest components of $v$ and $u$, respectively.

Notice that the largest component of $v$ contains $u$. Similarly, the largest component of $u$ contains $v$. Now, we partition trees with two *balanced*$_1$ vertices into two classes. A tree with two *balanced*$_1$ vertices $u$ and $v$ is said to belong to Class 1, if $d(u) = d(v) = d$. Moreover, for every $i \leq d$, $comp_i(v) \cong comp_i(u)$ (See Figure 2). Also, it is said to belong to Class 2 if it does not belong to Class 1.

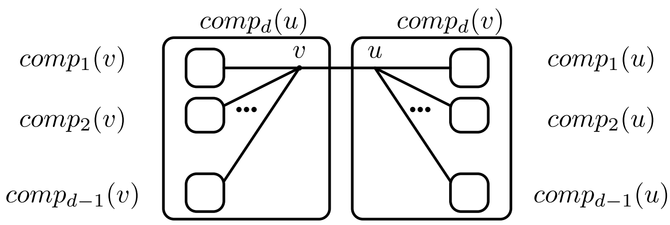



FIGURE 2. A tree with two *balanced*$_1$ vertices $u$ and $v$ of class 1.

Any balanced vertex $v$ of $T$ is called eq-balanced, if for some $i$ $|comp_i(v)| = \frac{n-1}{2}$. Moreover, a balanced tree $T$ is called eq-balanced if it has an eq-balanced vertex.

Note that $n$ is an odd number when $T$ is an eq-balanced tree.

THEOREM 2.7. *Let $T$ be a tree with an eq-balanced parent vertex. Then $drn(T) \leq 2$.*

In this Theorem, we consider subtree $T - l_v$ where $l_v$ is an adjacent leaf to $v$. We show $T - l_v$ is a tree with two $balanced_1$ vertices. If the subtree $T - l_v$ of $T$ belongs to Class 2, then we easily can conclude that $T$ is determined by two dacards $C_v$ and $C_{l_v}$. And if $T - l_v$ belongs to Class 1, then we use from another dacards for reconstruction.

Clearly, it follows from Theorems 2.5 and 2.7 that for any tree $T$ with a balanced parent vertex, $drn(T) \leq 2$.

## References


1. K. J. Asciak, M. Francalanza, J. Lauri and W. Myrvold, *A survey of some open questions in reconstruction numbers*, Ars Combin. **97** (2010) 443–456.
2. M. D. Barrus and D. B. West, *Degree-associated reconstruction number of graphs*, *Discrete Math.* **310** (20) (2010) 2600–2612.
3. J. A. Bondy, *On Ulam's conjecture for separable graphs*, Pacific J. Math. **31** (1969) 281–288.
4. J. A. Bondy and R. L. Hemminger, *Graph reconstruction–a survay*, J. Graph Theory **1** (3) (1977) 227–268.
5. F. Harary and M. plantholt, *The graph reconstruction number*, J. Graph Theory **9** (4) (1985) 451–454.
6. P. J. Kelly, *Acongruence theorem for trees*, Pacific J. Math. **7** (1957) 961–968.
7. S. Monikandan and S. Sundar Raj, *Adversary degree associated reconstruction number of graphs*, Discrete Math. Algorithms Appl. **7** (1) (2015) 1450069.
8. S. Ramachandran, *Reconstruction number for Ulam's conjecture*, Ars Combin. **78** (2006) 289–296
9. S. M. Ulam, *A collection of mathematical problems*, Interscience Publ., New York, 1960.


E-mail: m_shadravan@sbu.ac.ir
E-mail: borzooei@sbu.ac.ir

# Planarity of Perpendicular Graph of Modules

Maryam Shirali*
Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran

Abstract. Let $R$ be a ring and $M$ be an $R$-module. Two modules $A$ and $B$ are called orthogonal, written $A \perp B$, if they do not have non-zero isomorphic submodules. We consider an associated graph $\Gamma_\perp(M)$ to $M$ with vertices $\mathcal{M}_\perp = \{(0) \neq A \lneq M \mid \exists (0) \neq B \lneq M$ such that $A \perp B\}$, and for distinct $A, B \in \mathcal{M}_\perp$, the vertices $A$ and $B$ are adjacent if and only if $A \perp B$. The main object of this article is to study the interplay of module-theoretic properties of $M$ with graph-theoretic properties of $\Gamma_\perp(M)$. In this article, we investigate the planarity of perpendicular graph of $R$-module $M$.
**Keywords:** Perpendicular graph, Orthogonal submodules, Planar graph, Semi-artinian module.
**AMS Mathematical Subject Classification [2010]:** 05C25, 16P60, 16P40.

## 1. Introduction

In this paper, $R$ be a ring with identity and $M$ be an $R$-module, $\mathcal{M}_\perp = \{(0) \neq A \lneq M \mid \exists (0) \neq B \lneq M$ such that $A \perp B\}$ is the set of all vertices of perpendicular graph. As [3], we say that two modules $A$ and $B$ are orthogonal, written $A \perp B$, if they do not have non-zero isomorphic submodules. The perpendicular graph of $M$, denoted by $\Gamma_\perp(M)$, is an undirected simple graph with the vertex set $\mathcal{M}_\perp$ in which every two distinct vertices $A$ and $B$ are adjacent if and only if $A \perp B$ (See [3] for more details). We can see that every two non-isomorphic simple submodules of $M$ are mutually orthogonal. A module $M$ is called *atomic* if $M \neq 0$ and for any $x, y \in M \setminus \{0\}$, $xR$ and $yR$ have non-zero isomorphic submodules. A module $M$ has *finite type dimension* $n$, denoted by $\text{t.dim}(M) = n$, if $M$ contains an essential direct sum of $n$ pairwise orthogonal atomic submodules of $M$. If no such $n$ exists, we say that the type dimension of $M$ is infinite and write $\text{t.dim}(M) = \infty$. If $\text{t.dim}(M) = 0$, then $M = 0$. See [2] for a systematic study of type dimension and all related concepts. In [5, Proposition 2.5] we can see that, $R$ is left semi-artinian ring, if every left module over $R$ has a non-zero socle. We say that $G$ is connected if there is a path between any two distinct vertices. In [3], we have shown that $\Gamma_\perp(M)$ is connected graph and also, we showed that graph $\Gamma_\perp(M)$ is empty if and only if $M$ is atomic module.

A *complete graph* is a graph in which every pair of distinct vertices are adjacent. A complete graph with $n$ vertices is denoted by $K_n$. By a complete subgraph we mean a subgraph which is complete as a graph. A *bipartite graph* (or bigraph) is a graph whose vertices can be divided into two disjoint sets $V_1$ and $V_2$ (that is, $V_1$ and $V_2$ are each independent sets) such that every edge connects a vertex in $V_1$ to one in $V_2$. Assume that $K_{m,n}$ denoted the complete bipartite graph on

*Speaker

two non-empty disjoint sets $V_1$ and $V_2$ with $|V_1| = m$ and $|V_2| = n$ (here $m$ and $n$ may be infinite cardinal number). A $K_{1,n}$ graph is often called a *star graph.* A *clique* of a graph is a maximal complete subgraph and the number of vertices in the largest clique of graph $G$, denoted by $\omega(G)$, is called the *clique number* of $G$. Let $\chi(G)$ denote the *chromatic number* of the graph $G$, that is, the minimal number of colors need to color the vertices of $G$ so that no two adjacent vertices have the same color. Obviously $\omega(G) \leq \chi(G)$. A graph is said to be planar if it can be drawn in the plane so that its edge interest only at their ends. A remarkably simple characterization of planar graphs was given by Kuratowski in [1]. Kuratowskis Theorem says that a graph is planar if and only if it contains no subdivision of $K_5$ or $K_{3,3}$. The reader is referred to [6] for undefined terms and concepts in graph theory.

## 2. Main Results

We investigate the planarity of perpendicular graph of $R$-module $M$. Before we state and prove our first main result, we express an auxiliary lemma.

Lemma 2.1. *Let $A$, $B$ and $C$ are submodules of $M$ as $R$-module. Then the following facts hold.*

1) *If $A \perp B$, then $A \cap B = 0$.*
2) *If $B \cong C$ and $A \perp B$, then $A \perp C$.*

Proposition 2.2. *Let $M$ be semi-artinian $R$-module such that $\Gamma_\perp(M) \neq \emptyset$. The following statements are equivalent.*

1) $\text{t.dim}(M) = 2$*;*
2) $\Gamma_\perp(M)$ *is a bipartite graph;*
3) *$M$ has only two non-isomorphic simple submodules;*
4) *$M$ has no triangle.*

Proof. ($1 \Longleftrightarrow 2$) See [3, Theorem 4.6].
($2 \Rightarrow 3$) Suppose that $\Gamma_\perp(M)$ is a bipartite graph with two parts $V_1$ and $V_2$. Since $M$ is semi-artinian module thus every non-zero submodule of $M$ contains a simple submodule of $M$. But, if $M$ has only a simple submodule $S$, then every vertex of $\Gamma_\perp(M)$ contains $S$. Thus $\Gamma_\perp(M) = \emptyset$, which is a contradiction. Now, if $M$ has more than two non-isomorphic simple submodules then assume that $S_1$, $S_2$ and $S_3$ are non-isomorphic simple submodules of $M$. Since $\Gamma_\perp(M)$ is a complete bipartite graph, by Pigeon Hole Principal, two of the non-isomorphic simple submodules should belong to one of $V_i$'s, which is a contradiction. Hence $M$ has two non-isomorphic simple submodules.
($3 \Rightarrow 2$) Suppose that $S_1$ and $S_2$ are non-isomorphic simple submodules. Since $M$ is semi-artinian module, so every non-zero submodules of $M$ contains $S_1$ or $S_2$. Set $V_1 = \{N \in \Gamma_\perp(M) | S_1 \subset N\}$ and $V_2 = \{N \in \Gamma_\perp(M) | S_2 \subset N\}$. Clearly, $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = \mathcal{M}_\perp$ and the elements of $V_i$'s are not adjacent, for $i = 1, 2$. Now suppose that $A \in V_1$ so there exists $B \lneq M$ such that $A \perp B$. But $S_1 \subset A$ and since $M$ is semi-artinian module, we must have $S_2 \subset B$. This implies that $\Gamma_\perp(M)$ is a bipartite graph.
($3 \Rightarrow 4$) Let $S_1$ and $S_2$ be the only two non-isomorphic simple submodules of $M$. Then for three vertices $N, K$ and $L$ of $\Gamma_\perp(M)$, at least two of them are contain one

of $S_1$ or $S_2$ (since $M$ is a semi-artinian module) and hence they are not adjacent. Therefore there is no triangle in $\Gamma_\perp(M)$.
($4 \Rightarrow 3$) It is clear. □

LEMMA 2.3. *If $\Gamma_\perp(M)$ is a planar graph, then the following hold:*

1) *The number of non-isomorphic simple submodules of $M$ is at most* 4.
2) $\omega(\Gamma_\perp(\mathrm{M})) \leq 4$.

PROOF. (1) By Kuratowskis Theorem, it is clear.
(2) If $\omega(\Gamma_\perp(\mathrm{M})) \geq 5$, then $\Gamma_\perp(M)$ contains subgraph $K_5$ which is a contradiction with Kuratowskis Theorem. □

The converse part (1) of Lemma 2.3 is not true, for example if $M = \mathbb{Z}_{840}$ as $\mathbb{Z}$-module, then $\Gamma_\perp(M)$ contains subgraph $K_{3,3}$, such that the number of non-isomorphic simple submodules of $M$ is 4. Also the converse part (2) of Lemma 2.3, is not true, for example $M = \mathbb{Z}_{216}$ as $\mathbb{Z}$-module, we can see that $\omega(\Gamma_\perp(\mathrm{M})) = 2$, but $\Gamma_\perp(M)$ is not planar, because $\Gamma_\perp(M) = K_{3,3}$.

PROPOSITION 2.4. [4, Proposition 2.3] *Let $M$ be an $R$-module and $S$ be a simple submodule of $M$ and $\omega(\Gamma_\perp(M)) < \infty$. Then the following hold:*

1) *The number of non-isomorphic simple submodules of $M$ is finite.*
2) $\chi(\Gamma_\perp(M)) < \infty$.

PROPOSITION 2.5. *Let $M$ be an $R$-module such that* $\mathrm{Soc}(M) \neq 0$. *If $\Gamma_\perp(M)$ is planar, then $\chi(\Gamma_\perp(M)) < \infty$.*

PROOF. By Lemma 2.3 and Proposition 2.4 is given. □

THEOREM 2.6. *Let $M$ be semi-artinian $R$-module such that $\Gamma_\perp(M) \neq \emptyset$ and $M$ have two non-isomorphic simple submodules. Then $\Gamma_\perp(M)$ is planar graph if and only if $\Gamma_\perp(M)$ is star graph.*

PROOF. Let $M$ be a semi-artinian $R$-module such that $\Gamma_\perp(M)$ is planar. Since $M$ have exactly two non-isomorphic simple submodules $S_1$ and $S_2$, then by Proposition 2.2, $\Gamma_\perp(M)$ is bipartite graph. Thus by [3, Proposition 4.2], $\Gamma_\perp(M)$ is complete bipartite graph with two non-empty disjoint sets $V_1$ and $V_2$ where

$$V_1 = \{K \in \mathcal{M}_\perp | S_1 \subset K\},$$

and

$$V_2 = \{K \in \mathcal{M}_\perp | S_2 \subset K\}.$$

Then $\Gamma_\perp(M)$ is planar if and only if either $|V_1| \leq 2$ or $|V_2| \leq 2$, by Kuratowski Theorem. Assume that $|V_1| \leq 2$. We put $V_1 = \{S_1, A_1\}$. It is clear that $A_1$ is a simple submodule of $M$. Hence $A_1$ and $S_1$ are isomorphic simple submodules. By [3, Proposition 2.4] $A_1 = S_1$ therefore $|V_1| = 1$, i.e., $\Gamma_\perp(M)$ is a star graph. On the similar way, if $|V_2| \leq 2$ then $\Gamma_\perp(M)$ is a star graph. Hence $\Gamma_\perp(M)$ is planar if and only if $\Gamma_\perp(M)$ is a star graph. □

REMARK 2.7. If $M$ have exactly 3 non-isomorphic simple submodules. In this case, $\Gamma_\perp(M)$ may or may not be planar. For example, Let $R = \mathbb{Z}$ and consider $M_1 = \mathbb{Z}_{30}$ and $M_2 = \mathbb{Z}_{360}$ as $R$-modules. It is clear that $M_1$ and $M_2$ have exactly 3 non-isomorphic simple submodules, such that $\Gamma_\perp(M_1)$ is planar but $\Gamma_\perp(M_2)$ is not planar(because $\Gamma_\perp(M_2)$ contains the subgraph $K_{3,3}$ which has $\mathbb{Z}_2, \mathbb{Z}_4$ and $\mathbb{Z}_8$ in one part and $\mathbb{Z}_3, \mathbb{Z}_{15}$ and $\mathbb{Z}_{45}$ in another part and hence $\Gamma_\perp(M_2)$ is not planar). If $M$ have exactly four non-isomorphic simple submodules. In this case, $\Gamma_\perp(M)$ may or may not be planar. For example, Let $R = \mathbb{Z}$ and consider $M_1 = \mathbb{Z}_{210}$ and $M_2 = \mathbb{Z}_{840}$ as $R$-modules. It is clear that $M_1$ and $M_2$ have exactly four non-isomorphic simple submodules, such that $\Gamma_\perp(M_1)$ is planar but $\Gamma_\perp(M_2)$ is not planar(because $\Gamma_\perp(M_2)$ contains the subgraph $K_{3,3}$ which has $\mathbb{Z}_2, \mathbb{Z}_4$ and $\mathbb{Z}_8$ in one part and $\mathbb{Z}_3$, $\mathbb{Z}_{15}$ and $\mathbb{Z}_{21}$ in another part and hence $\Gamma_\perp(M_2)$ is not planar).

Example 2.8. Let $n$ be a natural number and $n = p_1^{n_1} p_2^{n_2} \dots p_m^{n_m}$, where $p_i$'s are distinct prime numbers and $n_i$'s are natural numbers. Then $\Gamma_\perp(\mathbb{Z}_n) \neq \emptyset$ is a planar graph if and only if one of the following hold:

1) $n = p_1^{n_1} p_2^{n_2}$ such that two cases may happen:
(Case 1) If $n_1 \geq 3$ then $n_2 \leq 2$.
(Case 2) If $n_1 \ngeq 3$ then $n_2 \in \mathbb{N}$.
2) $n = p_1^{n_1} p_2^{n_2} p_3^{n_3}$ such that two cases may happen:
(Case 1) If $n_1 \geq 3$ then $n_2, n_3 \leq 2$.
(Case 2) If $n_1 \ngeq 3$ then two cases may happen:
(Case a) If $n_2 \geq 3$ then $n_3 \leq 2$.
(Case b) If $n_2 \ngeq 3$ then $n_3 \in \mathbb{N}$.
3) $n = p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4}$ such that two cases may happen:
(Case 1) If $n_1 \geq 3$ then $n_2, n_3, n_4 \leq 2$.
(Case 2) If $n_1 \ngeq 3$ then two cases may happen:
(Case a) If $n_2 \geq 3$ then $n_3, n_4 \leq 2$.
(Case b) If $n_2 \ngeq 3$ then $n_3, n_4 \in \mathbb{N}$.

## References


1. J. Bosak, *The graphs of semigroups*, In: Theory of Graphs and Application, Academic Press, New York, (1964) pp. 119–125.
2. J. Dauns and Y. Zhou, *Classes of Modules*, Pure and Applied Mathematics (Boca Raton), Vol. 281, Chapman & Hall/CRC, Boca Raton, FL, 2006.
3. M. Shirali, E. Momtahan and S. Safaeeyan, *Perpendicular graph of modules*, Hokkaido Math. J. **49** (3) (2020) 463–479.
4. M. Shirali, E. Momtahan and S. Safaeeyan, *Further study of perpendicular graph of modules*, to appear.
5. B. Stenström, *Rings of Quotients*, Springer-Verlag, Berlin, Heidelberg, New York, 1975.
6. D. B. West, *Introduction to Graph Theory*, 2nd ed., Prentice-Hall, Inc., Upper Saddle River, NJ, 2001.


E-mail: maryam.shirali98@yahoo.com

# Relationship between $k$-Matching and Coefficient of Characteristic Polynomial of Graphs

Meysam Taheri-Dehkordi*
Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Kashan, Kashan, 87317-53153, Iran
and Gholam Hosein Fath-Tabar
Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Kashan, Kashan, 87317-53153, Iran

Abstract. In this paper we establish a formula for the number of $k$-matching in graphs with girth of at least $k+2$, in terms of coefficient of characteristic polynomial.
**Keywords:** Characteristic polynomial, $k$-Matching.
**AMS Mathematical Subject Classification [2010]:** 05C31, 05C70.

## 1. Introduction

All graphs in this paper are simple and connected. For such a graph $G$, $m$ and $n$ are assumed the number of its vertices and edges, respectively.

Let $G$ be a graph. The characteristic polynomial $\chi(G;\lambda)$ of $G$ that is $\chi(G;\lambda) = \det(\lambda I - A(G))$, where $I$ is the identity matrix. Let us suppose that the characteristic polynomial of $G$ is

$$\chi(G;\lambda) = \lambda^n + C_1\lambda^{n-1} + C_2\lambda^{n-2} + C_3\lambda^{n-3} + \cdots + C_n.$$

In this form we know that $-C_1$ is the sum of the roots of $\chi(G,\lambda)$, that is the sum of the eigenvalues. This is also the trace of $A(G)$ which, as we have already noted, is zero thus $C_1 = 0$; and we know that $-C_2$ is the number of edge of $G$ and $-C_3$ is twice the number of triangle in $G$.

Proposition 1.1. [3] *Let $A$ be the adjacency matrix of a graph $G$, then*

$$\det(A) = \sum (-1)^{r(H)} 2^{S(H)},$$

*where the summation is over all spanning elementary subgraph $H$ of $G$ and $r(H) = n - c$ and $S(H) = m - n + c$, where $c$ is the number of connected components of $H$, and $m, n$ are the number of edges and vertices of $H$, respectively.*

Proposition 1.2. [2] *The coefficient of the characteristic polynomial are given by*

$$(-1)^i C_i = \sum (-1)^{r(H)} 2^{S(H)}.$$

*Speaker

We define a matching in $G$ to be a spanning subgraph whose component are vertices and edges; A $k$-matching in $G$ is a matching with $k$ edges.

We use the $\rho(G,k)$ to denote the number of $k$-matching in $G$ and assumed that $\rho(G,0)=1$.

The matching polynomial of graph $G$ is defined by

$$\mu(G,x)=\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k\rho(G,k)x^{n-2k}.$$

It is obvious that from definition of $\rho(G,k)$ that $\rho(G,1)=m$.

Lemma 1.3.

$$\rho(G,2)=\binom{m}{2}-\sum_{i=1}^{n}\binom{d_i}{2},$$

*and*

$$\rho(G,3)=\binom{m}{3}(m-2)\sum_{i}\binom{d_i}{2}+2\sum_{i}\binom{d_i}{3}+\sum_{ij}(d_i-1)(d_j-1)-N_T,$$

*where $N_T$ is the number of triangles in $G$ and $d_i$ is the degree of vertex $V_i$ of $G$.*

The number of $k$-matching for $k=4,5,6$ can be founded in [**1, 5, 6**].

The number of $k$-matching calculated in the mentioned works shows when $k$ is grow up, the formula for the number of $k$-matching gets very long and complicated. Also for $k\geq 4$, calculations related to $\rho(G,k)$ are very long. Especially for graphs that are not regular. There is a relationship between the coefficients of characteristic polynomial and the number of 5 and 6 matching in regular graphs with girth 5 [**4**]. In this paper we obtain a relationship between $k$-matching and coefficient of characteristic polynomial in graphs.

## 2. Main Results

Proposition 2.1. *Let $G$ be a graph then*

$$\rho(G,2)=C_4+2S_q,$$

*where $S_q$ is the number of squares in $G$.*

Proof. By proposition 1.2, we have

$$C_4=\sum(-1)^{r(H)}2^{S(H)},$$

where $H$ is an elementary subgraph of $G$ with 4 vertices. The subgraph $H$ can have two cases as is Figure 1.

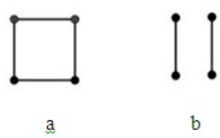



Figure 1. Two possible modes for $H$.

Let $N_a = \sum (-1)^{r(H)} 2^{S(H)}$, $N_b = \sum (-1)^{r(H)} 2^{S(H)}$; where $H$ is a subgraph of $G$ isomorphic to graphs $a$ and $b$ are shown in Figure 1. So we have $C_4 = N_a + N_b$ and $N_a = -2 \times S_q$, $N_b = \rho(G, 2)$. Thus $\rho(G, 2) = C_4 + 2S_q$. □

Proposition 2.2. *Let $G$ be a graph with girth at least five, then*

$$\rho (G, 3) = C_6 + 2h,$$

*where $h$ is the number of hexagons in $G$.*

Proof. We know that

$$C_6 = \sum (-1)^{r(H)} 2^{S(H)},$$

where $H$ is an elementary subgraph of $G$ with 6 vertices. The possible cases of $H$ are shown in Figure 2.

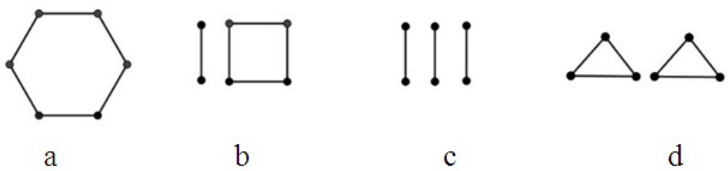



Figure 2. The possible cases of $H$.

According to the premise of the proposition $N_b = N_d = 0$, $N_c = \rho (G, 3)$, so

$$C_6 = \left((-1)^5 \times 2^1\right) h + \rho (G, 3),$$
$$\rho (G, 3) = 2h + C_6.$$

□

Theorem 2.3. *Let $G$ be a graph with girth at least $k + 2$, then*

$$\rho (G, k) = 2N_{2k} + C_{2k},$$

*where $N_{2k}$ is the number of cycles with length of $2k$ in $G$.*

## 3. Examples

In this section we will give examples of some graphs and calculate some of their matching.

**3.1. The 3-Matching of Heawood Graph.** Heawood graph is a 3-regular graph with 14 vertices and 21 edges as shown in Figure 3.

Let $G$ be the Heawood graph. We obtain the characteristic polynomial and we find $\rho (G, k)$, for $1 \leq k \leq 4$.

The characteristic polynomial of the Heawood graph is

$$\chi (H; \lambda) = \lambda^{14} - 21\lambda^{12} + 168\lambda^{10} - 700\lambda^8 + 1680\lambda^6 - 2352\lambda^4 + 1792\lambda^2 - 576.$$

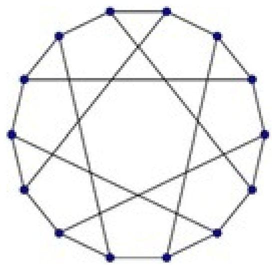


Figure 3. The Heawood graph.

Then

$$\rho(G,1) = 24,$$
$$\rho(G,2) = C_4 = 168,$$
$$\rho(G,3) = C_6 + 2h = |-700 + (2 \times (28))| = 644.$$

**3.2. The 3-Matching in Fullerenes Graph.** A fullerene graph is a planar, 3-regular and 3-connected graph, with $n$ vertices and $\frac{3n}{2}$ edge. Twelve of whose faces are pentagons, and any remaining faces are hexagons. If $G$ be a fullerene graph with $n$ vertices, then $G$ has $h = \frac{n}{2} - 10$ hexagones.

Theorem 3.1. *For any fullerene graph $G$ with $n$ vertices, $\rho(G,3) = C_6 + n - 20$.*

Proof. It is obtained directly from the definition. □

**3.3. The 4-Matching in an Arbitrary Graph.** As we said, if $G$ is not regular, it is very difficult to calculate the $\rho(G,k)$ for $k \geq 4$ with using previous methods. We calculate the $\rho(G,4)$ for a graph with 32 vertices which is shown in Figure 4.

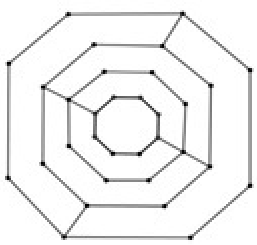


Figure 4

The characteristic polynomial of this graph is

$$\chi(G;\lambda) = \lambda^{32} - 38\lambda^{30} + 645\lambda^{28} - 6468\lambda^{26} + 42704\lambda^{24} - 195796\lambda^{22} + \cdots,$$

so

$$\begin{aligned}\rho(G,1) =& 3,\\ \rho(G,2) =& C_4 = 645,\\ \rho(G,3) =& C_6 = 6468,\\ \rho(G,4) =& 2N_8 + C_8 = 2\times 4 + 42704 = 42712.\end{aligned}$$

## References


1. A. Behmaram, *On the number of* 4*-matching in graphs*, MATCH Commun. Math. Comput. Chem. **62** (2009) 381–388.
2. N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, 2nd ed., Cambridge, 1974.
3. F. Harary, *The determinant of the adjacency matrix of a graph*, SIAM Rev. **4** (3) (1962) 202–210.
4. F. Taghavee and G. H. Fath-Tabar, *Relationship between coefficients of characteristic polynomial and matching polynomial of regular graphs and its applications*, Iranian J. Math. Chem. **8** (1) (2017) 7–23.
5. R. Vesalian and F. Asgari, *Number of* 5*-matchings in graphs*, MATCH Commun. Math. Comput. Chem. **69** (2013) 33–46.
6. R. Vesalian, R. Namazi and F. Asgari, *Number of* 6*-matchings in graphs*, MATCH Commun. Math. Comput. Chem. **73** (2015) 239–265.


E-mail: m.taheri@uast.ac.ir
E-mail: fathtabar@kashanu.ac.ir

# Total Double Roman Domination Number

Doost Ali Mojdeh
Department of Mathematics, University of Mazandaran, Babolsar, Iran
and Atieh Teymourzadeh*
Department of Mathematics, University of Mazandaran, Babolsar, Iran

Abstract. A double Roman dominating function on a graph $G$ is a function $f : V \to \{0, 1, 2, 3\}$ such that the following conditions hold. If $f(v) = 0$, then vertex $v$ must have at least two neighbors in $V_2$ or one neighbor in $V_3$ and if $f(v) = 1$, then vertex $v$ must have at least one neighbor in $V_2 \cup V_3$. The weight of a double Roman dominating function is the sum $w_f = \sum_{v\in V(G)} f(v)$. A total double Roman dominating function ($TDRDF$) on a graph $G$ with no isolated vertex is a $DRDF$ $f$ on $G$ with the additional property that the subgraph of $G$ induced by the set $\{v \in V : f(v) \neq 0\}$ has no isolated vertices. The total double Roman domination number $\gamma_{tdR}(G)$ is the minimum weight of a $TDRDF$ on $G$. We initiate the improvement of the upper bounds of $\gamma_{dR}(G)$ and we show that $\gamma_{tdR}(G) \leq \frac{4n}{3}$, for any graph with $\delta(G) \geq 2$.
**Keywords:** Total double Roman domination, Upper bound.
**AMS Mathematical Subject Classification [2010]:** 05C65.

## 1. Introduction

Let $G = (V, E)$ be a graph of order $n$ with $V = V(G)$ and $E = E(G)$. The open neighborhood of a vertex $v \in V(G)$ is the set $N(v) = \{u : uv \in E(G)\}$. The closed neighborhood of a vertex $v \in V(G)$ is $N[v] = N(v) \cup \{v\}$. We denote the degree of $v$ by $d_G(v) = |N(v)|$. By $\Delta = \Delta(G)$ and $\delta = \delta(G)$, we denote the maximum degree and minimum degree of a graph $G$, respectively. We write $K_n$, $P_n$ and $C_n$ for the complete graph, path and cycle of order $n$, respectively. A set $S \subseteq V$ in a graph $G$ is called a dominating set if $N[S] = V$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set in $G$, and a dominating set of $G$ of cardinality $\gamma(G)$ is called a $\gamma$-set of $G$. A total dominating set, abbreviated $TD$-set, of $G$ is a set $S$ of vertices of $G$ such that every vertex of $G$ is adjacent to a vertex in $S$. The total domination number of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a $TD$-set of $G$. A $TD$-set of $G$ of cardinality $\gamma_t(G)$ is a called a $\gamma_t(G)$-set [3, 5]. Given a graph $G$ and a positive integer $m$, assume that $g : V(G) \to \{0, 1, 2, \ldots, m\}$ is a function, and suppose that $(V_0, V_1, V_2, \ldots, V_m)$ is the ordered partition of $V$ induced by $g$, where $V_i = \{v \in V : g(v) = i\}$ for $i \in \{0, 1, \ldots, m\}$. So we can write $g = (V_0, V_1, V_2, \ldots, V_m)$. A Roman dominating function on graph $G$ is a function $f : V \to \{0, 1, 2\}$ such that if $v \in V_0$ for some $v \in V$, then there exists a vertex $w \in N(v)$ with $w \in V_2$. The weight of a Roman dominating function is the sum $w_f = \sum_{v\in V(G)} f(v)$, and the minimum weight of $w_f$ for every Roman dominating function $f$ on $G$ is called the Roman domination number of $G$, denoted by $\gamma_R(G)$ [2, 8]. A double Roman dominating function on

*Speaker

a graph $G$ is a function $f : V \to \{0, 1, 2, 3\}$ such that the following conditions are met:
(a) if $f(v) = 0$, then vertex $v$ must have at least two neighbors in $V_2$ or one neighbor in $V_3$,
(b) if $f(v) = 1$, then vertex $v$ must have at least one neighbor in $V_2 \cup V_3$.

The weight of a double Roman dominating function is the sum

$$w_f = \sum_{v \in V(G)} f(v),$$

and the minimum weight of $w_f$ for every double Roman dominating function $f$ on $G$ is called double Roman domination number of $G$. We denote this number with $\gamma_{dR}(G)$ and a double Roman dominating function of $G$ with weight $\gamma_{dR}(G)$ is called a $\gamma_{dR}(G)$-function of $G$. Double Roman domination was studied in [1, 6, 7] and elsewhere. The total double Roman dominating function ($TDRDF$) on a graph $G$ with no isolated vertex is a $DRDF$ $f$ on $G$ with the additional property that the subgraph of $G$ induced by the set $\{v \in V : f(v) \neq 0\}$ has no isolated vertices. The total double Roman domination number $\gamma_{tdR}(G)$ is the minimum weight of a $TDRDF$ on $G$. A $TDRDF$ on $G$ with weight $\gamma_{tdR}(G)$ is called a $\gamma_{tdR}(G)$-function, [4].

## 2. Main Results

In this section we show that $\gamma_{tdR}(G) \leq \frac{4n}{3}$. Before presenting the proof of the main result, we give some lemmas that are useful for investigation the main results. For integers $m$ and $k$ where $m \geq 3$ and $k \geq 1$, let $C_{m,k}$ be the graph obtained from a cycle $C_m : x_1x_2 \dots x_mx_1$ and a path $y_1y_2 \dots y_k$ by adding the edge $x_1y_1$. Let $\mathcal{Q}$ be the family of all connected graphs $G$ with $\delta(G) \geq 2$ and $\gamma_{tdR}(G) \leq \frac{4n}{3}$. Suppose that $A$ denotes the set of vertices of degree at least 3 in $G$, and let $B = V(G) - A$. A path $P$ of $G$ is called maximal if $V(P) \subseteq B$ and each end-vertex of $P$ is adjacent to a vertex of $A$. For each $i \geq 1$, let $\mathcal{P}_i = \{P|\ P$ is a maximal path with $|V(P)| = i$ $\}$. Let $\mathcal{P} = \bigcup_{i \geq 1} \mathcal{P}_i$. For $P \in \mathcal{P}$, let $X_P = \{u \in A|\ u$ is adjacent to an end-vertex of $P\}$.

Proposition 2.1. [4, Proposition 3] *For $n \geq 2$,*

$$\gamma_{tdR}(P_n) = \begin{cases} 6, & \text{if } n = 4, \\ \lceil \frac{6n}{5} \rceil, & \text{otherwise.} \end{cases}$$

We state a result from Proposition 2.1.

Lemma 2.2. *For $n \geq 3$ other than $n = 4$, $\gamma_{tdR}(P_n) \leq \frac{4n}{3}$.*

We state a result from [4].

Proposition 2.3. [4, Proposition 2] *For $n \geq 3$,*

$$\gamma_{tdR}(C_n) = \left\lceil \frac{6n}{5} \right\rceil.$$

By Proposition 2.3, we have:

Lemma 2.4. *For $n \geq 3$, $\gamma_{tdR}(C_n) \leq \frac{4n}{3}$.*

LEMMA 2.5. *Let $Q \in \mathcal{Q}$ and $u \in V(Q)$. If $G$ is a graph obtained from $Q$ and $C_{m,k}$ for some integers $m \geq 3$ and $k \geq 1$, by adding the edge $uy_k$, then $\gamma_{tdR}(G) \leq \frac{4|V(G)|}{3}$.*

PROOF. Let $f$ be a $\gamma_{tdR}(Q)$-function and $g$ be a $\gamma_{tdR}(C_{m,k})$-function. Then the function $h$ defined by $h(x) = f(x)$ for $x \in V(Q)$ and $h(x) = g(x)$ otherwise, is a $TDRDF$ of $G$. By a simple calculation we can see that $\gamma_{tdR}(C_{m,k}) \leq m + k + 1 \leq \frac{4|V(C_{m,k})|}{3}$. The fact $Q \in \mathcal{Q}$ and $\gamma_{tdR}(C_{m,k}) \leq \frac{4|V(C_{m,k})|}{3}$ imply that $\gamma_{tdR}(G) \leq \omega(f) + \omega(g) \leq \frac{4|V(Q)|}{3} + \frac{4|V(C_{m,k})|}{3} = \frac{4|V(G)|}{3}$. □

LEMMA 2.6. *Let $Q \in \mathcal{Q}$ and $u \in V(Q)$. If $G$ is a graph obtained from $Q$ and a cycle $C_m = x_1, \ldots, x_m x_1$, by adding the edge $ux_1$, then $\gamma_{tdR}(G) \leq \frac{4|V(G)|}{3}$.*

PROOF. Let $f$ be a $\gamma_{tdR}(Q)$-function and let $g$ be a $\gamma_{tdR}(C_m)$-function. Then the function $h$ defined by $h(x) = f(x)$ for $x \in V(Q)$ and $h(x) = g(x)$ otherwise, is a $TDRDF$ of $G$. By a simple calculation we can see that $\gamma_{tdR}(C_m) \leq m + 1 \leq \frac{4|V(C_m)|}{3}$. The fact $Q \in \mathcal{Q}$ and by $\gamma_{tdR}(C_m) \leq \frac{4|V(C_m)|}{3}$ imply that $\gamma_{tdR}(G) \leq \omega(f) + \omega(g) \leq \frac{4|V(Q)|}{3} + \frac{4|V(C_m)|}{3} = \frac{4|V(G)|}{3}$. □

THEOREM 2.7. *If $G$ is a simple graph of order $n$ with $\delta(G) \geq 2$, then*

$$\gamma_{tdR}(G) \leq \frac{4n}{3}.$$

PROOF. Suppose $G$ is a simple graph with $\delta(G) \geq 2$ and order $n$. The proof is given by induction on $n$. The result follows immediately for $n < 11$. Suppose $n \geq 11$ and let the result hold for all graphs of order less than $n$. Let $G$ be a graph of order $n \geq 11$. First $|A| = 2$, $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then by a simple calculation we can see $\gamma_{tdR}(G) \leq \frac{4n}{3}$. Now we consider the following cases.

**Case 1.** There exists $u \in A$ is adjacent to a path $P_1 \in \mathcal{P}_k$ where $k \geq 3$.

Let $P_1 = x_1 \ldots x_k$ and let $\{ux_1, ax_k\} \subseteq E(G)$ where $a \in A$. Assume that $G'$ is the graph obtained from $G$ by removing the vertices $x_1, x_2, x_3$ and joining $u$ to $x_4$ when $k \geq 4$. By the induction hypothesis, there exists a total double Roman dominating function $f$ of $G'$ such that $\omega(f) \leq \frac{4(n-3)}{3}$. If $k = 3$, then define the function $g$ by $g(x_1) = 1$, $g(x_3) = 0$, $g(x_2) = 3$ and $g(x) = f(x)$ otherwise. Now assume that $k > 3$.

If we assume that $f(u) = 3$, $f(x_4) = 0$, then define the function $g$ by $g(x_1) = 0$, $g(x_2) = 1$, $g(x_3) = 3$ and if $f(u) = 3$, $f(x_4) > 0$, then define the function $g$ by $g(x_2) = 0$, $g(x_1) = 1$, $g(x_3) = 3$ and if $f(u) = 2$, $f(x_4) = 0$, then define the function $g$ by $g(x_1) = 0$, $g(x_2) = g(x_3) = 2$, and if $f(u) = 0$, $f(x_4) = 2$, then define the function $g$ by $g(x_1) = g(x_2) = 2$, $g(x_3) = 0$ and $g(x) = f(x)$ and if $f(u) = 1$, $f(x_4) = 2$, then define the function $g$ by $g(x_1) = 3$, $g(x_2) = 0$, $g(x_3) = 1$ and if $f(u) = 2$, $f(x_4) = 1$, then define the function $g$ by $g(x_1) = g(x_3) = 2$, $g(x_2) = 0$ and $g(x) = f(x)$ otherwise. On the other hand define the function $g$ by $g(x_1) = 0, g(x_3) = 1$, $g(x_2) = 3$ and $g(x) = f(x)$ otherwise. Clearly, $g$ is a $TDRDF$ of $G$, and $\gamma_{tdR}(G) \leq \omega(f) + 4 \leq \frac{4(n-3)}{3} + 4 \leq \frac{4n}{3}$.

**Case 2.** There exists $u \in A$ is adjacent to maximal path $P_1 \in \mathcal{P}_2$.

**Subcase 2.1** Let $u$ be not adjacent to maximal path $P_2 \in \mathcal{P}_1$ such that $P_1, P_2$ be adjacent to an vertex $x \in A$. Let $P_1 = x_1x_2$ and let $\{ux_1, ax_2\} \subseteq E(G)$ where

$a \in A$. Assume $G'$ is the graph obtained from $G$ by removing the vertices $u, x_1, x_2$ and joining $a$ to each vertex $z \in N_G(u) - \{x_1, N_G(a)\}$.
**Subcase 2.2.** Let $u$ be adjacent to maximal path $P_2 \in \mathcal{P}_1$ such that $P_1, P_2$ are adjacent to an vertex $a \in A$. Assume that $P_1 = x_1x_2$, $P_2 = y$.
**1.** Let $u, a$ be adjacent to $c \in A$ where $deg(c) = 3$. Then assume that $G'$ is the graph obtained from $G$ by removing vertex $c$.
**2.** Let $u$ be adjacent to two maximal paths $P_3, P_4 \in \mathcal{P}_1$ where $P_2, P_3, P_4$ have no common vertex except in $u$. Then assume that $G'$ is the graph obtained from $G$ by removing the vertices maximal paths $P_2, P_3, P_4$, and if $u$ is not adjacent to $a$, then joining $u$ to $a$.
**3.** Let $u$ be adjacent to maximal path $P_3 \in \mathcal{P}_2$ where $P_1, P_3$ have no common vertex except in $u$. Then assume that $G'$ is the graph obtained from $G$ by removing the vertices maximal paths $P_1, P_3$, all $ua_i$s where $a_i \in A$ for $i \geq 1$.
**4.** Let $u$ be adjacent to maximal path $P_3 \in \mathcal{P}_1$. Then assume that $G'$ is the graph obtained from $G$ by removing the vertices maximal paths $P_2, P_3$, all $ua_i$s where $a_i \in A$ for $i \geq 1$, and if $u$ is not adjacent to $a$, then joining $u$ to $a$.
On the other hand assume that $G'$ is the graph obtained from $G$ by removing the vertices maximal paths $V(P_i)$s that $P_i$s are adjacent to $a, u$ for $i \geq 1$ and removing the vertices $u, a$.
**Case 3.** There exists $u \in A$ is adjacent to two maximal paths $P_1, P_2 \in \mathcal{P}_1$.
**Subcase 3.1** Let $u$ be adjacent to maximal paths $P_i = x_i \in \mathcal{P}_1$ where $i \geq 1$, $P_i$s have no common vertex except in $u$.
First let $\{ux_1, ax_1, ua, ux_2, x_2b\} \subseteq E(G)$ where $P_1 = x_1$, $P_2 = x_2$, $deg(a) = 3$, $a, b \in A$. Then assume that $G'$ is the graph obtained from $G$ by removing the vertices $x_1, x_2, u$ and joining $a$ to $b$.

On the other hand assume that $G'$ is the graph obtained from $G$ by removing the vertices $V(P_i)$s, $u$ for $i \geq 1$.
**Subcase 3.2** Let $u$ be adjacent to two maximal paths $P_1, P_2 \in \mathcal{P}_1$, $P_1$ be adjacent to $P_2$ in $a$ where $a \in A$. First assume that $deg(u) = 3$ and $deg(a) = 4$ and $P_1 = x_1$ and $P_2 = x_2$ and $\{ux_1, ax_1, ux_2, ax_2, ua\} \subseteq E(G)$ and $a$ is adjacent to maximal path $P' \in \mathcal{P}_1$ or an vertex $x \in A$. Then assume that $G'$ is the graph obtained from $G$ by removing the vertices $u, x_1, x_2, a, x'$ when $a$ is adjacent to $P'$ or by removing the vertices $u, x_1, x_2, a$ when $a$ is adjacent to an vertex $x \in A$.

On the other hand assume that $G'$ is the graph obtained from $G$ by removing the vertices $u$, $V(P_i)$s that $(P_i = x_i)$s are adjacent to $a, u$ and joining $a$ to each vertex $z \in N_G(u) - \{x_i, N_G(a)\}$ for $i \geq 1$.
**Case 4.** There exists $u \in A$ is adjacent to maximal path $P_1 \in \mathcal{P}_1$, to vertices $b_1, b_2 \in A$.
Let $P_1 = x$ and let $\{ux, ax, ub_1, ub_2\} \subseteq E(G)$ where $a \in A$. If $a = b_1$, $deg(a) = 3$, $a$ is adjacent to $c \in A$, then assume that $G'$ is the graph obtained from $G$ by removing the vertices $u, x, a$ and joining $c$ to vertex $b_2$. On the other hand by assume that $G'$ is the graph obtained from $G$ by removing the vertices $u, x, a$.

By the induction hypothesis, there exists a total double Roman dominating function $f$ of $G'$ such that $\omega(f) \leq \frac{4n(G')}{3}$. Then $f$ can be extended to a $TDRDF$ of $G$ of weight at most $\omega(f) + \frac{4n(G-G')}{3}$ and thus $\gamma_{tdR}(G) \leq \omega(f) + \frac{4n(G-G')}{3} \leq$

$\frac{4n(G')}{3} + \frac{4n(G-G')}{3} = \frac{4n}{3}$.

**Case 5.** $V(G) = A$.

Let $x, y, z \in V(G)$ such that $y$ be adjacent to $x, z$. Assume that $yv_i \in E(G)$ where $v_i \in V(G - \{x, y\})$ for $i \geq 1$. Now assume that $G'$ is the graph obtained from $G$ by removing $yv_i$s. By Case 4, $\gamma_{tdR}(G') \leq \frac{4n(G')}{3}$ and Since $\gamma_{tdR}(G) \leq \gamma_{tdR}(G-e)$ for every $e \in E(G)$,thus $\gamma_{tdR}(G) \leq \gamma_{tdR}(G-e) \leq \gamma_{tdR}(G') \leq \frac{4n(G)}{3}$.

According to the pervious Cases, Lemma 2.5, and Lemma 2.6, we may assume that $G$ is a graph obtained from a graph $H$ with $u \in V(H)$ and a cycle $C_m = x_1, \ldots, x_m x_1$, by identifying vertices $u$ and $x_1$. Let $z$ denote the vertex resulting by identifying $u$ and $x_1$. Then there exists two following Cases.

**Case 6.** $m \notin \{3, 5\}$.

Let $f$ be a $\gamma_{tdR}(H)$-function and let $g$ be a $\gamma_{tdR}$-function of the path of order $m-1$ induced by $x_2x_3 \ldots x_m$. Then the function $h$ defined by $h(x) = f(x)$ for $x \in V(H) - \{u\}$, $h(z) = f(u)$ and $h(x) = g(x)$ otherwise, is a $TDRDF$ of $G$ and $\gamma_{tdR}(G) \leq \frac{4n(G)}{3}$.

**Case 7.** $m \in \{3, 5\}$.

**1.** Let vertex $z$ be adjacent to maximal path $P = y$, $w$ is adjacent to $y$ where $z, w \in A$.

First assume that $\deg(w) = 3$, $w$ is adjacent to $z$. Assume that $G'$ is the graph obtained from $G$ by removing the vertices $z, x_2, \ldots, x_i, y, w$. If assume that $\deg(w) > 3$ or $w$ is not adjacent to $z$, then assume that $G'$ is the graph obtained from $G$ by removing the vertices $z, x_2, \ldots, x_i, y$.

**2.** Let vertex $z$ be adjacent to $a_i$ where $a_i \in A$, $i \geq 1$.

Assume that $G'$ is the graph obtained from $G$ by removing the vertices $z, x_2, \ldots, x_i$.

By the induction hypothesis, there exists a total double Roman dominating function $f$ of $G'$ such that $\omega(f) \leq \frac{4n(G')}{3}$. Then $f$ can be extended to a $TDRDF$ of $G$ of weight at most $\omega(f) + \frac{4n(G-G')}{3}$ and thus $\gamma_{tdR}(G) \leq \omega(f) + \frac{4n(G-G')}{3} \leq \frac{4n(G')}{3} + \frac{4n(G-G')}{3} = \frac{4n}{3}$. □

## References


1. R. A. Beeler, T. W. Haynes and S. T. Hedetniemi, *Double Roman domination*, Discrete Appl. Math. **211** (2016) 23–29.
2. E. J. Cockayne, P. M. Dreyer, Jr., S. M. Hedetniemi and S. T. Hedetniemi, *On Roman domination in graphs*, Discrete Math. **278** (1-3) (2004) 11–22.
3. T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Monographs and Textbooks in Pure and Applied Mathematics, 208. Marcel Dekker, Inc., New York, 1998.
4. G. Hao, L. Volkmann and D. A. Mojdeh, *Total double Roman domination in graphs*, Commun. Comb. Optim. **5** (2020) 27–39.
5. M. A. Henning and A.Yero, *Total Domination in Graphs*, Springer, New York, 2013.
6. N. Jafari Rad and H. Rahbani, *Some progress on the double Roman domination in graphs*, Discuss. Math. Graph Theory. **39** (2019) 41–53.
7. D. A. Mojdeh, A. Parsian and I. Masoumi, *Characterization of double Roman trees*, Ars Combin. (2018) to appear.
8. D. A. Mojdeh, A. Parsian and I. Masoumi, *Strong Roman domination number of complementary prism graphs*, Turk. J. Math. Comput. Sci. **11** (1) (2019) 40–47.

E-mail: damojdeh@umz.ac.ir
E-mail: atiehteymourzadeh@gmail.com

# Exatremal Polyomino Chains with Respect to Total Irregularity

Zahra Yarahmadi*
Department of Mathematics, Faculty of Science, Khorramabad Branch, Islamic Azad University, Khorramabad, I. R. Iran

Abstract. The total irregularity of a given simple graph $G$ is calculated by the formula $irr_t(G) = \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} \mid deg_G u - deg_G v \mid$ in which $deg_G v$ is the degree of a vertex $v$ in $G$. The aim of this paper is computing the total irregularity of polyomino chains. Upper and lower bounds for the total irregularity of polyomino chains together the fist and second extremal polyomino chain with respect to this graph invariant will be also presented.
**Keywords:** Total irregularity, Polyomino chain.
**AMS Mathematical Subject Classification [2010]:** 05C07, 05C35.

## 1. Introduction and Preliminaries

Let $G$ be a simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. If $a$ and $b$ are two adjacent vertices, then the edge connecting them is denoted by $e = ab$. The degree of a vertex $a$ is denoted by $deg_G a$. The subscript $G$ will be omitted, when the graph under consideration is clear from the context. The number $|deg_G a - deg_G b|$ is an important parameter associated to the edge $e$. This number is called the imbalance of the edge $e = ab$. In [2], Albertson defined the irregularity of $G$ as $irr(G) = \frac{1}{2}\sum_{e=uv\in E(G)} \mid deg_G u - deg_G v \mid$ .

The total irregularity of a graph $G$ was introduced by Abdo et al. [1] as $irr_t(G) = \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} \mid deg_G u - deg_G v \mid$ . They obtained all graphs $G$ such that $irr_t(G)$ are maximum possible value for them, and proved that among all trees of the same order the star has the maximal total irregularity.

It is usual to assume that a graph invariant $f$ is a measure of irregularity, when $f(G) = 0$ if and only if $G$ is regular. Since the irregularity and total irregularity are zero if and only if $G$ is regular, they are measures of irregularity for graphs. Furthermore, $irr_t(G)$ is an upper bound of $irr(G)$. Dimitrov [3], compared these two important measures of irregularity and proved that $irr_t(G) \leq n^2\frac{irr(G)}{4}$, when $G$ is an $n$-vertex connected graph. Moreover, for an arbitrary $n$-vertex tree $G$, we have $irr_t(G) \leq (n-2)irr(G)$.

A plane graph is a graph can be embedded on a sphere in such a way that edges intersect each other only in vertices of the graph. A connected graph $G$ is called 2-connected, if for each vertex $a$, $G - a$ is connected. A finite $2-$connected plane graph such that each interior face is surrounded by a regular square of length one is said to be a polyomino system. Polyominoes have a long and rich history, we convey for the origin polyominoes, Klarner [4]. A polyomino chain is a polyomino

*Speaker

system, in which the joining of the centers of its adjacent regular forms a path $c_1c_2\dots c_n$, where $c_i$ is the center of the $i-$th square.

Let $\mathbf{B_n}$ be the set of polyomino chains with $n$ squares. For $B_n \in \mathbf{B_n}$, it is easy to see that $|V(B_n)| = 2n+2$ and $|E(B_n)| = 3n+1$.

In the sequel, some important concepts about polyomino chains are presented that will be used later. A square of a polyomino chain has either one or two neighboring squares. If a square has one neighboring square, it is called terminal, and if it has two neighboring squares such that it has a vertex of degree 2, it is called kink. In Figure 1, the kinks are marked by $K$.

The linear chain $L_n$ with $n$ squares is a polyomino chains without kinks, see Figure 2.

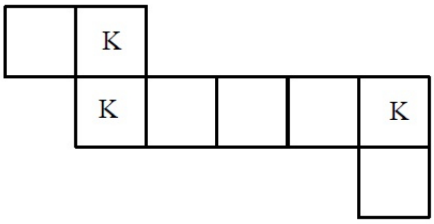



FIGURE 1. The kinks.

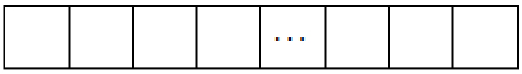



FIGURE 2. A linear chain.

A maximal linear chain in a polyomino chain is called a segment, if it includes the kinks and/or terminal squares at its end. The length of a segment $S$, $l(S)$, is the number of squares in $S$. Note that for each segment $S$ of a polyomino chain with $n \geq 2$ squares, $2 \leq l(S) \leq n$. In Figure 3, the squares on each segments of a polyomino chain is shown by directional lines.

A zigzag chain $Z_n$ with $n$ squares is a polyomino with $n-2$ kinks and in another word, a polyomino chain is a zig-zag chain if and only if the length of each segment is 2, Figure 4.

The present author [7], computed the first and second Zagreb indices of an arbitrary polyomino chain and determined the extremal polyomino chains with respect to Zagreb indices. For more information on polyomino chain, we refer to [5, 6, 8].

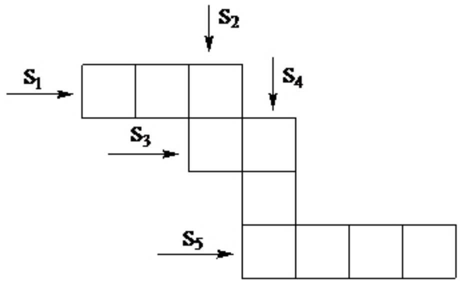



FIGURE 3. Segments of a polyomino chain.

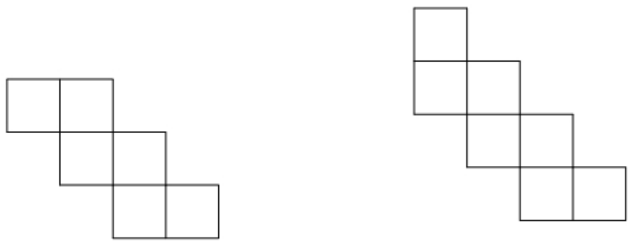


FIGURE 4. The zigzag chains $Z_6$ and $Z_7$.

## 2. Main Results

The aim of this section is to calculate the first and second extremal polyomino chains with respect to total irregularity. In what follows, we describe two types of transformations on polyomino chain, which help us to obtain extremal polyomino chains.

A transformation of type $\alpha$ for a polyomino chain is defined as follows: Let $B_n \in \mathbf{B_n}$, we choose a segment with maximum length containing at least one terminal square. Suppose that the length of this segment is t, denoted by $L_t$. Remove a terminal square of $B_n$ (which is not in $L_t$ ) and add it to terminal square of $L_t$, for obtaining $L_{t+1}$. This new polyomino chain is denoted by ${B^1}_n$. Notice that ${B^1}_n$ is not uniquely constructed, but by continuing this transformation to finite number we will find a linear chain with $n$ squares, $L_n$.

We now define a transformation of type $\beta$ for polyomino chains. To do this, we assume that $B_n \in \mathbf{B_n}$ and suppose $L$ is a segment of maximum length which contains at least one terminal square. We omit a terminal square of $L$ and add this square to another terminal square, to construct zigzag subgraph, step by step. The graph constructed from this transformation is denoted by ${B^{(1)}}_n$. Notice that ${B^{(1)}}_n$ is not uniquely constructed, but by continuing this transformation to finite number we will find a zigzag chain with $n$ squares, $Z_n$. It is obvious that if $B_n$ is a zigzag chain then ${B^{(1)}}_n = B_n$.

The total irregularity of $G$ is defined as $irr_t(G) = \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} \mid deg_G u - deg_G v \mid$, where $deg_G v$ is the degree of the vertex $v$ of $G$. It is easy to know that, for any $B_n \in \mathbf{B_n}$,

$$\{deg_{B_n}u \mid u \in V(B_n)\} = \{2, 3, 4\},$$

and the sequence of degree of vertices is as follows; $2, 2, \ldots, 2, 3, 3, \ldots, 3, 4, 4, \ldots, 4$. If set $n_2 = |\{u \in V(B_n)|deg_{B_n}u = 2\}|$, $n_3 = |\{u \in V(B_n)|deg_{B_n}u = 3\}|$ and $n_4 = |\{u \in V(B_n)|deg_{B_n}u = 4\}|$, it is easy to see that, $n_2 \geq 4$ and $|V(B_n)| = n_2 + n_3 + n_4$. By above argument one can see the following example.

Example 2.1. The total irregularity of linear and zigzag chains are computed as follows:

i) $irr_t(L_n) = 4n - 4$,
ii) $irr_t(Z_n) = n^2 + 2n - 4$.

Theorem 2.2. *Let $B_n \in \boldsymbol{B_n}$ and ${B^1}_n$ (${B^{(1)}}_n$) be a polyomino chain which is instructed by a transformation of type $\alpha$ (type $\beta$). Then,*

$$irr_t({B^1}_n) \leq irr_t(B_n) \leq irr_t({B^{(1)}}_n).$$

Corollary 2.3. *For any $B_n \in \boldsymbol{B_n}$, $irr_t(L_n) \leq irr_t(B_n) \leq irr_t(Z_n)$, with right (left) equality if and only if $B_n \cong Z_n$ ($B_n \cong L_n$).*

A semi linear polyomino chain $L'_n$ with $n$ squares is a polyomino chain, such that it has one kink. It is easy to see that $L'_n$ has 2 segments, which length are $r$ and $n - r + 1$ for $2 \leq r \leq n - 1$. We denote the set of all semi linear polyomino chains, by $\mathbf{L'_n}$.

Corollary 2.4. *For any $B_n \in \boldsymbol{B_n}$ and $B_n \neq L_n$ and $L'_n \in \boldsymbol{L'_n}$, following inequality is hold: $irr_t(L'_n) \leq irr_t(B_n)$ with equality if and only if $B_n \in \boldsymbol{L'_n}$.*

A semi zigzag polyomino chain $\widehat{Z}_n$ is a polyomino chain with exactly one segment of length 3 other segments have length 2. We denote the family of semi zigzag chains with $n$ squares by $\widehat{\mathbf{Z}}_\mathbf{n}$. By straightforward proof all semi zigzag polyomino chain with $n$ squares has the same total irregularity.

Corollary 2.5. *For any $B_n \in \boldsymbol{B_n}$ and $B_n \neq Z_n$ and $\widehat{Z}_n \in \widehat{\boldsymbol{Z}}_{\boldsymbol{n}}$, following inequality is hold: $irr_t(B_n) \leq irr_t(\widehat{Z}_n)$ with equality if and only if $B_n \in \widehat{\boldsymbol{Z}}_{\boldsymbol{n}}$.*

## References


1. H. Abdo, S. Brandt and D. Dimitrov, *The total irregularity of a graph*, Discrete Math. Theor. Comput. Sci. **16** (2014) 201–206.
2. M. Albertson, *The irregularity of a graph*, Ars Comb. **46** (1997) 219–225.
3. D. Dimitrov and R. Skrekovski, *Comparing the irregularity and the total irregularity of graphs*, Ars Math. Contemp. **9** (2015) 45–50.
4. D. A. Klarner, *Polyominoes,* In: J. E. Goodman and J. O'Rourke (Eds.), Handbook of Discrete and Computational Geometry, CRC Press LLC, Boca Raton, Florida, 1997.
5. L. Xu and S. Chen, *The PI index of polyomino chains*, Appl. Math. Lett. **21** (2008) 1101–1104.
6. J. Yang, F. Xia and S. Chen, *On the Randić index of of polyomino chains*, Appl. Math. Sci. **5** (5) (2011) 255–260.
7. Z. Yarahmadi, A. R. Ashrafi and S. Moradi, *Extremal polyomino chains with respect to Zagreb indices*, Appl. Math. Lett. **25** (2012) 166–171.
8. Y. Zeng and F. Zhang, *Extremal polyomino chains on k-matchings and k-independent sets*, J. Math. Chem. **42** (2) (2007) 125–140.


E-mail: z.yarahmadi@gmail.com

# Contributed Talk

## Logic

# NIP Theories and Actions

Alireza Mofidi*
Department of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), Hafez Avenue, 15194 Tehran, Iran
and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P. O. Box 19395-5746, Tehran, Iran

Abstract. The class of NIP theories is one of the most important classes of first order theories studied in mathematical logic and model theory. In recent years, the machinery of modern stability theory has been used to analyze several aspects of this class. We will consider this class from the point of view of dynamics of actions which naturally exist in there and prove some results on the entropy of those actions.
**Keywords:** NIP theories, Model theory (mathematical logic), Dynamics of group actions.
**AMS Mathematical Subject Classification [2010]:** 03C45, 03C95, 03C98.

## 1. Introduction

There are certain fundamental classes of theories studied in model theory. The class of stable theories is an important one of them which had received a lot of attentions in the classical model theory. One can see [5] as one of the main sources about stable class. Then, some other classes started to appear as central areas of research. The class of NIP theories is one of the most important classes of first order theories studied in the nowadays model theory. In recent years, the machinery of modern stability theory has been used to analyze several aspects of this class. The interested reader can refer to for example [1] and [2] for more details about the NIP theories. Also in [4], the notion of measures in the context of models and definable sets are introduced.

From another perspective, the theory of dynamical systems is recently involved with model theory, in particular stable theories and NIP theories, in several directions.

We will consider the class of NIP theories from the perspective of dynamics of actions of model theoretic objects and prove some results on the entropy of those actions. These result also helps one to have more connections between model theory and other fields of mathematics.

## 2. Some Definitions and Main Results

We work in the setting of first order logic. Assume that $T$ is a first order theory. Let $\phi(x, y)$ be a formula in the theory $T$. By $|x|$ and $|y|$ we mean the arity of

*Speaker

tuples $x$ and $y$. We use the notation $\mathbb{N}$ for the set of natural numbers starting from 1.

Definition 2.1. We say that a first order formula $\phi$ in the theory $T$ has the *order property* if there exists a model $M$ of $T$ such that for every $n \in \mathbb{N}$, there exist some elements $a_1, \ldots, a_n$, which are all $|x|$-tuples, and $b_1, \ldots, b_n$, which are all $|y|$-tuples, such that witness the order property of length $n$ for $\phi$ which means that we have $M \models \phi(a_i, b_j)$ if and only if $i \leqslant j$.

Definition 2.2. A first order theory is called a *stable* theory if no formula in that has the order property.

Definition 2.3. We say that $\phi$ has the *independence property or IP* if there exists some model $M$ of the theory $T$ such that for every natural number $n \in \mathbb{N}$, there exist $a_1, \ldots, a_n$, which are all $|x|$-tuples, such that witness independence property of length $n$ for $\phi$, which means that for every $J \subseteq \{1, \ldots, n\}$, there exists some $|y|$-tuple $b_J$ such that $M \models \phi(a_i, b_J)$ if and only if $i \in J$. A theory is called *NIP* if no formula in it has IP.

Definition 2.4. A first order theory is called a *NIP* theory if no formula in that theory has the independence property.

Remark 2.5. The class of NIP theories include the class of stable theories.

Definition 2.6. By a *Keisler measure* $\mu$ on $M^n$ over parameter set $A$ we mean a finitely additive probability measure on the set of definable sets with parameters from $A$ namely, $Def_A(M^n)$. When $n = 1$, we use $M$ instead of $M^1$ in all of the above notations.

Note that one can extract a countable additive Borel probably measure from each Keisler measure.

Entropy is an important numerical invariant associated to a (measure) dynamical system. We briefly review the definition of measure theoretic entropy.

Definition 2.7. Assume that $(M, \mathcal{A}, \mu)$ is a measure space and $f$ a measurable map on $M$ such that $\mu$ is $f$-invariant. Let $P = \{P_1, \ldots, P_n\}$ be an arbitrary measurable partition. Then, we define the *entropy* of $P$ with respect to $f$ with

$$h(f, P) := \lim_{n \to \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} f^{-i}(P)\right),$$

where

$$H\left(\bigvee_{i=0}^{n-1} f^{-i}(P)\right) := -\sum p_i \log p_i,$$

where $\bigvee_{i=0}^{n-1} f^{-i}(P)$ is the join of all partitions $f^{-i}(P)$'s and $p_i$'s are the measures of the atoms of the partition $\bigvee_{i=0}^{n-1} f^{-i}(P)$. The entropy of $f$ is defined as $h(f) := \sup_P h(f, P)$ where "sup" varies over all finite measurable partitions.

There are several characterizations for NIP theories in the paper [3] given by the author. In here we elaborate those studies and consider both automorphism groups and definable groups.

In the following statement, we call a group $G$ acting on space of types and model a nice group if it has the property that every indiscernible sequence is the orbit of some element of $M$ under the action of some subgroup of $G$ containing at least one member of infinite order. For example if $G = \mathrm{Aut}(M)$ and $M$ is homogeneous enough, then $G$ would be nice.

THEOREM 2.8. *Assume that the theory $T$ is NIP (does not have the independence property) and $G$ be either a definable group in $M$ acting on $S_G(M)$ or a subgroup of $\mathrm{Aut}(M)$ acting on $S(M)$, for $M$ a saturated enough model. If the theory $T$ is NIP, then the entropy of $\tau_g$ is zero for every $g \in G$ and $\mu$ which is $\tau_g$-invariant where $\tau_g$ represents the action by $g$. Conversely, assume that $G$ is a nice group acting on the type space in a way that each of the above mentioned entropies is zero (for action of $G$) for actions on one variable spaces, namely with arity of the variable equals 1. Then $T$ is NIP.*

PROOF. Since by assumption the measures of elements of $Def_M(G^N)$ (where these elements are seen as the clopen subsets of $S_G(M)$), approximate the measure of every measurable set, then it is enough to show that every partition consisting of elements of $Def_M(G^N)$ has zero entropy. Assume that $P = \{\phi_1(x, a_1), \ldots, \phi_r(x, a_r)\}$ is a set of clopen subsets of $S_G(M)$ partitioning $S_G(M)$ where $a_i \in M$ for each $i$. So

$$P' := \{\phi_1(G, a_1), \ldots, \phi_r(G, a_r)\},$$

is a definable partition of $G$. Now we use the definability of group operation in $G$ and define

$$\psi_i(x, y, z) := (y \in G) \wedge (\exists w \in G\ (\phi_i(w, z) \wedge x = y.w)),$$

for every $i = 1, \ldots, r$. Thus we have $\psi_i(x, s, a) = s.\phi_i(G, a)$ for every $s \in G$, $a \in M$ and $i$. For every $1 \leqslant i \leqslant r$ define

$$\mathcal{U}_{\phi_i} := \{\psi_i(x, g^{-j}, a_i) : j = 0, \ldots, n\} \cup \{\neg\psi_i(x, g^{-j}, a_i) : j = 0, \ldots, n\}.$$

Let $S_{\phi_i}$ be the set of complete types (maximal sets of consistent formulas) in the family $\mathcal{U}_{\phi_i}$. Then we have $\bigvee_{i=0}^{n-1} \tau_g^{-i}(P') \subseteq \{\bigcap_{i=1}^{r} q_i(M) : q_i \in S_{\phi_i} \text{ for each } i\}$. It follows that $|\bigvee_{i=0}^{n-1} \tau_g^{-i}(P')| \leqslant \prod_{i=1}^{r} |S_{\phi_i}|$. Note that

$$|\bigvee_{i=0}^{n-1} \tau_g^{-i}(P')| = |\bigvee_{i=0}^{n-1} \pi_g^{-i}(P)|.$$

Using the assumption, $|S_{\phi_i}|$ is bounded from above by a function of order $O(n^{t_i})$. It follows from a standard computation that for every $n$, $H(\bigvee_{i=0}^{n-1} \pi_g^{-i}(P)) \leqslant t \log(\sqrt[t]{c}\ n)$ for some $c$ where $t = t_1 + \cdots + t_r$. Now an easy calculation follows that $h(\pi_g, P) = 0$. The proof for the automorphisms is similar.

For the converse, by using a well-known theorem of Shelah, it would be sufficient to work with the space of one variable types $S^1(M)$. Using the assumption of being nice for $G$ and standard techniques, it is not hard to see that existence of independence property implies the existence of an instance of a formula $\phi(x, \bar{y})$, say $\phi(x, \bar{c})$, and an indiscernible sequence $b_i$'s, which is the orbit of some element under the action of a subgroup of $G$, with at least one member of infinite order, such that the indiscernible sequence has infinite alteration in the mentioned instance of the formula. Now it is not very difficult to observe that infinite alteration will give rise to the existence of a partition, say $\{\phi(S(M), \bar{c}), \neg\phi(S(M), \bar{c})\}$, on which the entropy of the action of some infinite order element of $G$ is positive, where by $\phi(S(M), \bar{c})$ we mean those types containing the formula $\phi(x, \bar{c})$. It follows that the entropy of the full action is also positive which leads to a contradiction. Therefore, independence property does not exist and we have NIP. □

## Acknowledgement

The author is indebted to Institute for Research in Fundamental Sciences, IPM, for support. This research was in part supported by a grant from IPM (No. 99030116)

## References

1. E. Hrushovski, Y. Peterzil and A. Pillay, *Groups, measures and the NIP*, J. Amer. Math. Soc. **21** (2) (2008) 563–596.
2. E. Hrushovski and A. Pillay, *On NIP and invariant measures*, J. Eur. Math. Soc. (JEMS) **13** (4) (2011) 1005–1061.
3. A. Mofidi, *On some dynamical aspects of NIP theories*, Arch. Math. Logic **57** (1-2) (2018) 37–71.
4. H. J. Keisler, *Measures and forking*, Ann. Pure Appl. Logic **45** (2) (1987) 119–169.
5. S. Shelah, *Classification Theory and the Number of Non-Isomorphic Models*, 2nd ed., North-Holland, Amsterdam, 1990.

E-mail: mofidi@aut.ac.ir

# Contributed Posters

## Algebra

# Some Results on Finitistic $n$-Self-Cotilting Modules

Mostafa Amini*
Faculty of Mathematical Sciences, University of Payame Noor, Tehran, Iran

ABSTRACT. Let $R$ be a ring, ${}_RU$ a module and $n$ a non-negative integer. In this paper, we obtain some another properties of finitistic $n$-self-cotilting modules. For instance, if ${}_RU$ is finitistic $n$-self-cotilting, then $k$-cop$_R$($n$-cop$_R(U)$) =k-cop$_R(U)$ for every $k \geq 1$. Some applications are also given.
**Keywords:** $n$-Finitely $U$-copresented module, Finitistic $n$-self-cotilting module.
**AMS Mathematical Subject Classification [2010]:** 13D02, 13E15, 16E10.

## 1. Introduction

Tilting (cotilting) modules were introduced by S. Brenner and M. Butler [3] as a natural generalization of injective cogenerators. Since then, Tilting (Cotilting) Theory is attracting the attention of many researchers in different aspects of mathematics, including mainly Representation Theory of (finite dimensional, Artin) algebras, Categories of Modules and Commutative Algebra. This theory has played an important role in relative homological algebra, recently. There are several papers devoted to tilting and cotilting modules, their generalizations and their applications in the representation of modules (e.g. see [1, 2, 4, 5, 8]).

Throughout this paper, all rings are associative with non-zero identity and all modules are unitary left modules. Let $R$ be a ring, $U$ an $R$-module and $n$ a non-negative integer. We denote by $Prod_RU$ the set of $R$-modules isomorphic to direct summands of a finite direct product of copies of $U$. For any homomorphism $f$, $Kerf$, $Imf$ and $Cokerf$ denote the kernel of $f$, image of $f$ and the cokernel of $f$, respectively. An $R$-module $L$ is called $n$-finitely $U$-copresented if there exists a long exact sequence of $R$-modules

$$0 \longrightarrow L \xrightarrow{\alpha_0} U^{X_0} \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-1}} U^{X_{n-1}},$$

such that $X_i$ is a finite set for every $0 \leq i \leq n-1$. The class of all $n$-finitely $U$-copresented $R$-modules is denoted by $n$-$cop_R(U)$. Note that 1-cop$_R(U)$ is the class of all finitely $U$-*cogenerated* modules and is denoted by $Cogen_R(U)$. The $R$-module $U$ is called $n$-$w_f$-*quasi-injective* if every exact sequence $0 \to L \to U^X \to M \to 0$ with $M \in n$-$cop(U)$ and $X$ a finite set stays exact under the functor $\mathrm{Hom}_R(-, U)$. An $R$-module $U$ is called finitistic $n$-*self-cotilting* if it is $n$-$w_f$-quasi-injective and $n$-$cop(U) = (n+1)$-$cop(U)$.

The notion of finitistic $n$-self-cotilting first was introduced by Breaz in [2]. He showed that finitistic $n$-self-cotilting modules can be characterized by using dual conditions of some generalizations for star modules. The classical star modules

*Presenter

were introduced by Menini and Orsatti [6] to study equivalences between module subcategories. We refer the reader to Colby and Fullers monograph [4] for more details on the classical star modules.

In this paper, we prove some other results about finitistic $n$-self-cotilting modules which were not considered by Breaz in [2]. For any class $\mathcal{C}$ of $R$-modules, we say that $\mathcal{C}$ is closed under $n$-kernels if for any exact sequence

$$0 \longrightarrow M \longrightarrow C_1 \longrightarrow C_2 \longrightarrow \cdots \longrightarrow C_n,$$

with $C_i \in \mathcal{C}$, for every $1 \leq i \leq n$, we have $M \in \mathcal{C}$. Let $k\text{-cop}_R(n\text{-cop}_R(U))$, for every $k \geq 1$, denote the class of all $R$-module $M$ such that there is an exact sequence

$$0 \longrightarrow M \longrightarrow C_1 \longrightarrow C_2 \longrightarrow \cdots \longrightarrow C_k,$$

with all ${}_RCi$ in $n\text{-cop}_R(U)$.

In section 2, it is shown that $k\text{-cop}_R(n\text{-cop}_R(U)) =\text{k-cop}_R(U)$ for every $k \geq 1$ and so, in particular, $n\text{-cop}_R(U)$ is closed under $n$-kernels and direct summands. Let $\xi : A \to R$ be a ring homomorphism and $U$ be an $R$-module. Then, it is proved that for any finitistic $n$-self-cotilting module ${}_AU$, one may have ${}_A\text{Hom}_A(R, U) \in n\text{-cop}_A(U)$ if and only if $n\text{-cop}_R(\text{Hom}_A(R, U)) = \{{}_RM \mid \; {}_AM \in n\text{-cop}_A(U)\}$.

## 2. Main Results

We begin this section by recalling the following definition.

Definition 2.1. [2, Definition 2.1] Let $U$ be an $R$-module. We say that an $R$-module $U$ is a finitistic $n$-self-cotilting module if it is $n$-$w_f$-quasi-injective and $n\text{-cop}_R(U) = (n + 1)\text{-cop}_R(U)$.

The following lemma will be used in this paper, frequently.

Lemma 2.2. *Let $R$ be a ring and $U$ an $R$-module. Then, the following statements are equivalent.*

(i) *${}_RU$ is a finitistic $n$-self-cotilting module.*

(ii) *Let $0 \to L \to M \to N \to 0$ be a short exact sequence with $L, M \in n\text{-cop}_R(U)$. Then $N \in n\text{-cop}_R(U)$ if and only if the sequence stays exact under the functor $\text{Hom}_R(-, U)$.*

Proof. (i) $\Longrightarrow$ (ii) It follows by [2, Proposition 3.7].

(ii)$\Longrightarrow$(i) It is easy to see that ${}_RU$ is $n$-$w_f$-quasi-injective. It remains to show that $n\text{-cop}_R(U) \subseteq (n + 1)\text{-cop}_R(U)$. If $L \in n\text{-cop}_R(U)$, then we obtain an exact sequence

$$0 \longrightarrow L \longrightarrow U^X \longrightarrow L^{'} \longrightarrow 0, \qquad (1)$$

where $X$ is a finite set. By [9, 14.3], we can assume that $\text{Hom}_R(L, U) \neq 0$ and so there exists a monomorphism $0 \to L \to U^{\text{Hom}_R(L,U)}$. Hence with no loss of generality, we may assume that $X \subseteq \text{Hom}_R(L, U)$ so that the sequence (1) stays exact under the functor $\text{Hom}_R(-, U)$. As $L, U^X \in n\text{-cop}_R(U)$, we have $L^{'} \in n\text{-cop}_R(U)$ by assumption. Therefore, $L \in (n + 1)\text{-cop}_R(U)$, as desired. □

Now, we use Lemma 2.2 to prove the following theorem which shows that the class of all $k$-finitely copresented modules by the class of $n$-finitely $U$-copresented modules equals with the class of $k$-finitely $U$-copresented modules, where $U$ is a finitistic $n$-self-cotilting module.

THEOREM 2.3. *Let $R$ be a ring and ${}_RU$ a module. If ${}_RU$ is finitistic $n$-self-cotilting, then $k$-$\mathrm{cop}_R(n$-$\mathrm{cop}_R(U))$ $=k$-$\mathrm{cop}_R(U)$ for every $k \geq 1$. Moreover, $n$-$\mathrm{cop}_R(U)$ is closed under $n$-kernels and direct summands.*

PROOF. It is easy to check that $k$-$\mathrm{cop}_R(U) \subseteq k$-$\mathrm{cop}_R(n$-$\mathrm{cop}_R(U))$. It remains to show that $k$-$\mathrm{cop}_R(n$-$\mathrm{cop}_R(U)) \subseteq k$-$\mathrm{cop}_R(U)$. To complete the proof, we proceed by induction on $k$. In case $k = 1$, the conclusion is clear. So we assume that $j$-$\mathrm{cop}_R(n\text{-}cop_R(U)) \subseteq j$-$\mathrm{cop}_R(U)$ for every $1 \leq j \leq k$. Let ${}_RM \in j$-$\mathrm{cop}_R(n\text{-}cop_R(U))$ be an $R$-module such that

$$0 \longrightarrow M \xrightarrow{i} C_1 \longrightarrow \cdots \longrightarrow C_{k+1},$$

is exact with all ${}_RC_i \in n$-$\mathrm{cop}_R(U)$. Suppose that ${}_RM_1 = \mathrm{Coker}(i)$. Then, there exists an exact sequence of the following form.

$$0 \longrightarrow M \xrightarrow{i} C_1 \xrightarrow{\pi} M_1 \longrightarrow 0.$$

Note that ${}_RM_1 \in k$-$\mathrm{cop}_R(U)$ by the induction hypothesis so that we have an exact sequence

$$0 \longrightarrow M_1 \xrightarrow{\alpha} U^X \longrightarrow M_1^{'} \longrightarrow 0,$$

where $X$ is a finite set and ${}_RM_1^{'} \in (k-1)$-$\mathrm{cop}_R(U)$. Since ${}_RC_1 \in n$-$\mathrm{cop}_R(U)$ and ${}_RU$ is a finitistic $n$-self-cotilting module, by Lemma 2.2, there exists an exact sequence

$$0 \longrightarrow C_1 \xrightarrow{\beta} U^Y \longrightarrow C_1^{'} \longrightarrow 0,$$

where $Y$ is a finite set and ${}_RC_1^{'} \in n$-$\mathrm{cop}_R(U)$. Now we can construct the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & & M & \xrightarrow{i} & C_1 & \xrightarrow{\pi} & M_1 \longrightarrow 0 \\
 & & \beta i \downarrow & & \gamma \downarrow & & \alpha \downarrow \\
0 \longrightarrow & & U^Y & \xrightarrow{(1,0)} & U^Y \oplus U^X & \xrightarrow{\delta} & U^X \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & & M^{'} & \longrightarrow & C_1^{''} & \longrightarrow & M_1^{'} \longrightarrow 0, \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}$$

where $\gamma(x) = (\beta(x), \alpha\pi(x))$ for every $x \in C_1$ and $\delta$ is the second projection map. Note that the sequence $0 \to C_1 \to U^Y \to C_1^{'} \to 0$ stays exact under the functor $\mathrm{Hom}_R(-, U)$. Since ${}_RU$ is a finitistic $n$-self-cotilting module, the sequence $0 \to C_1 \to U^Y \oplus U^X \to C_1^{''} \to 0$ also stays exact under the functor $\mathrm{Hom}_R(-, U)$ by the construction. Since ${}_RC_1 \in n$-$\mathrm{cop}_R(U)$, by Lemma 2.2, we have ${}_RC_1^{''} \in n$-$\mathrm{cop}_R(U)$. It follows from the bottom row that ${}_RM^{'} \in k$-$\mathrm{cop}_R(n$-$\mathrm{cop}_R(U))$ (because ${}_RM_1^{'} \in (k-1)$-$\mathrm{cop}_R(U)$). Thus, by the induction hypothesis, we have ${}_RM^{'} \in k$-$\mathrm{cop}(U)$. Finally, we obtain that ${}_RM \in (k+1)$-$\mathrm{cop}_R(U)$ from the left column. The last part of the theorem follows by [2, Propositions 3.3 and 3.7 (a)]. □

Proposition 2.4. *Let $\xi : A \to R$ be a ring homomorphism. Then for any ${}_AU$ and ${}_RM$, If ${}_AM \in \mathrm{Cogen}_A U$, then ${}_RM \in \mathrm{Cogen}_R \mathrm{Hom}_A(R, U)$. Moreover, ${}_AHom_A(R, U) \in \mathrm{Cogen}_A U$ if and only if*

$$\mathrm{Cogen}_R \mathrm{Hom}_A(R, U) = \{{}_RM \mid \ {}_AM \in \mathrm{Cogen_A U}\}.$$

Proof. Given ${}_RM$ and a monomorphism $0 \to {}_AM \to {}_AU^{\lambda}$, where $\lambda$ is a cardinal number. We obtain an $R$-monomorphism $0 \to \mathrm{Hom}_A(R, M) \to \mathrm{Hom}_A(R, U^{\lambda})$. On the other hand, since $\mathrm{Hom}_R(R, M) \subseteq \mathrm{Hom}_A(R, M)$, there exists a monomorphism $0 \to {}_RM \to {}_R\mathrm{Hom}_A(R, M)$. Hence the first statement follows. Thus

$$\mathrm{Cogen}_R \mathrm{Hom}_A(R, U) \supseteq \{{}_RM \mid \ {}_AM \in \mathrm{Cogen}_A U\}.$$

Now, from the monomorphisms $0 \to {}_A\mathrm{Hom}_A(R, U) \to {}_AU^{\Gamma}$ ($\Gamma$ is a cardinal number) and $0 \to {}_RM \to {}_R\mathrm{Hom}_A(R, U^{\lambda})$, we obtain the monomorphism $0 \to {}_AM \to {}_AU^{\Gamma\lambda}$ and this proves the remaining part. □

Now, we prove the next theorem which generalizes Proposition 2.4.

Theorem 2.5. *Let $\xi : A \to R$ be a ring homomorphism. Then for any finitistic $n$-selfcotilting module ${}_AU$, ${}_A\mathrm{Hom}_A(R, U) \in n\text{-}\mathrm{cop}_A(U)$ if and only if*

$$n - \mathrm{cop}_R(\mathrm{Hom}_A(R, U)) = \{{}_RM \mid \ {}_AM \in n - \mathrm{cop}_A(U)\}.$$

Proof. The sufficiency is easy. Now, we show the necessity. Take any ${}_RM$ such that ${}_AM \in n\text{-}\mathrm{cop}_A(U)$. By assumption, ${}_A\mathrm{Hom}_A(R, U) \in n\text{-}\mathrm{cop}_A(U)$. It is clear that $n\text{-}\mathrm{cop}_A(U) \subseteq \mathrm{Cogen}_A U$.

Thus, by Proposition 2.4, ${}_RM \in \mathrm{Cogen}_R \mathrm{Hom}_A(R, U)$. Hence we have an exact sequence of $R$-modules $0 \to M \to V^{X_1} \to M_1 \to 0$ which stays exact under the functor $\mathrm{Hom}_R(-, \mathrm{Hom}_A(R, U))$, where $V = \mathrm{Hom}_A(R, U)$ and $X_1$ is a finite set. By [7, Theorem 2.76], the exact sequence of induced $A$-modules $0 \to M \to V_1 \to M_1 \to 0$ stays exact under the functor $\mathrm{Hom}_A(-, U)$. Since ${}_AU$ is a finitistic $n$-self-cotilting module and ${}_AV_1$, ${}_AM \in n\text{-}\mathrm{cop}_A(U)$, we see that ${}_AM_1 \in n\text{-}\mathrm{cop}_A(U)$ by Lemma 2.2. It follows that ${}_RM_1$ is also an $R$-module such that ${}_AM_1 \in n\text{-}\mathrm{cop}_A(U)$. by repeating the process to the $R$-module ${}_RM_1$, we get ${}_RM \in n\text{-}\mathrm{cop}_R(\mathrm{Hom}_A(R, U))$. On the other hand, suppose that ${}_RM \in n\text{-}\mathrm{cop}_R(\mathrm{Hom}_A(R, U))$. Then we have an exact sequence of $R$-modules

$$0 \longrightarrow M \longrightarrow V^{X_1} \longrightarrow \cdots \longrightarrow V^{X_n},$$

where $X_i$ is a finite set for every $1 \leq i \leq n$. Thus we obtain an exact sequence of induced $A$-modules

$$0 \longrightarrow M \longrightarrow V^{X_1} \longrightarrow \cdots \longrightarrow V^{X_n}.$$

As ${}_AU$ is a finitistic $n$-self-cotilting module, $n\text{-}\mathrm{cop}_A(U)$ is closed under direct summands and $n$-kernels by Theorem 2.3. Hence ${}_AM \in n\text{-}\mathrm{cop}_A(U)$, as desired. □

## References

1. S. Bazzoni, *A characterization of n-cotilting and n-tilting modules*, J. Algebra **273** (1) (2005) 359–372.
2. S. Breaz, *Finitistic n-self-sotilting modules*, Comm. Algebra **37** (9) (2009) 3152–3170.
3. S. Brenner and M. C. R. Butler, *Generalizations of the Bernstein-Gelfand-Ponomarev reflection functors, Representation Theory II*, Proc. Second Internat. Conf. Carleton Univ. Ottawa, Ont. 1979. pp. 103–169. Lecture Notes in Math. 832, Springer, Berlin-New York, 1980.
4. R. R. Colby and K. R. Fuller, *Equivalence and Duality for Module Categories*, Cambridge Univ. Press, Cambridge, 2004.
5. R. Colpi, *Tilting modules and ∗-modules*, Comm. Algebra **21** (4) (1993) 1095-1102.
6. C. Menini and A. Orsatti, *Representable equivalences between categories of modules and applications*, Rend. Sem. Mat. Univ. Padova **82** (1989) 203–231.
7. J. J. Rotman, *An Introduction to Homological Algebra*, Academic Press, New York, 2009.
8. J. Wei, *n-Star modules and n-tilting modules*, J. Algebra **283** (2) (2005) 711–722.
9. R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach Science Publishers, Philadelphia, 1991.

E-mail: amini.pnu1356@gmail.com

# Torsion Submodule of a Finitely Generated Module over an Integral Domain

Somayeh Hadjirezaei*
Faculty of Mathematical Sciences, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran

Abstract. Let $R$ be a commutative ring and $M$ be an $R$-module. In this paper, we introduce Fitting ideals of $M$. Then we obtain a constructive description of $\mathrm{T}(M)$ which asserts the relation between torsion submodule and Fitting ideals of $M$.
**Keywords:** Torsion submodule, Fitting ideals, Integral domain.
**AMS Mathematical Subject Classification [2010]:** 13C05, 13D05.

## 1. Introduction

Let $R$ be a commutative ring with identity and $M$ be a finitely generated $R$-module. For a set $\{x_1, \ldots, x_n\}$ of generators of M there is an exact sequence

$$0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0\ ,$$

where $R^n$ is a free $R$-module with basis $\{e_1, \ldots, e_n\}$, the $R$-homomorphism $\varphi$ is defined by $\varphi(e_j) = x_j$ and $N$ is the kernel of $\varphi$. Let $N$ be generated by $u_\lambda = a_{1\lambda}e_1 + \cdots + a_{n\lambda}e_n$,where $\lambda$ in some index set $\Lambda$. Put $A := (a_{ij})$, $1 \leq i \leq n$, $j \in \Lambda$. We call $A$ a matrix presentation of $M$ and will denote the columns of $A$ by $\mathbf{a}_\lambda$, $\lambda \in \Lambda$. In the following we regard the elements of $R^n$ as being $n \times 1$ column vectors. Thus $N$ is a submodule of $R^n$ which is generated by columns of the matrix $A = (\mathbf{a}_\lambda)_{\lambda\in\boldsymbol{\Lambda}}$. For every $\mu = \{j_1, \ldots, j_q\} \subseteq \Lambda$, let $\mathrm{I}_\mu(N)$ be the ideal generated by subdeterminants of size $q$ of the matrix $(a_{ij} : 1 \leq i \leq n, j \in \mu)$. For each $q \geq 0$, the $(n-q)$th Fitting ideal of the module $M$, denoted by $\mathrm{Fitt}_{n-q}(M)$, is defined by $\sum_{\mu\subseteq\Lambda} i_\mu(N)$, where the summation is taken over all subsets $\mu$ of $\Lambda$ with cardinal $q$.

For $i > n$, $\mathrm{Fitt}_i(M)$ is defined $R$. It is known that $\mathrm{Fitt}_i(M)$ is the ideal determined by $M$, that is, it is determined uniquely by M and it does not depend on the choice of the set of generators of $M$ [2]. It follows, by the definition, that $\mathrm{Fitt}_i(M) \subseteq \mathrm{Fitt}_{i+1}(M)$ for every $i$. The most important Fitting ideal of $M$ is the first of the $\mathrm{Fitt}_i(M)$ that is nonzero. We shall denote this Fitting ideal by $i(M)$.

Fitting ideals are strong tools to charachterize modules and deal with splitting problems.

Buchsbaum and Eisenbud have shown in [1] that for a finitely generated $R$-module $M$, $i(M) = R$ if and only if $M$ is a projective $R$-module of constant rank. Also, a lemma of Lipman asserts that if $R$ is a quasilocal ring, $M = R^m/K$ and $i(M)$ is the $(m-q)$th Fitting ideal of $M$ then $i(M)$ is a regular principal ideal if and only if $K$ is finitely generated free and $M/\mathrm{T}(M)$ is free of rank $m-q$ [4].

*Presenter

## 2. Main Results

THEOREM 2.1. [3] *Let $M = \langle x_1, \dots, x_n\rangle$ be a regular $R$-module with a matrix presentation $A$. Then*

$$\mathrm{T}(M) = \{\sum_{i=1}^{n} b_i x_i; \quad \mathrm{rank}((b_1, \dots, b_n)^t|A) = \mathrm{rank}\, A\}.$$

Let $M$ be generated by the set $\{x_1, \dots, x_n\}$ and

$$0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0\ ,$$

be an exact sequence, where $R^n$ is a free $R$-module with the set $\{e_1, \dots, e_n\}$ of basis, the $R$-homomorphism $\varphi$ is defined by $\varphi(e_j) = x_j$ and $N$ is the kernel of $\varphi$. Let $N$ be generated by $u_\omega = a_{1\omega}e_1 + \cdots + a_{n\omega}e_n$, where $\omega$ in some index set $\Omega$. Assume that $A$ is the following matrix.

$$A = \begin{pmatrix} \cdots & \cdots & a_{1\omega} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & a_{n\omega} & \cdots \end{pmatrix}.$$

Let I$(M)$ =Fitt$_{n-t+1}(M)$, for some positive integer $t$. We call the set

$$\{i_1, \dots, i_t; i_1 < i_2 < \cdots < i_t\} \subseteq \{1, 2, \dots, n\},$$

a system of $t$ elements of $\{1, 2, \cdots, n\}$ and $\mu = \{j_1, \dots, j_{t-1}\} \subseteq \Omega$ a system of $t-1$ elements of $\Omega$. For two systems $\mu = \{j_1, \dots, j_{t-1}\}$ and $\lambda = \{i_1, \dots, i_t\}$ of $t-1$ and $t$ elements, respectively and for an element $a_1e_1 + \cdots + a_ne_n$ in $R^n$, Consider the $t \times t$ determinant

$$\begin{vmatrix} a_{i_1} & a_{i_1 j_1} & \dots & a_{i_1 j_{t-1}} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i_t} & a_{i_t j_1} & \dots & a_{i_t j_{t-1}} \end{vmatrix}.$$

Let $g_{i_k}^{\mu,\lambda}$ be the cofactor of this determinant with respect to $a_{i_k}$.

Assume that $L$ is the submodule of $R^n$ generated by elements $a_1e_1+\cdots+a_ne_n$ such that $a_{i_1}g_{i_1}^{\mu,\lambda} + \cdots + a_{i_t}g_{i_t}^{\mu,\lambda} = 0$, for all systems $\mu = \{j_1, \dots, j_{t-1}\}$ and $\lambda = \{i_1, \dots, i_t\}$ of $\{1, 2, \dots, n\}$.

Now assume that $R$ is an integral domain. let $\mu_0 = \{j_1, \dots, j_{t-1}\}$ be a system of $\{1, \dots, n\}$ such that there exists a system $\lambda_0 = \{i_1, \dots, i_t\}$, where at least one of $g_{i_j}^{\mu_0,\lambda_0}$ is nonzero for $1 \le j \le t$. We claim that

$$L = \{\sum_{j=1}^{n} a_j e_j : \sum_{j=1}^{t} a_{i_j} g_{i_j}^{\mu_0,\lambda} = 0, \text{for all systems } \lambda \text{ of t elements}\}.$$

Let $\mu = \{k_1, \dots, k_{t-1}\}$ and $\lambda = \{l_1, \dots, l_t\}$ be another systems. As Fitt$_{n-t}(M) = 0$, by McCoy's Theorem, there exist some elements $0 \neq b_i, 1 \le i \le t-1$, such that $b_iA_{k_i} = b_{i1}A_{j_1} + \cdots + b_{i(t-1)}A_{j_{t-1}}$, where $A_i$ is the $i$-th column of the matrix $A$. Let $\alpha_{k_i}$ be the $k_i$-th column of $A$ containing the rows $l_2, \dots, l_t$. We have

$$b_1 \dots b_{t-1} g_{l_1}^{\mu,\lambda} = \begin{vmatrix} b_1\alpha_{k_1}^j & \dots & b_{t-1}\alpha_{k_{t-1}}^j \end{vmatrix} =$$

$$\begin{vmatrix} \sum_{i=1}^{t-1} b_{1i}a_{l_2j_i} & \dots & \sum_{i=1}^{t-1} b_{(t-1)i}a_{l_2j_i} \\ \vdots & \vdots & \vdots \\ \sum_{i=1}^{t-1} b_{1i}a_{l_tj_i} & \dots & \sum_{i=1}^{t-1} b_{(t-1)i}a_{l_tj_i} \end{vmatrix} =$$

$$\left| \begin{pmatrix} a_{l_2j_1} & \dots & a_{l_2j_{t-1}} \\ \vdots & \vdots & \vdots \\ a_{l_tj_1} & \dots & a_{l_tj_{t-1}} \end{pmatrix} \begin{pmatrix} b_{11} & \dots & b_{(t-1)1} \\ \vdots & \vdots & \vdots \\ b_{1(t-1)} & \dots & b_{(t-1)(t-1)} \end{pmatrix} \right| = detB^t g_{l_1}^{\mu_0,\lambda} = (detB) g_{l_1}^{\mu_0,\lambda},$$

where

$$B = \begin{pmatrix} b_{11} & \dots & b_{(t-1)1} \\ \vdots & \vdots & \vdots \\ b_{1(t-1)} & \dots & b_{(t-1)(t-1)} \end{pmatrix}.$$

Thus $b_1 \cdots {}_{t-1}\, g_{l_1}^{\mu,\lambda} = (detB) g_{l_1}^{\mu_0,\lambda}$. Similarly, we have $b_1 \cdots {}_{t-1}\, g_{l_i}^{\mu,\lambda} = (detB) g_{l_i}^{\mu_0,\lambda}$ for every $1 \le i \le t$. Since $R$ is an integral domain, neither $b_1 \cdots {}_{t-1}$ nor $detB$ is the zero element of $R$ that do not depend on the index $i$. Hence $\sum_{i=1}^{t} a_{l_i} g_{l_i}^{\mu,\lambda} = 0$ if and only if $\sum_{i=1}^{t} a_{l_i} g_{l_i}^{\mu_0,\lambda} = 0$. Thus

$$L = \{\sum_{j=1}^{n} a_j e_j : \sum_{j=1}^{t} a_{i_j} g_{i_j}^{\mu_0,\lambda} = 0, \text{for all systems } \lambda \text{ of t elements } \}.$$

Let $\lambda_i = \{i_1, \dots, i_t\}, 1 \le i \le k = \begin{pmatrix} n \\ t \end{pmatrix}$ and $C = \begin{pmatrix} \beta_{\lambda_1} \\ \vdots \\ \beta_{\lambda_k} \end{pmatrix}_{k \times n}$, where $\beta_{\lambda_i} = \begin{pmatrix} 0 \dots g_{i_1}^{\mu_0,\lambda_i} \dots g_{i_t}^{\mu_0,\lambda_i} 0 \dots 0 \end{pmatrix}$. In fact $(a_1, \dots, a_n)^t$ is an element of $L$ if and only if $(a_1, \dots, a_n)^t$ is a solution of the equation $CY = 0$. As $S^{-1}R$ is a field, this set equation is solved easily. Therefore, if $C$ be the above matrix we have

$$T(M) = \{\sum_{i=1}^{n} a_i x_i; (a_1, \dots, a_n)^t \quad \text{is a solution of the equation } CY = 0\}.$$

EXAMPLE 2.2. Let $R$ be an integral domain and $M$ be an $R$-module generated by four elements $x_1, x_2, x_3, x_4$. Let $I(M) = Fitt_2(M)$. Thus by the notation of Theorem 2.1, $t = 3$. Without loss of generality let $\mu_0 = \{1, 2\}$. Assume that $\lambda_1 = \{2, 3, 4\}, \lambda_2 = \{1, 3, 4\}$ and $\lambda_3 = \{1, 2, 4\}$. We have $g_1^{\mu_0,\lambda_2} = g_2^{\mu_0,\lambda_1}, g_1^{\mu_0,\lambda_3} = -g_3^{\mu_0,\lambda_1}$, $g_1^{\mu_0,\lambda_4} = g_4^{\mu_0,\lambda_1}, g_2^{\mu_0,\lambda_3} = g_3^{\mu_0,\lambda_2}, g_2^{\mu_0,\lambda_4} = -g_4^{\mu_0,\lambda_2}, g_3^{\mu_0,\lambda_4} = g_4^{\mu_0,\lambda_3}$. Therefore, we should solve the following equation.

$$\begin{pmatrix} g_4^{\lambda_1} & g_4^{\lambda_2} & g_4^{\lambda_3} & 0 \\ g_3^{\lambda_1} & g_3^{\lambda_2} & 0 & g_4^{\lambda_3} \\ g_2^{\lambda_1} & 0 & g_3^{\lambda_2} & -g_4^{\lambda_2} \\ 0 & g_2^{\lambda_1} & -g_3^{\lambda_1} & g_4^{\lambda_1} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = 0. \tag{1}$$

After solving this set equation in $S^{-1}R$, we can obtain the solution of this equation in $R$. Thus

$$T(M) = \{\sum_{i=1}^{4} a_i x_i; (a_1, a_2, a_3, a_4)^t \quad \text{is a solution of the Eq. (1)}\}.$$

## References

1. D. A. Buchsbaum and D. Eisenbud, *What makes a complex exact?* J. Algebra **25** (1973) 259–268.
2. H. Fitting, *Die Determinantenideale eines Moduls*, Jahresber. Dtsch. Math.-Ver. **46** (1936) 195–228.
3. S. Hadjirezaei, *On the relation between torsion submodule and fitting ideals*, submitted.
4. J. Lipman, *On the Jacobian ideal of the module of differentials*, Proc. Amer. Math. Soc. **21** (1969) 423–426

E-mail: s.hajirezaei@vru.ac.ir

# On the Structure of a Module and its Torsion Submodule

Somayeh Hadjirezaei*
Faculty of Mathematical Sciences, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran

Abstract. Let $R$ be a commutative ring and $M$ be a finitely generated $R$-module. In this paper we investigate the structure of an $R$-module and the torsion submodule, using Fitting ideals and comaximal ideals.
**Keywords:** Decomposition, Fitting ideal, Torsion submodule, Comaximal ideals.
**AMS Mathematical Subject Classification [2010]:** 13C05, 13D05.

## 1. Introduction

Throughout this paper $R$ is a commutative and unitary ring. Assume that $M$ is a finitely generated $R$-module and $\phi : G \longrightarrow F$ is a map of free modules. Let $I_j(\phi)$ be the image of the map $\Lambda^j G \otimes \Lambda^j F^* \longrightarrow R$ induced by $\Lambda^j \phi : \Lambda^j G \longrightarrow \Lambda^j F$. If we choose some bases for $G$ and $F$, $\phi$ can be represented by a matrix and it is seen that $I_j(\phi)$ is generated by the minors of size $j$ of this matrix. We consider the covention that the subdeterminant of size 0 is one. Therefore $I_0(\phi) = R$ and in general we have $I_j(\phi) = R$ if $j \leq 0$.

It is not necessary to suppose that $G$ is free to make the preceding construction. We can replace $G$ by any free module mapping onto $G$ without changing $I_j(\phi)$.

This ideals of minors turn out to define invariants of a module that generalize the usual invariants for finitely generated abelian groups, the fact that first was observed by Fitting [2].

Let $\phi : G \longrightarrow F \longrightarrow M \longrightarrow 0$ and $\phi' : G' \longrightarrow F' \longrightarrow M \longrightarrow 0$ be two presentation with $F$ and $F'$ finitely generated free modules of rank $r$ and $r'$. For every $j, 0 \leq j$, we have $I_{r-j}(\phi) = I_{r'-j}(\phi')$. So we define $\mathrm{Fitt}_j(M) = I_{r-j}(\phi)$.

## 2. Torsion Submodule and Fitting Ideals

Fitting ideals give us some good information about a module. We will show that, if we know the Fitting ideals of a module, we can determine some intersting properties of the module and even sometimes to know the structure of it.

We define the torsion submodule of a module $M$ to be the submodule of $M$ consisting of all elements of $M$ that are annihilated by a regular element of $R$ and will denote this submodule by $\mathrm{T}(M)$. Let $rank(A)$ denotes the largest integer $t$ of a matrix $A$ such that there exists a nonzero subdeterminant of size $t$ of the matrix $A$.

The most important Fitting ideal of $M$ is the first of the $\mathrm{Fitt}_i(M)$ that is nonzero. We denote this Fitting ideal by $i(M)$.

*Presenter

Theorem 2.1. *Let $(R, \mathfrak{m})$ be a local ring which is not a field and let $M$ be a finitely generated non-torsionfree $R$-module. If $i(M) = \mathfrak{m}$, then $\mathrm{T}(M)$ is a vector space over the field $R/\mathfrak{m}$.*

Proof. Let $M$ be generated by elements $x_1, \ldots, x_n$ and let

$$0 \longrightarrow N \longrightarrow R^n \xrightarrow{\phi} M \longrightarrow 0\ ,$$

be an exact sequence and $A = (a_{ij}), 1 \leq i \leq n, j \in \wedge$, be the matrix presentation of $\phi$. Since $\mathfrak{m}$ is a maximal ideal of $R$, then $\mathrm{rank}(A) = 1$ so that $\mathfrak{m} =< a_{ij}, 1 \leq i \leq n, j \in \wedge >$. Assume that $K$ is a submodule of $R^n$ generated by the elements $a_1e_1 + \cdots + a_ne_n$ such that $a_ia_{jt} = a_ja_{it}$, where $t \in \wedge$. We claim that $\phi(K) = \mathrm{T}(M)$. Let $x = \sum_{i=1}^n a_ix_i \in \phi(K)$. Thus $a_ia_{jt} = a_ja_{it}$, for every $i, j, t$, $1 \leq i, j \leq n$ and $t \in \wedge$. Hence $a_{11}x = \sum_{i=1}^n a_{11}a_ix_i$. On the other hand, $N$= Ker$(\phi)$, hence $a_{11}x_1 + \cdots + a_{n1}x_n = 0$. Thus $a_{11}x_1 = -a_{21}x_2 - \cdots - a_{n1}x_n$. Therefore $a_{11}x = \sum_{i=1}^n a_{11}a_ix_i = a_{11}a_1x_1 + \sum_{i=2}^n a_{11}a_ix_i = \sum_{i=2}^n -a_1a_{i1}x_i + \sum_{i=2}^n a_{11}a_ix_i = \sum_{i=2}^n (a_{11}a_i - a_{i1}a_1)x_i = 0$. Thus $a_{11}x = 0$. By the same argument we have $a_{ij}x = 0$, for every $i, j$. Hence $\mathfrak{m}\varphi(K) = 0$. Since $0 \neq \mathrm{T}(M)$, hence $\mathfrak{m}$ contains a regular element. Thus $\phi(K) \subseteq \mathrm{T}(M)$. Now let $x = \sum_{i=1}^n a_ix_i \in \mathrm{T}(M)$. We have to show that $a_ia_{jt} = a_ja_{it}$, for every $i, j, t$. Since $x \in \mathrm{T}(M)$, hence there exists a regular element $q$ in $R$ such that $qx = \sum_{i=1}^n qa_ix_i = 0$. Thus $(qa_1, \ldots, qa_n)^t \in N$. So there exist some elements $c_k \in R$, $1 \leq k \leq n$ such that $qa_i = \sum_{k=1}^n c_ka_{ik}, 1 \leq i \leq n$. Let $t, 1 \leq t \leq n$, be arbitrary and fixed. We have $qa_ia_{jt} = \sum_{k=1}^n c_ka_{ik}a_{jt}$, for every $i, j, 1 \leq i, j \leq n$. Thus $q(a_ia_{jt} - a_ja_{it}) = \sum_{k=1}^n c_k(a_{ik}a_{jt} - a_{jk}a_{it})$. Since $\mathrm{rank}(\phi) = 1$, hence $a_{ik}a_{jt} - a_{jk}a_{it} = 0$ and so we have $q(a_ia_{jt} - a_ja_{it}) = 0$. Since $q$ is a regular element, $a_ia_{jt} - a_ja_{it} = 0$. Hence $\mathfrak{m}\,\mathrm{T}(M) = 0$ and so $\mathrm{T}(M)$ is an $R/\mathfrak{m}$-module, as desired. ☐

We have the following corollaries:

Corollary 2.2. *Let $M$ be a non-torsionfree $R$-module generated by $m$ elements. If $i(M) = \mathrm{Fitt}_{m-1}(M)$, then $i(M) \subseteq \mathrm{ann}(\mathrm{T}(M))$.*

Proof. By the same argument of the proof of previous Theorem. ☐

Corollary 2.3. *Let $(R, \mathfrak{m})$ be a local ring and let $M$ be a finitely generated non-torsionfree $R$-module with $i(M) = \mathfrak{m}$. Then $\cdot_R(\mathrm{T}(M)) = \mathrm{gldim}(R)$.*

Proposition 2.4. *Let $(R, \mathfrak{m})$ be a local ring and let $M$ be a finitely generated $R$-module such that $0 \neq \mathrm{T}(M)$ is a direct summand of $M$. Then $\mathrm{T}(M) \not\subseteq \mathfrak{m}M$.*

Proof. Let $M \cong \mathrm{T}(M) \oplus M/\mathrm{T}(M)$ and $\mathrm{T}(M)$ and $M/\mathrm{T}(M)$ have minimal generator sets with $m$ and $n$ elements, respectively. So $M$ and $M/\mathfrak{m}M$ have minimal generator sets with $m + n$ elements. Thus $M/\mathfrak{m}M \cong (R/\mathfrak{m})^{m+n}$. If $\mathrm{T}(M) \subseteq \mathfrak{m}M$, then the sequence

$$M/\mathrm{T}(M) \longrightarrow M/\mathfrak{m}M \longrightarrow 0\ ,$$

is exact. Hence, by [3, Lemma 2.5],

$$\mathrm{Fitt}_n(M/\mathrm{T}(M)) \subseteq \mathrm{Fitt}_n(M/\mathfrak{m}M) = \mathfrak{m}^{m+n}.$$

But since $M/\mathrm{T}(M)$ is generated by $n$ elements, hence $\mathrm{Fitt}_n(M/\mathrm{T}(M)) = R \subseteq \mathfrak{m}^{m+n}$, a contradiction. ☐

Example 2.5. Let $(R, \mathfrak{m})$ be a Noetherian local ring and $M$ be a finitely generated non-torsionfree $R$-module. By [4], if $\mathrm{I}(M) = \mathfrak{m}$ is a principal ideal then $M/T(M)$ is free. Therefore, the exact sequence

$$0 \longrightarrow \mathrm{T}(M) \longrightarrow M \longrightarrow M/T(M) \longrightarrow 0\ ,$$

splits and so $M \cong \mathrm{T}(M) \oplus M/\mathrm{T}(M)$ which implies that $\mathrm{T}(M)$ is a direct summand of $M$. Hence $\mathrm{T}(M) \nsubseteq \mathfrak{m}M$ by Proposition 2.4.

## 3. Comaximal Ideals and Decomposition of a Module

Decomposition of a module to the direct sum of submodules is one of the basic topics in theory of rings and modules. An $R$-module $M$ is called decomposable if there exist nonzero submodules $A$ and $B$ of $M$ such that $M = A \oplus B$, otherwise $M$ is called indecomposable.

In the previous section we investigate decomposition of a module using Fittinig ideals and in this section we investigate decomposition of a module using comaximal ideals.

Definition 3.1. Let $R$ be a ring and $I$ and $J$ be two ideals of $R$. We say that $I$ and $J$ are comaximal ideals if $I + J = R$, in other words $1 \in I + J$.

Example 3.2. Every two maximal ideal are comaximal ideals. For example $2\mathbb{Z}$ and $3\mathbb{Z}$ are comaximal ideals of $\mathbb{Z}$.

We recall the following well-known properties of comaximal ideals.

Proposition 3.3. *Let $I$ and $J$ be two comaximal ideals of a ring $R$. Then for every positive integers $m$ and $n$, $J^m$ and $I^n$ are comaximal ideals.*

Proposition 3.4. *Let $\{X_i\}_{i=1}^n$ be a set of pairwise comaximal ideals of $R$, where $n \geq 2$. Then*

$$\bigcap_{i=1}^{n} X_i = \prod_{i=1}^{n} X_i.$$

Proposition 3.5. *Let $\{X_i\}_{i=1}^n$ be a set of pairwise comaximal ideals of $R$. Then*

$$\sum_{i=1}^{n}(\bigcap_{j \neq i} X_j) = R = \sum_{i=1}^{n}(\prod_{j \neq i} X_j).$$

Proof. It is clear by induction on $n$ and Proposition 3.4. □

Lemma 3.6. *The following conditions are equivalent.*

1) *Every prime ideal of $R$ contains a uniqe minimal prime ideal of $R$.*
2) *Every two distinct minimal prime ideal of $R$, are comaximal.*

Proof. $1 \Rightarrow 2$) Let $P_1$ and $P_2$ be two distinct minimal prime ideal of $R$ and $P_1 + P_2 \neq R$. Thus there exists a maximal ideal $M$ such that $P_1 + P_2 \subseteq M$. We have

$$P_1, P_2 \subseteq P_1 + P_2 \subseteq M,$$

which is a contradiction, because M is a prime ideal that contains two distinct minimal prime ideals.

$2 \Rightarrow 1$) By Zorn's Lemma, $R$ contains a minimal prime ideal. Assume that $P$ is a prime ideal of $R$ such that $P$ contains two distinct minimial prime ideal $P_1$ and $P_2$. Hence $P_1 + P_2 = R \subseteq P$, a contradiction. □

Now, we bring up the main result of this section.

THEOREM 3.7. *Let $A = \{Q_i\}_{i\in I}$ be a set of minimal prime ideals of $R$ and $M$ be a finitely generated $R$-module with generating set $\{x_1, \ldots, x_n\}$. Let every prime ideal of $R$ contains a unique minimal prime ideal. Then for any $x_i \in M, 1 \leq i \leq n$, $Ann_R(x_i)$ contains a finite intersection of elements of $A$ if and only if $M = \oplus_{i\in I} Ann_M(Q_i)$.*

PROOF. By Lemma 3.6, every two minimal prime ideals of $R$ are comaximal. Thus $A = \{Q_i\}_{i\in I}$ is a set of pairwise comaximal ideals. By [1, Theorem 2.3], for any $i = 1, \ldots, n, Ann_R(x_i)$ contains a finite intersection of elements of $A$ if and only if $M = \oplus_{i\in I} Ann_M(Q_i)$. □

EXAMPLE 3.8. Let $R = \mathbb{Z}$, $M = \mathbb{Z}_2 \oplus \mathbb{Z}_3$. One can see that $A = \{2\mathbb{Z}, 3\mathbb{Z}, 5\mathbb{Z}\}$ is a set of minimal prime ideals of $\mathbb{Z}$. We have $2\mathbb{Z} \cap 3\mathbb{Z} \cap 5\mathbb{Z} = 30\mathbb{Z}$. It is clear that $30\mathbb{Z} \subseteq Ann_{\mathbb{Z}}(\mathbb{Z}_2 \oplus \mathbb{Z}_3)$. Hence, by Theorem 3.7, $M = Ann_M(2\mathbb{Z}) \oplus Ann_M(3\mathbb{Z}) \oplus Ann_M(5\mathbb{Z})$. In fact, we have

$$Ann_M(2\mathbb{Z}) = \mathbb{Z}_2 \oplus 0, \ Ann_M(3\mathbb{Z}) = 0 \oplus \mathbb{Z}_3, \ Ann_M(5\mathbb{Z}) = 0.$$

## References

1. G. Birkenmeier and C. Edward Ryan, *Module decomposition using pairwise comaximal ideals*, Comm. Algebra **44** (10) (2016) 4200–4211.
97–101.
2. H. Fitting, *Die determinantenideale eines moduls*, Jahresber. Deutsch. Math.-Verein **46** (1936) 195–228.
3. S. Hadjirezaei, *On finitely generated module whose first nonzero fitting ideal is prime*, Bull. Iranian Math. Soc. **46** (6) (2020) 1675–1681.
4. J. Lipman, *On the Jacobian ideal of the module of differentials*, Proc. Amer. Math. Soc. **21** (1969) 423–426.

E-mail: s.hajirezaei@vru.ac.ir

# Spectrum Topology on Lattice Equality Algebras

Sogol Niazian*
Faculty of Medicine, Tehran Medical Sciences, Islamic Azad University, Tehran, Iran

ABSTRACT. In this paper, we construct an spectrum topology on a lattice equality algebra (where spectrum is the set of all $\vee$-irreducible filters of an equality algebra) and prove this topology is a compact $T_0$-space and maximal spectrum (as a subspace of that) is a compact $T_1$ topological space.
**Keywords:** Equality algebra, Maximal filter, $\vee$-Irreducible filter, Spectrum topology.
**AMS Mathematical Subject Classification [2010]:** 03G10, 06B99, 06B75.

## 1. Introduction

Since the algebra of truth values is no longer than a residuated lattice, Novák and De Beats generalized residuated lattices and proposed a specific algebra called EQ-algebra [3]. If the product operation in EQ-algebra is replaced by another binary operation smaller or equal than the original one, then we will obtain a new algebra which was introduced by Jenei and called equality algebra [1]. As equality algebras could be candidates for a possible algebraic semantics in fuzzy type theory, their study is highly motivated.

DEFINITION 1.1. [1] An algebraic structure $(E; \wedge, \sim, 1)$ of type $(2, 2, 0)$ is called an *equality algebra* if, for all $u, v, w \in E$, it satisfies the following conditions.

(E1) $(E, \wedge, 1)$ is a commutative idempotent integral monoid.
(E2) $u \sim v = v \sim u$.
(E3) $u \sim u = 1$.
(E4) $u \sim 1 = u$.
(E5) $u \leq v \leq w$ implies $u \sim w \leq v \sim w$ and $u \sim w \leq u \sim v$.
(E6) $u \sim v \leq (u \wedge w) \sim (v \wedge w)$.
(E7) $u \sim v \leq (u \sim w) \sim (v \sim w)$.

The operation $\wedge$ is called *meet* and $\sim$ is an *equality* operation. On the equality algebra, we write $u \leq v$ if and only if $u \wedge v = u$. It is easy to see that "$\leq$" is a partial order relation on $E$. Also, we define the operation "$\to$" on $E$ as $u \to v = u \sim (u \wedge v)$. Equality algebra $(E; \wedge, \sim, 1)$ is denoted by $\mathcal{E}$ unless otherwise state.

An equality algebra $\mathcal{E}$ is *bounded* if there is an element $0 \in E$ such that $0 \leq u$ for all $u \in E$. In a bounded equality algebra $\mathcal{E}$, we define the negation "$^-$ " on $E$ by $u^- = u \to 0 = u \sim 0$ for all $u \in E$. Equality algebra $\mathcal{E}$ is called *prelinear* if 1 is the unique upper bound of the set $\{u \to v,\ v \to u\}$ for all $u, v \in E$. A *lattice equality algebra* is an equality algebra which is a lattice.

*Presenter

Proposition 1.2. [1, 4] *Let* $(E; \wedge, \sim, 1)$ *be an equality algebra. Then, for all* $u, v, w \in E$, *the following conditions hold.*

i) $u \to v = 1$ *if and only if* $u \leq v$.
ii) $1 \to u = u$, $u \to 1 = 1$, *and* $u \to u = 1$.
iii) $u \leq v \to u$.
iv) $u \leq v$ *implies* $v \to w \leq u \to w$ *and* $w \to u \leq w \to v$.
v) $u \to v = u \to (u \wedge v)$.
vi) *If* $\mathcal{E}$ *is a lattice equality algebra, then* $u \to v = (u \vee v) \to v$.

A non-empty subset $F$ of $E$ is called a *filter* of $\mathcal{E}$ if and only if, for all $u, v \in E$, $1 \in F$ and if $u \in F, u \to v \in F$, then $v \in F$. The set of all filters of $\mathcal{E}$ is denoted by $\mathcal{F}(\mathcal{E})$. Clearly, $1 \in F$ for any filter $F$ of $\mathcal{E}$. A filter $F$ of $\mathcal{E}$ is called a *proper filter* of $\mathcal{E}$ if $F \neq E$. Clearly, if $\mathcal{E}$ is a bounded equality algebra, then a filter of $\mathcal{E}$ is proper if and only if it is not containing 0. A proper filter is called a *maximal* filter if that is not included in any other proper filter of $\mathcal{E}$. We denote by $\mathcal{M}ax(\mathcal{E})$ the set of all maximal filters of $\mathcal{E}$.

Definition 1.3. [2] Let $X \subseteq E$. The smallest filter of $\mathcal{E}$ containing $X$ is called *the generated filter by* $X$ *in* $\mathcal{E}$ which is denoted by $\langle X \rangle$. Indeed, $\langle X \rangle = \bigcap\limits_{X \subseteq F \in \mathcal{F}(\mathcal{E})} F$. Also,

$$\langle X \rangle = \{u \in E \mid p_1 \to (p_2 \to (\cdots \to (p_n \to u)\ldots)) = 1, \text{ for some } n \in \mathbb{N} \text{ and } p_1, \ldots, p_n \in X\}.$$

From now on, let $\mathcal{E}$ be a lattice equality algebra unless otherwise state.

Definition 1.4. [2] Let $F \in \mathcal{F}(\mathcal{E})$ be proper. Then $F$ is called a $\vee-$irreducible filter if $u \vee v \in F$ implies $u \in F$ or $v \in F$ for all $u, v \in E$. We denote by $\mathcal{S}pec(\mathcal{E})$ the set of all $\vee$-irreducible filters of $\mathcal{E}$.

Theorem 1.5. [2] *Let* $F \in \mathcal{F}(\mathcal{E})$ *be proper. Then*

i) *For each* $p \notin F$, *there exists* $P \in \mathcal{S}pec(\mathcal{E})$ *such that* $F \subseteq P$ *and* $p \notin P$.
ii) *There exists a maximal filter of* $\mathcal{E}$ *that contains* $F$.

Theorem 1.6. *Any maximal filter of* $\mathcal{E}$ *is a* $\vee$*-irreducible filter of* $\mathcal{E}$. *Indeed, we have* $\mathcal{M}ax(\mathcal{E}) \subseteq \mathcal{S}pec(\mathcal{E})$.

## 2. Main Results

Definition 2.1. Let $X \subseteq E$ and $p \in E$. Then the set of all $\vee$-irreducible filters of $\mathcal{E}$ containing $X$ is denoted by $V(X) = \{P \in \mathcal{S}pec(\mathcal{E}) | \ X \subseteq P\}$ and $V(p) = \{P \in \mathcal{S}pec(\mathcal{E}) | \ p \in P\}$.
The complement of $V(X)$ in $\mathcal{S}pec(\mathcal{E})$ is denoted by $U(X)$. Indeed,

$$U(X) = \{P \in \mathcal{S}pec(\mathcal{E}) | \ X \nsubseteq P\}, \quad U(p) = \{P \in \mathcal{S}pec(E) | \ p \notin P\}.$$

Proposition 2.2. *Let* $X, Y \subseteq E$. *Then, we have the following statements.*

i) *If* $X \subseteq Y$, *then* $U(X) \subseteq U(Y)$.
ii) $U(X) = U(\langle X \rangle)$.
iii) $U(X) = \mathcal{S}pec(\mathcal{E})$ *if and only if* $\langle X \rangle = E$. *In particular,* $U(E) = \mathcal{S}pec(\mathcal{E})$.
iv) $U(X) = \emptyset$ *if and only if* $X = \emptyset$ *or* $X = \{1\}$.
v) $U(\bigcup_{i \in \Delta} X_i) = \bigcup_{i \in \Delta} U(X_i)$.
vi) $U(\langle X \rangle \cap \langle Y \rangle) = U(X) \cap U(Y)$.

vii) *$U(X) = U(Y)$ if and only if $\langle X\rangle = \langle Y\rangle$.*
viii) *If $p \in X$, then $U(p) \subseteq U(X)$.*

THEOREM 2.3. *Let $\tau = \{U(X) |\ X \subseteq E\}$. Then $\tau$ is a topology on $\mathcal{S}pec(\mathcal{E})$.*

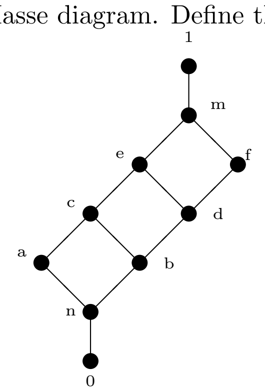
The topology induced by $\tau = \{U(X) |\ X \subseteq E\}$ on $\mathcal{S}pec(\mathcal{E})$ is called the *spectrum topology* and $U(X)$ is the open subsets of $\mathcal{S}pec(\mathcal{E})$ for any $X \subseteq E$. Also, $\beta = \{U(p)\}_{p\in E}$ is a basis for this topology.

THEOREM 2.4. *For $p \in E$, the following statements hold true.*
i) *$U(p)$ is compact in $(\mathcal{S}pec(\mathcal{E}), \tau)$.*
ii) *If $\mathcal{E}$ is bounded, then $(\mathcal{S}pec(\mathcal{E}), \tau)$ is a compact topological space.*

THEOREM 2.5. *$(\mathcal{S}pec(\mathcal{E}), \tau)$ is a $T_0$-topological space.*

In the following example, we can see that $(\mathcal{S}pec(\mathcal{E}), \tau)$ is not a $T_1$-topological space, in general.

EXAMPLE 2.6. Let $E = \{0, n, a, b, c, d, e, f, m, 1\}$ be a set with the following Hasse diagram. Define the operation $\sim$ on $E$ as follows.




| $\sim$ | 0 | $n$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $m$ | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | $m$ | $f$ | $e$ | $d$ | $c$ | $b$ | $a$ | $n$ | 0 |
| $n$ | $m$ | 1 | $f$ | $e$ | $d$ | $c$ | $b$ | $a$ | $n$ | $n$ |
| $a$ | $f$ | $f$ | 1 | $d$ | $e$ | $b$ | $c$ | $n$ | $a$ | $a$ |
| $b$ | $e$ | $e$ | $d$ | 1 | $f$ | $e$ | $d$ | $c$ | $b$ | $b$ |
| $c$ | $d$ | $d$ | $e$ | $f$ | 1 | $d$ | $e$ | $b$ | $c$ | $c$ |
| $d$ | $c$ | $c$ | $b$ | $e$ | $d$ | 1 | $f$ | $e$ | $d$ | $d$ |
| $e$ | $b$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 | $d$ | $e$ | $e$ |
| $f$ | $a$ | $a$ | $n$ | $c$ | $b$ | $e$ | $d$ | 1 | $f$ | $f$ |
| $m$ | $n$ | $n$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 | $m$ |
| 1 | 0 | $n$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $m$ | 1 |

| $\rightarrow$ | 0 | $n$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $m$ | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $n$ | $m$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $a$ | $f$ | $f$ | 1 | $f$ | 1 | $f$ | 1 | $f$ | 1 | 1 |
| $b$ | $e$ | $e$ | $e$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $c$ | $d$ | $d$ | $e$ | $f$ | 1 | $f$ | 1 | $f$ | 1 | 1 |
| $d$ | $c$ | $c$ | $c$ | $e$ | $e$ | 1 | 1 | 1 | 1 | 1 |
| $e$ | $b$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 | $f$ | 1 | 1 |
| $f$ | $a$ | $a$ | $a$ | $c$ | $c$ | $e$ | $e$ | 1 | 1 | 1 |
| $m$ | $n$ | $n$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 | 1 |
| 1 | 0 | $n$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $m$ | 1 |

Then $(E, \sim, \wedge, 1)$ is an equality algebra and

$$\mathcal{S}pec(\mathcal{E}) = \{\underbrace{\{1\}}_{P_1}, \underbrace{\{f, m, 1\}}_{P_2}, \underbrace{\{a, c, e, m, 1\}}_{P_3}\},$$

$$\begin{aligned} U(0) &= \mathcal{S}pec(\mathcal{E}) = U(n) = U(b) = U(d), \\ U(a) &= \{P_1, P_2\} = U(c) = U(e), \\ U(f) &= \{P_1, P_3\}, \quad U(m) = \{P_1\}, \quad U(1) = \emptyset. \end{aligned}$$

Hence, $\tau = \{\emptyset, \{P_1\}, \{P_1, P_2\}, \{P_1, P_3\}, \mathcal{S}pec(\mathcal{E})\} = \beta$. Let $P_1, P_3 \in \mathcal{S}pec(\mathcal{E})$. As there is no open subset $U \in \tau$ such that $P_3 \in U$ and $P_1 \notin U$, we get $(\mathcal{S}pec(\mathcal{E}), \tau)$ is not a $T_1$-space. Also, it is not a Hausdorff space.

Let $\mathcal{E}$ be a bounded equality algebra. The set of all $u \in E$ such that $u \vee u^- = 1$ and $u \wedge u^- = 0$ is denoted by $B(\mathcal{E})$.

THEOREM 2.7. *Let $\mathcal{E}$ be a bounded equality algebra. The following statements hold true.*

i) *$B(\mathcal{E}) = E$ implies $(\mathcal{S}pec(\mathcal{E}), \tau)$ is a Hausdorff space.*
ii) *If $(\mathcal{S}pec(\mathcal{E}), \tau)$ is connected, then $B(\mathcal{E}) = \{0, 1\}$.*

The converse of Theorem 2.7(ii) is not necessarily true. Since $\mathcal{E}$ is the equality algebra as in Example 2.6. Then $B(\mathcal{E}) = \{0, 1\}$ and $(\mathcal{S}pec(E), \tau)$ is not connected; because, if we suppose $U_1 = \{P_1, P_2\}$ and $U_2 = \{P_1, P_3\}$, then $\mathcal{S}pec(\mathcal{E}) = U_1 \cup U_2$.

By Theorem 1.6, $\mathcal{M}ax(\mathcal{E}) \subseteq \mathcal{S}pec(\mathcal{E})$. Thus we can consider the spectrum topology on $\mathcal{M}ax(\mathcal{E})$ that is called *maximal spectrum* of $\mathcal{E}$. For $X \subseteq E$ and $u \in E$, define

$$V_M(X) = V(X) \cap \mathcal{M}ax(\mathcal{E}), \quad V_M(u) = V(u) \cap \mathcal{M}ax(\mathcal{E}),$$

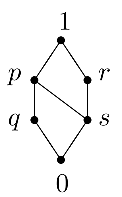
$$U_M(X) = U(X) \cap \mathcal{M}ax(\mathcal{E}), \quad U_M(u) = U(u) \cap \mathcal{M}ax(\mathcal{E}).$$

Then $\{U_M(X) |\ X \subseteq E\}$ and $\{U_M(u) |\ u \in E\}$ are the family of open sets and basis for the topology on $\mathcal{M}ax(\mathcal{E})$. Also, $\mathcal{M}ax(\mathcal{E})$ is a compact $T_0$-space.

THEOREM 2.8.

i) *The topological space $(\mathcal{M}ax(\mathcal{E}), \tau)$ is a $T_1$-space.*
ii) *The topological space $(\mathcal{S}pec(\mathcal{E}), \tau)$ is a $T_1$-space if and only if $\mathcal{S}pec(\mathcal{E}) = \mathcal{M}ax(\mathcal{E})$.*

EXAMPLE 2.9. Let $(E = \{0, p, q, r, s, 1\}, \leq)$ be a lattice with the following Hasse diagram. Define the operation "$\sim$" on $E$ as follows.




| $\sim$ | 0 | $a$ | $b$ | $c$ | $d$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 1 | $d$ | $c$ | $b$ | $a$ | 0 |
| $a$ | $d$ | 1 | $a$ | $d$ | $c$ | $a$ |
| $b$ | $c$ | $a$ | 1 | 0 | $d$ | $b$ |
| $c$ | $b$ | $d$ | 0 | 1 | $a$ | $c$ |
| $d$ | $a$ | $c$ | $d$ | $a$ | 1 | $d$ |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | 1 |

| $\rightarrow$ | 0 | $a$ | $b$ | $c$ | $d$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| $a$ | $d$ | 1 | $a$ | $c$ | $c$ | 1 |
| $b$ | $c$ | 1 | 1 | $c$ | $c$ | 1 |
| $c$ | $b$ | $a$ | $b$ | 1 | $a$ | 1 |
| $d$ | $a$ | 1 | $a$ | 1 | 1 | 1 |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | 1 |

Then $(E, \sim, \wedge, 1)$ is a bounded equality algebra. Hence, we have

$$\mathcal{S}pec(\mathcal{E}) = \{\underbrace{\{r, 1\}}_{P}, \underbrace{\{p, q, 1\}}_{Q}\} = \mathcal{M}ax(\mathcal{E}),$$

and $\tau = \{\emptyset,\ \{P\},\ \{Q\},\ \mathcal{S}pec(\mathcal{E})\}$. It is easy to see that $(\mathcal{S}pec(\mathcal{E}), \tau)$ is a $T_1$-space.

THEOREM 2.10. *If $\mathcal{E}$ is prelinear, then $(\mathcal{M}ax(\mathcal{E}), \tau)$ is a Hausdorff space.*

## References

1. S. Jenei, *Equality algebras*, Studia Logica **100** (6) (2012) 1201–1209.
2. S. Niazian, M. Aaly Kologani and R. A. Borzooei, *Co-annihilators on equality algebras*, submitted.
3. V. Novák and B. De Baets, *EQ-algebras*, Fuzzy Sets and Systems **160** (20) (2009) 2956–2978.
4. F. Zebardast, R. A. Borzooei and M. Aaly Kologani, *Results on equality algebras*, Inform. Sci. **381** (2017) 270–282.

E-mail: s.niazian@iautmu.ac.ir

# Cofiniteness and Associated Primes of Local Cohomology Modules via Linkage

Maryam Jahangiri
Faculty of Mathematical Sciences and Computer, Kharazmi University, Tehran, Iran
and Khadijeh Sayyari*
Faculty of Mathematical Sciences and Computer, Kharazmi University, Tehran, Iran

Abstract. Let $R$ be a commutative Noetherian ring and $M$ be a finitely generated $R$-module. Considering the new concept of linkage of ideals over a module, we study associated prime ideals and cofiniteness of local cohomology modules of $M$ with respect to some linked ideals over it.
**Keywords:** Linkage of ideals, Local cohomology, Cohen-Macaulay modules.
**AMS Mathematical Subject Classification [2010]:** 13D45, 13C45, 13C14.

## 1. Introduction

Let $R$ be a commutative Noetherian ring with $1 \neq 0$, $\mathfrak{a}$ be an ideal of $R$ and $M$ be an $R$-module. For $i \in \mathbb{Z}$, the $i$-th local cohomology functor with respect to $\mathfrak{a}$ is defined to be the $i$-th right derive functor of the $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}(-)$, where $\Gamma_{\mathfrak{a}}(M) = \cup_{n \in \mathbb{N}_0} 0 :_M \mathfrak{a}^n$. There are lots of problems in the study of local cohomology modules and finiteness problems in this subject attract lots of interests. One of the main problems in this topic is finiteness of the set of associated prime ideals, i.e. Ass $H^i_{\mathfrak{a}}(M)$. Although, Singh showed that Ass $H^i_{\mathfrak{a}}(M)$ might be infinite, but in some cases it is finite.

Another important topic in commutative algebra and algebraic geometry is the theory of linkage. The significant work of Peskine and Szpiro stated this theory in the modern algebraic language. In a recent paper [4], inspired by the works in the ideal case, the authors present the concept of the linkage of ideals over a module. Let $M$ be a finitely generated $R$-module. Let $\mathfrak{a}$, $\mathfrak{b}$ and $I$ be ideals of $R$ with $I \subseteq \mathfrak{a} \cap \mathfrak{b}$ such that $I$ is generated by an $M$-regular sequence and $\mathfrak{a}M \neq M \neq \mathfrak{b}M$. Then, $\mathfrak{a}$ and $\mathfrak{b}$ are said to be linked by $I$ over $M$, denoted by $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$, if $\mathfrak{b}M = IM :_M \mathfrak{a}$ and $\mathfrak{a}M = IM :_M \mathfrak{b}$. Also, the ideal $\mathfrak{a}$ is said to be linked over $M$ if there exist ideals $\mathfrak{b}$ and $I$ of $R$ such that $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$. This is a generalization of the classical concept of linkage when $M = R$. In this paper, we consider the above generalization and study cofiniteness and associated prime ideals of local cohomology modules $H^i_{\mathfrak{a}}(M)$ where $\mathfrak{a}$ is a linked ideal over $M$. More precisely, we show that if $R$ is Cohen-Macaulay and $t \in \mathbb{N}$ then, for any ideal $\mathfrak{a}$ of $R$, Ass $H^t_{\mathfrak{a}}(R)$ is finite if and only if Ass $H^t_{\mathfrak{a}}(R)$ is finite for any linked ideal $\mathfrak{a}$ of $R$ (Theorem 2.4). Then, we study finiteness of some Ext modules and, as a

*Presenter

corollary, we show that if $\mathfrak{a} \sim_{(I;R)} \mathfrak{b}$ then $H^i_{\mathfrak{a}}(R)$ is $\mathfrak{a}$-cofinite and $\mathfrak{b}$-cofinite if and only if it is $I$-cofinite (Corollary 2.8).

Also, using attached prime ideals of local cohomology modules, we present some necessary and sufficient conditions for the finitely generated $R$-module $M$ to be Cohen-Macaulay in terms of the existence of some special linked ideals over it (Theorem 2.9).

Throughout the paper, $R$ denotes a non-trivial commutative Noetherian ring, $\mathfrak{a}$ and $\mathfrak{b}$ are non-zero proper ideals of $R$ and $M$ will denote a finitely generated $R$-module.

## 2. Associated Prime Ideals and Cofiniteness

In this section, we study finiteness of the set of associated prime ideals of local cohomology modules and the cofinite property of these modules over some linked ideals.

We begin by the definition of one of our main tools.

Definition 2.1. *Assume that $\mathfrak{a}M \neq M \neq \mathfrak{b}M$ and let $I \subseteq \mathfrak{a} \cap \mathfrak{b}$ be an ideal generated by an $M$-regular sequence. Then we say that the ideals $\mathfrak{a}$ and $\mathfrak{b}$ are linked by $I$ over $M$, denoted by $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$, if $\mathfrak{b}M = IM :_M \mathfrak{a}$ and $\mathfrak{a}M = IM :_M \mathfrak{b}$. The ideals $\mathfrak{a}$ and $\mathfrak{b}$ are said to be geometrically linked by $I$ over $M$ if $\mathfrak{a}M \cap \mathfrak{b}M = IM$. Also, we say that the ideal $\mathfrak{a}$ is linked over $M$ if there exist ideals $\mathfrak{b}$ and $I$ of $R$ such that $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$. $\mathfrak{a}$ is $M$-selflinked by $I$ if $\mathfrak{a} \sim_{(I;M)} \mathfrak{a}$.*

Definition 2.2. Assume that $I$ is an ideal of $R$ which is generated by an $M$-regular sequence. Set

$$S_{(I;M)} := \{\mathfrak{a} \triangleleft R | I \subsetneqq \mathfrak{a}, \mathfrak{a} = IM :_R (IM :_M \mathfrak{a})\}.$$

Note that $S_{(I;R)}$ actually contains all linked ideals by $I$.

The following theorem provides an equivalent condition for the finiteness of $\operatorname{Ass} H^t_{\mathfrak{a}}(M)$.

Theorem 2.3. *Let $R$ be a local ring and $M$ be a Cohen-Macaulay $R$-module and set $t \in \mathbb{N}$. Then, the following statements are equivalent.*

(i) *For any ideal $\mathfrak{a}$ of $R$, $\operatorname{Ass} H^t_{\mathfrak{a}}(M)$ is a finite set.*
(ii) *For any ideal $\mathfrak{a} \in S_{(I;M)}$ and all $I$ generated by an $M$-regular sequence of length $t-1$, $\operatorname{Ass} H^t_{\mathfrak{a}}(M)$ is a finite set.*

Proof. (ii) $\to$ (i): Let $\mathfrak{a}$ be an ideal and assume that, for all ideals $I$ generated by an $M$-regular sequence of length $t-1$ and all ideals $\mathfrak{b} \in S_{(I;M)}$, $\operatorname{Ass} H^t_{\mathfrak{b}}(M)$ is a finite set. By [1, Lemma 2.4], we may assume that $\operatorname{ht}_M \mathfrak{a} = t-1$. Using [6, 2.11], there exist a radical ideal $\mathfrak{a}' \supseteq \mathfrak{a}$ and an ideal $\mathfrak{a} \supseteq I$ such that $\operatorname{grade}_M \mathfrak{a}' = \operatorname{grade}_M \mathfrak{a} = t-1$ and $\mathfrak{a}' \in S_{(I;M)}$. In view of the structure of $\mathfrak{a}'$ (in the proof of [6, 2.11(i)], and the Cohen-Macaulayness of $M$, we have

$$\operatorname{Ass} \frac{R}{\mathfrak{a}'} = \{\mathfrak{p} | \mathfrak{p} \in \operatorname{Min} \operatorname{Ass} \frac{M}{\mathfrak{a}M}, \operatorname{ht}_M \mathfrak{p} = \operatorname{ht}_M \mathfrak{a}\}.$$

Set $\mathfrak{b} := \cap_{\mathfrak{p} \in \operatorname{Min} \operatorname{Ass} \frac{M}{\mathfrak{a}M} - \operatorname{Ass} \frac{R}{\mathfrak{a}'}} \mathfrak{p}$. Then $\operatorname{ht}_M \mathfrak{b} > t-1$ and $\sqrt{\mathfrak{a} + \operatorname{Ann} M} = \mathfrak{a}' \cap \mathfrak{b}$. We claim that $\operatorname{ht}_M \mathfrak{a}' + \mathfrak{b} > t$. For that, if $\operatorname{ht}_M \mathfrak{a}' + \mathfrak{b} = t$ then there exists

$\mathfrak{q} \in \operatorname{Min}\operatorname{Ass}\frac{R}{\mathfrak{a}'+\mathfrak{b}}$ with $\operatorname{ht}_M\mathfrak{q} = t$. Hence $\mathfrak{q} \in \operatorname{Min}\operatorname{Ass}\frac{R}{\mathfrak{b}} \cap V(\mathfrak{a}')$ and there exists $\mathfrak{p} \in \operatorname{Ass}\frac{R}{\mathfrak{a}'}$ such that $\mathfrak{p} \subseteq \mathfrak{q}$, which is a contradiction.

Now, the Mayer-Vietoris sequence

$$0 \longrightarrow H^t_{\mathfrak{a}'}(M) \oplus H^t_{\mathfrak{b}}(M) \longrightarrow H^t_{\mathfrak{a}}(M) \longrightarrow H^{t+1}_{\mathfrak{a}'+\mathfrak{b}}(M),$$

in conjunction with [3, Theorem 1], proves the claim. □

Corollary 2.4. *Let $R$ be a Cohen-Macaulay local ring and let $t \in \mathbb{N}$. Then, the following statements are equivalent.*

(i) *For any ideal $\mathfrak{a}$ of $R$,* $\operatorname{Ass} H^t_{\mathfrak{a}}(R)$ *is a finite set.*
(ii) *For any linked ideal $\mathfrak{a}$,* $\operatorname{Ass} H^t_{\mathfrak{a}}(R)$ *is a finite set.*

Proposition 2.5. *Let $R$ be a UFD and $\mathfrak{a}$ be a linked ideal. Then,* $\operatorname{Ass} H^2_{\mathfrak{a}}(R)$ *is finite. If, in addition,* $\dim R < 4$ *then* $\operatorname{Ass} H^i_{\mathfrak{a}}(R)$ *is a finite set for all $i \in \mathbb{N}_0$.*

Definition 2.6. The $R$-module $X$ is said to be $\mathfrak{a}$-cofinite if $\operatorname{Supp} X \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}^i_R(\frac{R}{\mathfrak{a}}, X)$ is a finitely generated $R$-module for all $i \in \mathbb{N}_0$.

The following theorem considers some equivalent conditions for the finiteness of some Ext modules.

Theorem 2.7. *Let $I$ be a non-prime ideal of $R$ which is generated by an $R$-regular sequence and $N$ be an $R$-module. Then, the following statements are equivalent.*

(i) $\operatorname{Ext}^i_R(\frac{R}{I}, N)$ *is finitely generated, for all $i \in \mathbb{N}_0$.*
(ii) $\operatorname{Ext}^i_R(\frac{R}{\mathfrak{p}}, N)$ *is finitely generated, for any ideal $\mathfrak{p} \in \operatorname{Ass}\frac{R}{I}$ and all $i \in \mathbb{N}_0$.*
(iii) $\operatorname{Ext}^i_R(\frac{R}{\mathfrak{a}}, N)$ *is finitely generated, for any linked ideal $\mathfrak{a}$ by $I$ and all $i \in \mathbb{N}_0$.*
(iv) $\operatorname{Ext}^i_R(\frac{R}{\mathfrak{a}}, N)$ *and* $\operatorname{Ext}^i_R(\frac{R}{\mathfrak{b}}, N)$ *are finitely generated, for some ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $\mathfrak{a} \sim_{(I;R)} \mathfrak{b}$ and all $i \in \mathbb{N}_0$.*

Proof. (i) → (ii) is clear by the fact that $\operatorname{Supp}\frac{R}{\mathfrak{p}} \subseteq \operatorname{Supp}\frac{R}{I}$ and [2, Proposition 1].

(ii) → (iii) Let $\mathfrak{a}$ be a linked ideal by $I$. As $\operatorname{Supp}\frac{R}{\sqrt{\mathfrak{a}}} = \operatorname{Supp}\frac{R}{\mathfrak{a}}$, in view of [2, Proposition 1], we can assume that $\mathfrak{a}$ is a radical linked ideal by $I$. Let $\operatorname{Min}\operatorname{Ass}\frac{R}{\mathfrak{a}} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ and $M := \frac{R}{\mathfrak{p}_1} \oplus \cdots \oplus \frac{R}{\mathfrak{p}_n}$. By [7, Proposition 5.p594], $\mathfrak{p}_j \in \operatorname{Ass}\frac{R}{I}$. So, $\operatorname{Ext}^i_R(M, N)$ is finitely generated for all $i \geq 0$. Now, the result follows from the fact that $\operatorname{Supp}\frac{R}{\mathfrak{a}} = \operatorname{Supp} M$.

(iii) → (i) Let $\operatorname{Ass}\frac{R}{I} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ and $M := \frac{R}{\mathfrak{p}_1} \oplus \cdots \oplus \frac{R}{\mathfrak{p}_n}$. By the assumption and [5, 2.3], $\operatorname{Ext}^i_R(M, N)$ is finitely generated, for all $i \geq 0$. Therefore, the result has desired from [2, Proposition 1].

(iv) → (i) Assume that $\operatorname{Ass}\frac{R}{I} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ and $M := \frac{R}{\mathfrak{p}_1} \oplus \cdots \oplus \frac{R}{\mathfrak{p}_n}$. By [4, 2.5(iii)], $\operatorname{Supp}\frac{R}{I} = \operatorname{Supp}\frac{R}{\mathfrak{a}} \bigcup \operatorname{Supp}\frac{R}{\mathfrak{b}}$. Let $1 \leq i \leq n$ and assume that $\mathfrak{p}_i \in \operatorname{Supp}\frac{R}{\mathfrak{a}}$. Then, $\operatorname{Supp}\frac{R}{\mathfrak{p}_i} \subseteq \operatorname{Supp}\frac{R}{\mathfrak{a}}$ and $\operatorname{Ext}^i_R(\frac{R}{\mathfrak{p}_i}, N)$ is finitely generated. Therefore, $\operatorname{Ext}^i_R(M, N)$ is finitely generated for all $i \geq 0$ and the result follows from the fact that $\operatorname{Supp}\frac{R}{I} = \operatorname{Supp} M$. □

The following corollary presents an equivalent condition for the $\mathfrak{a}$-cofiniteness of $H^i_{\mathfrak{a}}(R)$ in the case where $\mathfrak{a}$ is a linked ideal.

Corollary 2.8. *Let $i \in \mathbb{N}_0$ and $I$ be a non-prime ideal of $R$ such that $\mathfrak{a}$ is linked by $I$. If $H^i_{\mathfrak{a}}(R)$ is $I$-cofinite then $H^i_{\mathfrak{a}}(R)$ is $\mathfrak{a}$-cofinite. In particular, in the case where $i >$ grade $I$ and $\mathfrak{a} \sim_{(I;R)} \mathfrak{b}$, $H^i_{\mathfrak{a}}(R)$ is $\mathfrak{a}$-cofinite and $\mathfrak{b}$-cofinite if and only if it is $I$-cofinite.*

Let $(R, \mathfrak{m})$ be a local ring. For all $i \in \mathbb{N}$, the family $\{H^i_{\mathfrak{m}}(\frac{M}{\mathfrak{a}^n M})\}_{n\in\mathbb{N}}$ forms an inverse system. The inverse limit $F^i_{\mathfrak{a}}(M) := \varprojlim_n H^i_{\mathfrak{m}}(\frac{M}{\mathfrak{a}^n M})$ is called the $i$-th formal local cohomology module of $M$ with respect to $\mathfrak{a}$. Formal local cohomology were used by Peskine and Szepiro in order to solve a conjecture of Hartshorne.

The following theorem gives us a necessary and sufficient condition for $M$ to be Cohen-Macaulay in terms the existence of some special linked ideals over it.

Theorem 2.9. *Let $(R, \mathfrak{m})$ be local and $d := \dim M$. Then the following statements are equivalent.*

(i) *$M$ is Cohen-Macaulay.*

(ii) *There exist ideals $\mathfrak{a}, \mathfrak{b}$ and $I$ such that $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$ and $\operatorname{Att} H^d_{\mathfrak{a}}(M) \bigcap \operatorname{Att} H^d_{\mathfrak{b}}(M) \neq \varnothing$.*

(iii) *There exist ideals $\mathfrak{a}, \mathfrak{b}$ and $I$ such that $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$, $\operatorname{Ass} F^0_{\mathfrak{a}}(\frac{M}{IM}) \bigcap \operatorname{Ass} F^0_{\mathfrak{b}}(\frac{M}{IM}) \neq \varnothing$ and $|\operatorname{Ass} \frac{M}{IM}| = 1$.*

(iv) *There exist ideals $\mathfrak{a}, \mathfrak{b}$ and $I$ such that $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$, $\operatorname{Ass} \frac{M}{IM} = \operatorname{Min} \operatorname{Ass} \frac{M}{IM}$ and $\dim \frac{M}{\mathfrak{a}M} = 0$.*

(v) *There exist ideals $\mathfrak{a}, \mathfrak{b}$ and $I$ such that $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$, $\operatorname{Ass} \frac{M}{IM} = \operatorname{Min} \operatorname{Ass} \frac{M}{IM}$ and $\operatorname{Ass} F^0_{\mathfrak{a}}(M) = \operatorname{Ass} M$.*

## References


1. S. Asadollahi and P. Schenzel, *Some results on associated primes of local cohomology module*, Jpn. J. Math. **29** (2) (2003) 285–296.
2. D. Delfino and T. Marley, *Cofinite modules and local cohomology*, J. Pure Appl. Algebra **121** (1) (1997) 45–52.
3. M. Hellus, *On the set of associated primes of a local cohomology module*, J. Algebra **237** (1) (2001) 406–419.
4. M. Jahangiri and Kh. Sayyari, *Linkage of ideals over a module*, J. Algebr. Syst. **8** (2) (2021) 267–279.
5. M. Jahangiri and Kh. Sayyari, *Charaterization of some special rings via linkage*, J. Algebra Relat. Topics **8** (1) (2020) 67–81.
6. M. Jahangiri and Kh. Sayyari, *Cohomological dimension with respect to linked ideals*, J. Algebra Appl. (2020). DOI:10.1142/S0219498821501048
7. A. Martsinkovsky and J. R. Strooker, *Linkage of modules*, J. Algebra **271** (2) (2004) 587–626.


E-mail: jahangiri@khu.ac.ir
E-mail: sayyarikh@gmail.com

# Properties of Common Neighborhood Graph under Types Product of Cayley Graph

Worood Mohammed Salah Mahdi
Faculty of Computer Sciences and Mathematics, University of Kufa, Najaf, Iraq
and Hayder Baqer Ameen Shelash*
Faculty of Computer Sciences and Mathematics, University of Kufa, Najaf, Iraq

Abstract. Let $G$ be a finite group and $\Gamma_{G,S} = Cay(G,S)$ be a Cayley graph on $G$. The common neighborhood graph $\mathbf{Con}(\Gamma_{G,S})$ is a graph with vertex set $V(\mathbf{Con}\Gamma_{G,S}) = \{x, x \in V(\Gamma_{\{G,s\}})\}$ and the set of all edges defined by $E(\mathbf{Con}\Gamma_{G,S}) = \{\{x,y\} \mid N(x) \cap N(y) \neq \emptyset\}$. The neighborhood of a vertex $x$ is denoted by $N(x)$. In this paper, we establish some properties of the common neighborhood graph of on the cyclic group $C_n$ and dihedral group $D_{2n}$.

**Keywords:** Common neighborhood graph, Cayley graph, graph operation.
**AMS Mathematical Subject Classification [2010]:** 05C75, 05C50.

## 1. Introduction

Throughout this paper, all graphs $\Gamma(V,E)$ are assumed to be simple and connected. The set of all vertices of graph $\Gamma(V,E)$ is denoted by $V(\Gamma)$ and the set of all edges is denoted by $E(\Gamma)$. $C_n$, $K_n$ and $GP(n,1)$ are cycle, complete and prism graph with $n, n$ and $2n$ vertices, respectively.

Let $\Gamma(V,E)$ be a simple graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. The common neighborhood graph (congraph) of $\Gamma(V,E)$ is denoted by $\mathbf{Con}(\Gamma(V,E))$. It is a simple graph with the same vertex set and two vertices $v_i$ and $v_j$ are adjacent if and only $N(v_i) \cap N(v_j) \neq \emptyset$. Here, the neighborhood of a vertex $v$ is the set of all vertices $u$ such that they are the endpoints of the same edge and denoted by $N(v)$.

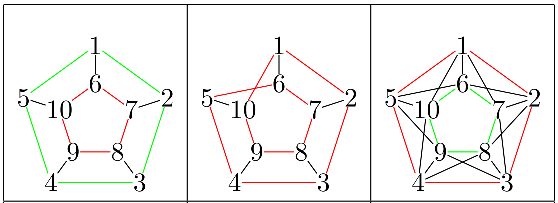
Suppose that $\Gamma_1(V_1,E_1)$ and $\Gamma_2(V_2,E_2)$ are two graphs. The direct product, Cartesian product and strong product of two graphs are denoted by $DiPro(\Gamma_1 \times \Gamma_2) = \Gamma_1 \times \Gamma_2$, $CarPro(\Gamma_1 \times \Gamma_2) = \Gamma_1 \boxdot \Gamma_2$ and $StrPro(\Gamma_1 \times \Gamma_2) = \Gamma_1 \boxtimes \Gamma_2$, respectively. The vertex and edge sets of these graphs are as follows:

$$\begin{aligned} V(\Gamma_1 \times \Gamma_2) &= V(\Gamma_1 \boxdot \Gamma_2) = V(\Gamma_1 \boxtimes \Gamma_2) = V(\Gamma_1) \times V(\Gamma_2), \\ E(\Gamma_1 \times \Gamma_2) &= \{\{(v_1,u_1),(v_2,u_2)\} \mid \{v_1,v_2\} \in E(\Gamma_1) \text{ and } \{u_1,u_2\} \in E(\Gamma_2)\}, \\ E(\Gamma_1 \boxdot \Gamma_2) &= \{\{(v_1,u_1),(v_2,u_2)\} \mid (v_1 = v_2 \; and \; \{u_1,u_2\} \in E(\Gamma_2)) \\ &\quad \text{or } (u_1 = u_2 \text{ and } \{v_1,v_2\} \in E(\Gamma_2))\}, \\ E(\Gamma_1 \boxtimes \Gamma_2) &= E(\Gamma_1 \times \Gamma_2) \cup E(\Gamma_1 \boxdot \Gamma_2). \end{aligned}$$

*Presenter

Let $S$ be a finite subset of a finite group $G$ with this property that $S$ satisfies the conditions $1 \notin S$ and $S = S^{-1}$. The Cayley graph $Cay(G, S)$ ia a simple graph with vertex set $G$, $g, h$ are adjacent if and only if there exists $s \in S$ such that $h = gs$. The subset $S$ is called the connection set this Cayley graph.

Suppose $\Omega = \{C_n, D_{2n}\}$, where $C_n = \langle a \mid a^n = e \rangle$ is the cyclic group of order $n$ and $D_{2n} = \langle a, b | a^n = b^2 = e, bab = a^{-1} \rangle$ denotes the dihedral group of order $2n$. The generalized Petersen graph $GP(n, 1)$, $n \geq 3$, was introduced by Coxeter (1950) and named by Watkins (1969). A Mobius ladder graph or Mobius wheel graph $M_n$ was introduced by Jakobson and Rivin in 1999 as in the flowing figures




$GP(5,1)$ | $M(5)$ | $QW(5)$

The graph $QW_n$ of order $2n$ is a graph with this condition that $|N(v)| = 5$, for all vertices $v$.

In [8] the Cayley graphs of the cyclic and dihedral groups with respect to the sets $S_1 = \{a, a^{-1}\}$, $S_2 = \{a, a^{-1}, a^{\frac{n}{2}}\}$, $S_3 = \{a^{\frac{n}{2}+1}, a^{\frac{n}{2}-1}, a^{\frac{n}{2}}\}$, $S_4 = \{a, a^{-1}\}$ and $S_5 = \{a, a^{-1}, b\}, S_6 = \{ab, a^{-1}b, b\}$ were computed.

Lemma 1.1. *The Cayley graph of these group can be described in the following simple forms:*

$$Cay(C_n, S) = \begin{cases} C_n, & S = S_1, \\ K_4, & S = S_2, S_3, n = 4, \\ M_{\frac{n}{2}}, & S = S_2, n \geq 6, \\ GP(\frac{n}{2}, 1), & S = S_3, 2 \nmid \frac{n}{2}, \\ M_{\frac{n}{2}}, & S = S_3, 2 \mid \frac{n}{2}, \end{cases}$$

*and*

$$Cay(D_{2n}, S) = \begin{cases} K_2 \cup K_2, & S = S_4, \ \ n = 2, \\ C_n \cup C_n, & S = S_4, \ \ n \geq 3, \\ C_4, & S = S_5, S_6, \ \ n = 2, \\ GP(n, 1), & S = S_5, \ \ n \geq 3, \\ M_n, & S = S_6, \ \ 2 \nmid n \geq 3, \\ GP(n, 1), & S = S_6, \ \ 2 \mid n \geq 4. \end{cases}$$

Theorem 1.2. *The following are hold:*

1) $Cay(C_n, S_1) \times Cay(C_2, S_1) \cong \begin{cases} C_{2n}, & 2 \nmid n, \\ C_n \cup C_n, & 2 \mid n. \end{cases}$
2) $Cay(C_n, S_1) \boxdot Cay(C_2, S_1) \cong GP(n, 1)$.
3) $Cay(C_n, S_1) \boxtimes Cay(C_2, S_1) = \begin{cases} K_{2n}, & n = 2, 3, \\ QW_n, & n \geq 4. \end{cases}$
4) $Cay(C_n, S_2) \times Cay(C_2, S_1) \cong \begin{cases} GP(n, 1), & 2 \mid \frac{n}{2}, \\ M_{(\frac{n}{2})} \cup M_{(\frac{n}{2})}, & 2 \nmid \frac{n}{2}. \end{cases}$
5) $Cay(C_n, S_2) \boxtimes Cay(C_2, S_1) = K_8, \ \ n = 4.$

6) $Cay(C_n, S_3) \times Cay(C_2, S_1) = GP(n, 1)$.
7) $Cay(C_n, S_3) \boxtimes Cay(C_2, S_1) = K_8$ $n = 4$.

LEMMA 1.3. [1] *The following are hold:*

1) $\mathbf{Con}(K_n) \cong K_n$.
2) $\mathbf{Con}(C_n) = \begin{cases} C_n, & 2 \nmid n \geq 3, \\ P_2 \cup P_2, & n = 4, \\ C_{\frac{n}{2}} \cup C_{\frac{n}{2}}, & 2 \mid n \geq 6. \end{cases}$
3) $\mathbf{Con}(G_1 \cup G_2) = \mathbf{Con}(G_1) \cup \mathbf{Con}(G_2)$.

## 2. Main Results

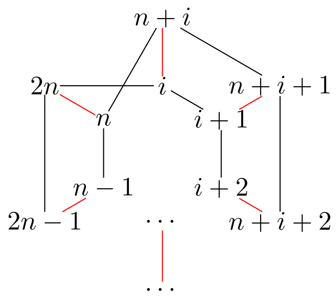
THEOREM 2.1. *Let $n$ be a positive integer. The common neighborhood graph of $M(n)$ is given by the following:*

1) *If $n$ is odd, then*

$$\mathbf{Con}(M(n)) = \begin{cases} K_i \cup K_i, & i = 3, 5, \\ Cay(C_n, \{a, a^{\frac{n-1}{2}}, a^{\frac{n+1}{2}}, a^{-1}\}) \cup Cay(C_n, \{a, a^{\frac{n-1}{2}}, a^{\frac{n+1}{2}}, a^{-1}\}), & n \geq 7. \end{cases}$$

2) *If $n$ is even, then* $\mathbf{Con}(M(n)) = Cay(C_{2n}, \{a^2, a^{n-1}, a^{n+1}, a^{-2}\})$.

PROOF.




From the structure of $\Gamma_{M(n)}$, we can see the set neighborhood of the vertices are equal to $N(v(i)) = \{v(i+1), v(i+n), v(i+2n-1)\}$ and the common neighborhood is equal to $N(v(i)) \cap N(v(j)) \neq \emptyset$ if and only if $j = \{i+2, i+n-1, i+n+1, i+2n-2\}$. It is also clear that,

$$\begin{aligned} N(v(i)) \cap N(v(i+2)) &= \{i+1, i+n, i+2n-1\} \cap \{i+3, i+n+3, i+1\} = \{i+1\}, \\ N(v(i)) \cap N(v(i+n-1)) &= \{i+1, i+n, i+2n-1\} \cap \{i+n, i-1, i+n-2\} = \{i+n\}, \\ N(v(i)) \cap N(v(i+n+1)) &= \{i+1, i+n, i+2n-1\} \cap \{i+2+n, i+1, i+n\} = \{i+n, i+1\}, \\ N(v(i)) \cap N(v(i+2n-2)) &= \{i+1, i+n, i+2n-1\} \cap \{i+2n-3, i+n-2, i+2n-1\} \\ &= \{i+2n-1\}. \end{aligned}$$

Now, we suppose that $n$ is an odd number. Then

$$N(v(i)) = \begin{cases} \underbrace{\{i+1, i+n, i+2n-i\}}_{\text{are even number}}, & if\ 2 \nmid i, \\ \underbrace{\{i+1, i+n, i+2n-i\}}_{\text{are odd number}}, & if\ 2 \mid i. \end{cases}$$

Hence the result. □

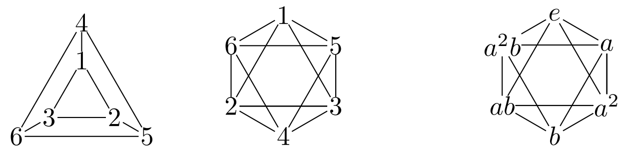
We now consider two cases that $n$ is odd or even.

Proposition 2.2. *Let $n$ be a positive integer. Then the common neighborhood graph of $GP(n,1)$ is given by the following:*

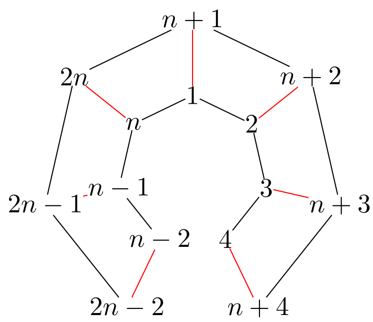
$$\mathbf{Con}(GP(n,1)) = \begin{cases} Cay(D_6, \{a, a^2, ab, a^2b\}), & 2 \nmid n, n = 3, \\ Cay(D_{2n}, \{a^2, a^{n-2}, a^{\frac{n-1}{2}-1}b, a^{n-1}b\}), & 2 \nmid n, n \geq 5, \\ Cay(D_{2\frac{n}{2}}, \{a, a^{\frac{n}{2}-1}, b, a^{\frac{n}{2}-1}b\}) \cup Cay(D_{2\frac{n}{2}}, \{a, a^{\frac{n}{2}-1}, b, a^{\frac{n}{2}-1}b\}), & 2 \mid n. \end{cases}$$

Proof. We give the proof for the case that $n = 3$. The neighborhood of vertices is given by following table:

| | | | | | |
|---|---|---|---|---|---|
| $N(1) = \{2,3,4\}$ | $N(2) = \{1,3,5\}$ | $N(1) \cap N(i) \neq \emptyset$ | $i = 2,3,5,6$ | $N(2) \cap N(i) \neq \emptyset$ | $i = 1,3,4,6$ |
| $N(3) = \{1,2,6\}$ | $N(4) = \{1,5,6\}$ | $N(3) \cap N(i) \neq \emptyset$ | $i = 1,2,4,5$ | $N(4) \cap N(i) \neq \emptyset$ | $i = 2,3,5,6$ |
| $N(5) = \{4,2,6\}$ | $N(5) = \{2,4,6\}$ | $N(5) \cap N(i) \neq \emptyset$ | $i = 1,3,4,6$ | $N(6) \cap N(i) \neq \emptyset$ | $i = 1,2,4,5$ |




$GP(3,1)$ $\quad$ $\mathbf{Con}(GP(3,1)) \cong$ $\quad$ $Cay(D_6, \{a, a^2, ab, a^2b\})$




$N(v(i)) \cap N(v(j)) \neq \emptyset$ and they are satisfying by following condition:

$$\begin{cases} j = \{i+n+1, i+n-2, i+2, i-2\}, & i \neq \{1, n+1, n, 2n, n-1, n+1, 2, n+2\}, \\ j = \{i+n+1, i+n-1, i+2, i+2+\frac{n-1}{2}\}, & i = \{2, n+2\}, \\ j = \{i+n+1, i+n-1, i-2, i-2-\frac{n-1}{2}\}, & i = \{n-1, 2n-1\}, \\ j = \{i+n+1, i+n+2+\frac{n-1}{2}, i+2, i+2+\frac{n-1}{2}\}, & i = \{1, n+1\}, \\ j = \{i+n+1, i+1, i-2, i+n-(\frac{n-1}{2}+2)\}, & i = \{n, 2n\}. \end{cases}.$$

Thus,

$$\mathbf{Con}(GP(n,1)) = \begin{cases} Cay(D_6, \{a, a^2, ab, a^2b\}), & n = 3, \\ Cay(D_{2n}, \{a^2, a^{n-2}, a^{\frac{n-1}{2}-1}b, a^{n-1}b\}), & 2 \nmid n, n \geq 5, \\ Cay(D_{2\frac{n}{2}}, \{a, a^{\frac{n}{2}-1}, b, a^{\frac{n}{2}-1}b\}) \cup Cay(D_{2\frac{n}{2}}, \{a, a^{\frac{n}{2}-1}, b, a^{\frac{n}{2}-1}b\}), & 2 \mid n. \end{cases}$$

□

Proposition 2.3. *Let $n$ be a positive integer. The following are hold:*

$$\mathbf{Con}(\Gamma(QW(n))) = \begin{cases} K_{2n}, & n = 3,4,5, \\ Cay(C_{2n}, \{a, a^2, a^{n-2}, a^{n-1}, a^n, a^{n+1}, a^{n+2}, a^{2n-2}, a^{2n-1}\}), & n \geq 6. \end{cases}$$

Proposition 2.4. *Let $n$ be an integer, then the common neighborhood graph can be given by the following:*

1) $\mathbf{Con}(\Gamma_{C_n,S}) =$

$$\begin{cases} C_n & S = S_1, 2 \nmid n \geq 3, \\ P_2 \cup P_2 & S = S_1, n = 4, \\ C_{\frac{n}{2}} \cup C_{\frac{n}{2}} & S = S_1, 2 \mid n \geq 6, \\ K_4 & S = S_2, S_3, n = 4, \\ Cay(C_{2\frac{n}{2}}, \{a^2, a^{\frac{n}{2}-1}, a^{\frac{n}{2}+1}, a^{n-2}\}) & S = S_2, 2 \mid n \geq 6, \\ Cay(D_{2\frac{n}{4}}, \{a, a^{\frac{n}{4}-1}, b, a^{\frac{n}{4}-1}b\}) \cup Cay(D_{2\frac{n}{4}}, \{a, a^{\frac{n}{4}-1}, b, a^{\frac{n}{4}-1}b\}) & S = S_3, 2 \nmid \frac{n}{2}, \\ Cay(C_{2\frac{n}{2}}, \{a^2, a^{\frac{n}{2}-1}, a^{\frac{n}{2}+1}, a^{n-2}\}) & S = S_3, 2 \mid \frac{n}{2}. \end{cases}$$

2) $\mathbf{Con}(\Gamma_{D_{2n},S}) =$

$$\begin{cases} C_n \cup C_n, & S = S_4,\ 2 \nmid n, \\ K_2 \cup K_2, & S = S_4,\ n = 2, \\ P_2 \cup P_2 \cup P_2 \cup P_2, & S = S_4,\ n = 4, \\ C_{\frac{n}{2}} \cup C_{\frac{n}{2}} \cup C_{\frac{n}{2}} \cup C_{\frac{n}{2}}, & S = S_4,\ 2 \mid n \geq 6, \\ P_2 \cup P_2, & S = S_5,\ n = 2, \\ C_6, & S = S_5,\ 2 \nmid n, n = 3, \\ Cay(D_{2n}, \{a^2, a^{n-2}, a^n b, a^{\frac{n-1}{2}-1}b\}), & S = S_5,\ 2 \nmid n, n \geq 5, \\ K_4 \cup K_4, & S = S_5, S_6,\ 2 \mid n, n = 4, \\ Cay(D_{2\frac{n}{2}}, \{a, a^{\frac{n}{2}-1}, b, a^{\frac{n}{2}-1}b\}) \cup Cay(D_{2\frac{n}{2}}, \{a, a^{\frac{n}{2}-1}, b, a^{\frac{n}{2}-1}b\}), & S = S_5, S_6,\ 2 \mid n, n \geq 6, \\ K_n \cup K_n, & S = S_6, n = 3, 5, \\ Cay(C_n, \{a, a^{\frac{n-1}{2}}, a^{\frac{n+1}{2}}, a^{n-1}\}) \cup Cay(C_n, \{a, a^{\frac{n-1}{2}}, a^{\frac{n+1}{2}}, a^{n-1}\}), & S = S_6, 2 \nmid n \geq 7. \end{cases}$$

PROOF. It is directly from Lemma 1.1 and Lemma 1.3. □

THEOREM 2.5. *The following are hold:*

1) $\mathbf{Con}(\Gamma_{(C_n,S_1)} \times \Gamma_{(C_2,S_1)}) = \begin{cases} \mathbf{Con}(C_{2n}), & 2 \nmid n, \\ \mathbf{Con}(C_n \cup C_n), & 2 \mid n. \end{cases}$
2) $\mathbf{Con}(\Gamma_{(C_n,S_1)} \boxdot \Gamma_{(C_2,S_1)}) = \mathbf{Con}(GP(n,1))$.
3) $\mathbf{Con}(\Gamma_{(C_n,S_1)} \boxtimes \Gamma_{(C_2,S_1)}) = \begin{cases} \mathbf{Con}(K_{2n}), & n = 2, 3, \\ \mathbf{Con}(QW(n)), & n \geq 4. \end{cases}$
4) $\mathbf{Con}(\Gamma_{(C_n,S_2)} \times \Gamma_{(C_2,S_1)}) = \begin{cases} \mathbf{Con}(GP(n,1)), & 2 \mid n, 2 \mid \frac{n}{2}, \\ \mathbf{Con}(M(\frac{n}{2}) \cup M(\frac{n}{2})), & 2 \mid n, 2 \nmid \frac{n}{2}. \end{cases}$
5) $\mathbf{Con}(\Gamma_{(C_n,S_3)} \times \Gamma_{(C_2,S_1)})) = \mathbf{Con}(GP(n,1))$.
6) $\mathbf{Con}(\Gamma_{(C_n,S_i)} \boxtimes \Gamma_{(C_2,S_1)}) = K_8,\ n = 4, i = 2, 3$.

In the next corollary, we will present some properties of the common neighborhood graph under some graph operations on Cayley graphs.

COROLLARY 2.6. *Let $\Gamma_1$ and $\Gamma_2$ be two graphs, then:*

1) $\mathbf{Con}(\Gamma_{(C_n,S_1)}) \times \mathbf{Con}(\Gamma_{(C_2,S_1)}) = \mathbf{Con}(\Gamma_{(C_n,S_1)} \times \Gamma_{(C_2,S_1)}),\ 2 \nmid n$.
2) $\mathbf{Con}(\Gamma_{(C_n,S_1)}) \times \mathbf{Con}(\Gamma_{(C_2,S_1)}) \neq \mathbf{Con}(\Gamma_{(C_n,S_1)} \times \Gamma_{(C_2,S_1)}),\ 2 \mid n$.
3) $\mathbf{Con}(\Gamma_{(C_n,S_1)}) \boxdot \mathbf{Con}(\Gamma_{(C_2,S_1)}) = \mathbf{Con}(\Gamma_{(C_n,S_1)} \boxdot \Gamma_{(C_2,S_1)}),\ 2 \nmid n$.
4) $\mathbf{Con}(\Gamma_{(C_n,S_1)}) \boxdot \mathbf{Con}(\Gamma_{(C_2,S_1)}) \neq \mathbf{Con}(\Gamma_{(C_n,S_1)} \boxdot \Gamma_{(C_2,S_1)}),\ 2 \mid n$.
5) $\mathbf{Con}(\Gamma_{(C_n,S_1)}) \boxtimes \mathbf{Con}(\Gamma_{(C_2,S_1)}) = \mathbf{Con}(\Gamma_{(C_n,S_1)} \boxtimes \Gamma_{(C_2,S_1)}),\ 2 \nmid n$.
6) $\mathbf{Con}(\Gamma_{(C_n,S_1)}) \boxtimes \mathbf{Con}(\Gamma_{(C_2,S_1)}) \neq \mathbf{Con}(\Gamma_{(C_n,S_1)} \boxtimes \Gamma_{(C_2,S_1)}),\ 2 \mid n$.
7) $\mathbf{Con}(\Gamma_{(C_n,S_2)}) \times \mathbf{Con}(\Gamma_{(C_2,S_1)}) \neq \mathbf{Con}(\Gamma_{(C_n,S_2)}) \times \Gamma_{(C_2,S_1)})$.
8) $\mathbf{Con}(\Gamma_{(C_4,S_i)}) \boxtimes \mathbf{Con}(\Gamma_{(C_2,S_1)}) = \mathbf{Con}(\Gamma_{(C_4,S_i)}) \times \Gamma_{(C_2,S_1)}),\ i = 2, 3$.

PROOF. (1) It is clear that

$$\mathbf{Con}((Cay(C_n, S_1)) \boxdot \mathbf{Con}(Cay(C_2, S_1)) = \begin{cases} GP(n,1), & 2 \nmid n, \\ K_2 \cup K_2 \cup K_2 \cup K_2, & n = 4, \\ GP(\frac{n}{2}, 1) \cup GP(\frac{n}{2}, 1), & 2 \mid n \geq 6. \end{cases}$$

(2) Note that

$$\mathbf{Con}((Cay(C_n,S_1))\boxtimes\mathbf{Con}(Cay(C_2,S_1))=\begin{cases} If\ 2\nmid n, then & \begin{cases} K_{2n}, & n=3,\\ QW_n, & n\geq 5.\end{cases}\\ If\ 2\mid n, then & \begin{cases} K_n\cup K_n, & n=2,\\ QW_{\frac{n}{2}}\cup QW_{\frac{n}{2}}, & n\geq 4.\end{cases}\end{cases}$$

If $n$ is an odd, then $\mathbf{Con}(Cay(C_n,S_1))=C_n$ and $\mathbf{Con}(Cay(C_2,S_1))=K_2$ and we can prove the result. So, if $n$ is even, then this means that $\mathbf{Con}(Cay(C_n,S_1))=C_{\frac{n}{2}}\cup C_{\frac{n}{2}}$,

From Proposition 2.4, $\mathbf{Con}(\Gamma(C_{2n}))=\Gamma(C_{2n})$ if $n$ is odd. Otherwise, there are two possibilities $n=4$ in which $\mathbf{Con}(\Gamma(P_2\cup P_2))=\Gamma(P_2\cup P_2)$, or, $n\geq 6$ and $\mathbf{Con}(\Gamma(C_n\cup C_n))=\Gamma(C_n\cup C_n)$.

(6) In this case $(P_2\cup P_2)\boxtimes C_2=K_4\cup K_4$ and $(C_{\frac{n}{2}}\cup C_{\frac{n}{2}})\boxtimes C_2=QW(\frac{n}{2})\cup QW(\frac{n}{2})$.

(7) $\mathbf{Con}(\Gamma_{(C_n,S_2)})\times\mathbf{Con}(\Gamma_{(C_2,S_1)})=\begin{cases} GP(n,1), & 2\mid\frac{n}{2}\\ M(\frac{n}{2})\cup M(\frac{n}{2}), & 2\nmid\frac{n}{2}\end{cases}$.

(8) It is clear that $\mathbf{Con}(\Gamma_{(C_4,S_i)})\boxtimes\mathbf{Con}(\Gamma_{(C_2,S_1)})=\mathbf{Con}(K_4)\boxtimes\mathbf{Con}(K_2)=K_8$, and $\mathbf{Con}(\Gamma_{(C_4,S_i)})\times\Gamma_{(C_2,S_1)})=\mathbf{Con}(K_8)=K_8$. □

## References


1. A. Alwardi, B. Arsic, I. Gutman and N. Soner, *The common neighborhood graph and its energy*, Iran. J. Math. Sci. Inform. **7** (2) (2012) 1–8.
2. J. A. Bondy and U. S. R. Murty, *Graph Theory*, Graduate Texts in Mathematics, Vol. 244, Springer, New York, 2008.
3. A. S. Bonifacio, R. R. Rosa and N. M. M. de Abreu, *Complete common neighborhood graphs*, Proceedings Congreso Latino-Iberoamericano de Investigaci on Operativa and Simposio Brasileiro de Pesquisa Operacional (2012) 4026–4032.
4. A. Cayley, *On the theory of groups*, Proc. London Math. Soc. **9** (1) (1878) 126–133.
5. M. Dehn, Duber die *Toplogie des dreidimensionalen Raumes*, Math. Ann. **69** (1) (1910) 137–168. (In German)
6. W. Dicks and M. J. Dunwoody, *Groups Acting on Graphs*, Cambridge Univ. Press, Cambridge, 1989.
7. J. Friedman, On Cayley graphs on the symmetric group generated by transpositions, Combinatorica **20** (4) (2000) 505–519.
8. W. M. Salah and H. B. Shelash, *Structure graph of the finite group*, submitted.



E-mail: msc500620@gmail.com

E-mail: hayder.ameen@uokufa.edu.iq

# Contributed Posters

## Analysis

# Some Results on Hermite-Hadamard Inequality with Respect to Uniformly Convex Functions

Hasan Barsam*
Department of Mathematics, Faculty of Science, University of Jiroft,
P.O. Box 78671-61167, Jiroft, Iran
and Ali Reza Sattarzadeh
Department of Mathematics, Graduate University of Advanced Technology, Kerman,
Iran

Abstract. In this note, we establish Hermit-Hadamard inequality for uniformly s-convex functions.
**Keywords:** Hermite-Hadamard, Hölder inequality, Uniformly s-convex.
**AMS Mathematical Subject Classification [2010]:** 26D15, 26D07, 39B62.

## 1. Introduction

Inequalities are very important and applicable tools in mathematics. Most of the well known inequalities are closely related to the concept of convexity. Indeed, using the notion of convex functions, the Hermit-Hadamard inequality has been obtained as follows:

For a convex function $g : I \subseteq \mathbb{R} \to \mathbb{R}$ so that $x, y \in I$ and $x < y$, the following inequality holds:

$$g(\frac{x+y}{2}) \leq \frac{1}{y-x}\int_x^y g(t)dt \leq \frac{g(x)+g(y)}{2}.$$

In recent years many researchers have improved the Hermit-Hadamard inequality and extended it to other functions such as $m$-convex functions and etc. (See [2, 4, 5]).

## 2. Preliminares

In this section, we consider the basic concepts and results, which are needed to obtain our main results.

In [1, Definition 10.5], the class of uniformly convex functions are defined. We generalize the definition of the uniformly convex functions in the following:

Definition 2.1. Assume that $g : \mathbb{R} \to \mathbb{R}$ is a function. Then $g$ is called a uniformly s-convex function with modulus $\psi : [0, +\infty) \to [0, +\infty]$ if $\psi$ is increasing, $\psi$ vanishes only at 0, and

$$g(tx + (1-t)y) + t^s(1-t)\psi(|x-y|) \leq t^s g(x) + (1-t)^s g(y),$$

*Presenter

for each $x, y \in [0, +\infty)$ and $t \in [0, 1]$. Furthermore, if $s = 1$, then $g$ is called a uniformly convex.

Example 2.2. [1] In view of the following equality,

$$(ta + (1-t)b)^2 + t(1-t)(a-b)^2 = ta^2 + (1-t)b^2,$$

for all $t \in (0, 1)$ and $a, b \in \mathbb{R}$, the function $g(t) = t^2$ for $t \in \mathbb{R}$ is a uniformly convex with modulus $\psi(t) = t^2$ for all $t \geq 0$.

In order to prove our main theorems, we need the following lemma that has been obtained in [3].

Lemma 2.3. [3] *Assume that $g : I^o \to \mathbb{R}$ is a differentiable function on $I^o$, $a, b \in I^o$ with $a < b$. If $g' \in L[a, b]$, then the following equality holds:*

$$\frac{g(a) + g(b)}{2} - \frac{1}{b-a} \int_a^b g(t)dt = \frac{b-a}{2} \int_0^1 (1-2t)g'(ta + (1-t)b)dt.$$

## 3. Main Results

The next theorem gives a generalization of the Hermite-Hadamard inequalities for uniformly s-convex functions:

Theorem 3.1. *Assume that $g : \mathbb{R} \to \mathbb{R}$ is a uniformly s-convex function. Then,*

$$2^{s-1} g(\frac{a+b}{2}) + \frac{1}{8(b-a)} \int_{a-b}^{b-a} \psi(|t|)dt \leq \frac{1}{b-a} \int_a^b g(t)dt$$
$$\leq \frac{g(a) + g(b)}{s+1} - \frac{1}{(s+1)(s+2)} \psi(|a-b|).$$

Proof. It is easy by some calculations. □

If in Theorem 3.1 we set $\psi(t) = \frac{\beta}{2}t^2, \beta > 0$ and $s = 1$ then, we obtain the following important inequality:

Corollary 3.2. *Let $g : \mathbb{R} \to \mathbb{R}$ be a uniformly convex function with modulus $\psi(t) = \frac{\beta}{2}t^2, \beta > 0$. Then,*

$$g(\frac{a+b}{2}) + \frac{\beta}{24}(b-a)^2 \leq \frac{1}{b-a} \int_a^b g(t)dt \leq \frac{g(a) + g(b)}{2} - \frac{\beta}{12}(b-a)^2.$$

Here, we give some applications of Lemma 2.3 related with Hermite-Hadamard's inequality for s-convex functions which are very interesting.

Theorem 3.3. *Assume that $g : I^o \to \mathbb{R}$ is a differentiable function on $I^o$, $a, b \in I^o$ with $a < b$. If $|g'|$ is a uniformly s-convex function on $I^o$, then the following inequality holds:*

$$|\frac{g(a) + g(b)}{2} - \frac{1}{b-a} \int_a^b g(t)dt| \leq \frac{(b-a)(2^s s + 1)}{2^{s+1}(s+1)(s+2)} (|g'(a)| + |g'(b)|)$$
$$- (b-a)(\frac{2^{s+1}(s-1) + (s+5)}{2^{s+2}(s+1)(s+2)(s+3)}) \psi(|a-b|).$$

Proof. In view of Lemma 2.3 and uniformly convexity of $|g'|$, one has

$$|\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)dt| \leq \frac{b-a}{2}\int_0^1 |(1-2t)||g'(ta+(1-t)b)|dt$$
$$\leq \frac{b-a}{2}\int_0^1 |1-2t|(t^s|g'(a)|+(1-t)^s|g'(b)|+t^s(t-1)\psi(|a-b|))dt$$
$$\leq \frac{b-a}{2}[\int_0^1 t^s|1-2t||g'(a)|dt+\int_0^1 |1-2t|(1-t)^s|g'(b)|dt$$
$$+\int_0^1 |1-2t|t^s(t-1)\psi(|a-b|))dt].$$

Since,

$$\int_0^1 t^s|1-2t|dt=\int_0^1 (1-t)^s|1-2t|dt=\frac{s}{(s+1)(s+2)}+\frac{1}{2^s(s+1)(s+2)},$$
$$\int_0^1 |1-2t|t^s(t-1)\psi(|a-b|)dt=\frac{2^{s+1}(1-s)-(s+5)}{2^{s+1}(s+1)(s+2)(s+3)})\psi(|a-b|).$$

So,

$$\leq \frac{b-a}{2}(|g'(a)|+|g'(b)|)(\frac{s}{(s+1)(s+2)}+\frac{1}{2^s(s+1)(s+2)})$$
$$+(b-a)(\frac{2^{s+1}(1-s)-(s+5)}{2^{s+2}(s+1)(s+2)(s+3)})\psi(|a-b|).$$

Hence, The proof is complete. □

Theorem 3.4. *Assume that $g: I^o \to \mathbb{R}$ is a differentiable mapping on $I^o$, $a, b \in I^o$ with $a < b$ and $p > 1$. If $|g'|^q$ is uniformly $s$-convex on $I^o$, then the following inequality holds:*

$$|\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)dt| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}(\frac{|g'(a)|^q+|g'(b)|^q}{s+1}$$
$$-\frac{1}{(s+1)(s+2)}\psi(|a-b|))^{\frac{1}{q}},$$

*where $\frac{1}{p}+\frac{1}{q}=1$.*

Proof. By Lemma 2.3 and Hölder's inequality, we conclude

$$|\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)dt| \leq \frac{b-a}{2}\int_0^1 |(1-2t)||g'(ta+(1-t)b)|dt$$
$$\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |g'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{b-a}{2}\frac{1}{(p+1)^{\frac{1}{p}}}\left(|g'(a)|^q\int_0^1 t^s dt+|g'(b)|^q\int_0^1 (1-t)^s dt+\psi(|a-b|)\int_0^1 t^s(t-1)dt\right)^{\frac{1}{q}}$$
$$\leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\frac{|g'(a)|^q+|g'(b)|^q}{s+1}-\frac{1}{(s+1)(s+2)}\psi(|a-b|)\right)^{\frac{1}{q}}.$$

Hence, the proof is complete. □

## References

[1] H. H. Bauschke and P. L. Combettes,*Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer-Verlag, New York, 2011.

[2] H. Barsam and M. Ramezani, *Some results on Hermite-Hadamard type inequalities with respect to fractional integrals*, Casp. J. Math. Sci. (In press).

[3] S. S. Dragomir and R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett. **11** (5) (1998) 91–95.

[4] U. S. Kirmaci and M. E. Ozdemir, *On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comput. **153** (2004) 361–368.

[5] M. E. Ozdemir, M. Avci and E. Set, *On some inequalities of Hermite–Hadamard type via $m$-convexity*, Appl. Math. Lett. **23** (9) (2010) 1065–1070.

E-mail: barsam65@gmail.com
E-mail: arsattarzadeh@gmail.com

# Power Bounded Weighted Composition Operators on the Bloch Space

Fakhroddin Falahat
Department of Mathematics, Faculty of Sciences, Shiraz Branch, Islamic Azad University, Shiraz, Iran
and Zahra Kamali*
Department of Mathematics, Faculty of Sciences, Shiraz Branch, Islamic Azad University, Shiraz, Iran

Abstract. In this paper, we investigate about power boundedness of weighted composition operators on Bloch space and we give some necessarily and sufficient conditions under which a weighted composition operator is power bounded on Bloch space.
**Keywords:** Weighted composition operator, Power bounded, Bloch space.
**AMS Mathematical Subject Classification [2010]:** 47B38, 46E15, 47A35.

## 1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of all holomorphic functions on $\mathbb{U}$. The *Bloch space* $\mathcal{B}$ is defined to be the space of all functions in $H(\mathbb{D})$ such that

$$\beta_f = \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| < \infty.$$

The *little Bloch space* $\mathcal{B}_0$ is the closed subspace of $\mathcal{B}$ consisting of all functions $f \in \mathcal{B}$ with

$$\lim_{|z|\to 1}(1 - |z|^2)|f'(z)| = 0.$$

It is easy to check that the Bloch space and the Little Bloch space are Banach spaces under the norm

$$||f|| = |f(0)| + \beta_f.$$

The following useful lemma determines that norm convergence implies pointwise convergence in the Bloch space.

Lemma 1.1. [8] *For all $f \in \mathcal{B}$ and for each $z \in \mathbb{D}$, we have*

$$|f(z)| \le ||f|| \log \frac{2}{1 - |z|^2}.$$

In geometric function theory, Bloch space is important, mainly because of its Möbius invariant property, i.e. for any automorphism of $\mathbb{D}$, $\varphi$, $||fo\varphi|| = ||f||$ for all $f \in \mathcal{B}$.

*Presenter

**1.1. Weighted Composition Operators on Bloch space.** Each $\psi \in H(\mathbb{D})$ and holomorphic self map $\varphi$ of $\mathbb{D}$, induces a linear weighted composition operator $C_{\psi,\varphi} : H(\mathbb{D}) \to H(\mathbb{D})$ by $C_{\psi,\varphi}(f)((z) = M_\psi C_\varphi(f)(z) = \psi(z)f(\varphi(z))$ for every $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$, where $M_\psi$ denotes the multiplication operator by $\psi$ and $C_\varphi$ is a composition operator. The mapping $\varphi$ is called composition map and $\psi$ is called the weight. For a positive integer $n$, the $nth$ iterate of $\varphi$ is denoted by $\varphi_n$, also $\varphi_0$ is the identity function. We note that

$$C^n_{\psi,\varphi}(f) = \prod_{j=0}^{n-1} \psi o \varphi_j (f o \varphi_n),$$

for all $f$ and $n \geq 1$.

For $\psi \in H(\mathbb{D})$ and analytic self map $\varphi$ of $\mathbb{D}$ define

$$\sigma_{\varphi,\psi} = \sup_{z\in\mathbb{D}} \frac{1}{2}(1-|z|^2)|\psi'(z)| \log \frac{1+|\varphi(z)|}{1-|\varphi(z)|},$$

$$\tau_{\varphi,\psi}(z) = \sup_{z\in\mathbb{D}} \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)||\psi(z)|.$$

The authors in [1] Showed that if $\sigma_{\varphi,\psi} < \infty$ and $\tau_{\varphi,\psi} < \infty$, then $C_{\psi,\varphi}$ is bounded on Bloch space. Also we have

$$||C_{\psi,\varphi}|| \leq \max\{||\psi||, \frac{1}{2}|\psi(0)| \log \frac{1+|\varphi(0)|}{1-|\varphi(0)|} + \sigma_{\varphi,\psi} + \tau_{\varphi,\psi}\}. \tag{1}$$

The holomorphic self maps of the unit disk are divided in two classes of elliptic and non-elliptic. The elliptic type is an automorphism and has a fixed point in $\mathbb{D}$. The non-elliptic one has a unique fixed point $p \in \overline{\mathbb{D}}$, called the Denjoy-Wollf point of $\varphi$, which is known as attractive fixed point, that is the sequence of iterates of $\varphi$, $\{\varphi_n\}_n$ converges to $p$ uniformly on compact subsets of $\mathbb{D}$. See [4] for more details. Note that the class of all holomorphic self maps of $\mathbb{D}$ is denoted by $S(\mathbb{D})$.

**1.2. Power Bounded Operators.** Let $L(X)$ be the space of all linear bounded operators from locally convex Hausdorff space $X$ into itself and $T \in L(X)$. $T$ is called *power bounded* if the sequence $\{T^n\}_{n=0}^{\infty}$ is bounded in $L(X)$. In this paper, we look for conditions under which the weighted composition operator $C_{\psi,\varphi}$ is power bounded on Bloch space. power bounded composition operators on Bloch spaces was investigated in [5]. The authors of [3] characterized power bounded weighted composition operators on spaces of holomorphic functions. Also E. Wolf studied when weighted composition operators acting between weighted Banach spaces are power bounded [6] is a good source to study about power bounded operators.

## 2. Main Results

The following proposition provides the necessary conditions for which the weighted composition operators to be power bounded.

Proposition 2.1. *Let $\psi \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Suppose $C_{\psi,\varphi}$ is a bounded weighted composition operator on $\mathcal{B}$ ($\mathcal{B}_0$). If $C_{\psi,\varphi}$ is power bounded then*

i) *$\{\prod_{j=0}^{n-1} \psi o \varphi_j\}$ is a bounded sequence in $\mathcal{B}$ ($\mathcal{B}_0$).*
ii) *If $z_0 \in \mathbb{D}$ is Denjoy-Wolff point of $\varphi$, then $|\psi(z_0)| \leq 1$.*

PROOF. Part (i) folows directly from $||\prod_{j=0}^{n-1}\psi o\varphi_j|| = ||C^n_{\varphi,\psi}(1)|| \leq ||C^n_{\varphi,\psi}||$.

By Lemma (1.1) norm bouded implies pointwise bounded in Bloch space, so we must have $|\prod_{j=0}^{n-1}\psi(\varphi_j(z_0))| = |\psi(z_0)|^n$ is bounded, it forces $|\psi(z_0)| \leq 1$. □

THEOREM 2.2. *Let $\psi \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$, such that $||\psi||_\infty \leq 1$ and $||\varphi||_\infty < 1$ and $z_0 \in \mathbb{D}$ is the Denjoy-wolff point of $\varphi$. If $C_{\psi,\varphi}$ is bounded on $\mathcal{B}$ ($\mathcal{B}_0$) and $\{\prod_{j=0}^{n-1}\psi o\varphi_j\}$ is a bounded sequence, then $C_{\psi,\varphi}$ is power bounded on $\mathcal{B}$ ($\mathcal{B}_0$).*

PROOF. We have

$$(2)\ ||C^n_{\psi,\varphi}|| \leq \max\{||\prod_{j=0}^{n-1}\psi o\varphi_j||, \frac{1}{2}\prod_{j=0}^{n-1}|\psi\varphi_j(0)|\log\frac{1+|\varphi_n(0)|}{1-|\varphi_n(0)|} + \tau_n + \sigma_n\},$$

where for all $n \in \mathbb{N}$:

$$\tau_n = \sup_{z\in\mathbb{D}}\frac{1-|z|^2}{1-|\varphi_n(z)|^2}|\varphi'_n(z)||\prod_{j=0}^{n-1}\psi(\varphi_j(z))|,$$

$$\sigma_n = \sup_{z\in\mathbb{D}}\frac{1}{2}(1-|z|^2)|(\prod_{j=0}^{n-1}\psi o\varphi_j)'(z)\log\frac{1+|\varphi_n(z)|}{1-|\varphi_n(z)|}.$$

Since $\varphi_n(0) \rightarrow z_0$ and norm bounded implies pointwise bounded, clearly $\{\frac{1}{2}\prod_{j=0}^{n-1}\psi(\varphi_j(0))\log\frac{1+|\varphi_n(0)|}{1-|\varphi_n(0)|}\}$ is a bounded sequence. By Shwarz-Pick Lemma [4], for all $z \in \mathbb{D}$, $\frac{1-|z|^2}{1-|\varphi_n(z)|^2}|\varphi'_n(z)| \leq 1$, so $\tau_n \leq ||\prod_{j=0}^{n-1}\psi o\varphi_j||\log\frac{2}{1-||\varphi||_\infty}$ and $\{\tau_n\}$ is bounded. On the other hand,

$$\sigma_n \leq \frac{1}{2}||\prod_{j=0}^{n-1}\psi o\varphi_j||\log\frac{2}{1-||\varphi||_\infty},$$

cosequently, $\sigma_n$ is bounded too. The proof is completed by (2). □

In the following proposition we give some necessary conditions of power boundedness of weighted composition operators in the case $\varphi$ has boundary Denjoy-Wolff point.

PROPOSITION 2.3. *Let $\psi \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ with $z_0 \in \partial\mathbb{D}$ as boundary Denjoy-Wolff point of it. If $C_{\psi,\varphi}$ is bounded on $\mathcal{B}$ ( $\mathcal{B}_0$) and one of the following conditions satisfy, then $C_{\varphi,\psi}$ is not power bounded on $\mathcal{B}$ ($\mathcal{B}_0$).*

i) *there exists $z \in \mathbb{D}$ such that $\prod_{j=0}^{n-1}\psi(\varphi_j(z))$ does not converges to zero,*
ii) *there exists $z \in \mathbb{D}$ and $N \in \mathbb{N}$ such that for all $j \geq N$, $|\psi(\varphi_j(z))| \geq 1$.*

PROOF. Let $f(z) = \log\log\frac{2}{z_0 - z}$. $f \in \mathcal{B}_0 \subseteq \mathcal{B}$, see [2]. Let $e_z$ be the linear functional for evaluation at $z$, that is, $e_z(f) = f(z)$ for all $f \in \mathcal{B}(\mathcal{B}_0)$. Since $\varphi_n(z) \rightarrow z_0$ as $n \rightarrow \infty$ and

$$|e_z(\prod_{j=0}^{n-1}\psi o\varphi_j f o\varphi_n)| = |\prod_{j=0}^{n-1}\psi(\varphi_j(z))|\log\log\frac{2}{z_0-\varphi_n(z)}|,$$

(i) and (ii) follows immediately. □

PROPOSITION 2.4. *Let $\varphi$ be a self map of $\mathbb{D}$ and $\lambda \in \mathbb{C}$.*

(i) *If* $\lambda C_\varphi$ *is power bounded on* $\mathcal{B}$ $(\mathcal{B}_0)$*, then* $|\lambda| \leq 1$.
(ii) *If* $|\lambda| < 1$ *then* $\lambda C_\varphi$ *is power bounded.*
(iii) *If* $|\lambda| = 1$ *then* $\lambda C_\varphi$ *is power bounded if and only if* $C_\varphi$ *is power bounded if and only if* $\varphi$ *has interior fixed point.*

Proof. Suppose $\lambda C_\varphi$ is power bounded, there exists $M > 0$ such that for all $n \in \mathbb{N}$ and $f \in \mathcal{B}$, we have $||\lambda^n C_{\varphi_n} f|| \leq M||f||$. Put $f \equiv 1$, so $|\lambda^n| \leq M$ and consequently, $|\lambda| \leq 1$.

Now suppose $\varphi$ is not an elliptic automorphism with boundary or interior Denjoy- Wollff point and $|\lambda| < 1$. By (1) we have

$$||\lambda^n C_{\varphi_n}|| \leq |\lambda|^n \max\{1, \frac{1}{2}\log\frac{1+|\varphi_n(0)|}{1-|\varphi_n(0)|} + \sup_{z\in\mathbb{D}} \frac{1-|z|^2}{1-|\varphi_n(z)|^2}|\varphi_n'(z)|\}.$$

By Shawrz-Pick lemma $\sup_{z\in\mathbb{D}} \frac{1-|z|^2}{1-|\varphi_n(z)|^2}|\varphi_n'(z)| \leq 1$ and then

$$||\lambda^n C_{\varphi_n}|| \leq |\lambda|^n \max\{1, \frac{1}{2}\log\frac{1+|\varphi_n(0)|}{1-|\varphi_n(0)|} + 1\}.$$

$|\lambda| < 1$ get us that $|\lambda^n| \log\frac{1+|\varphi_n(0)|}{1-|\varphi_n(0)|} \to 0$, (consider that it is true in both cases; interior Denjoy-Wollff or boundary Denjoy-Wollff point). Thus $||\lambda^n C_{\varphi_n}|| \to 0$ as $n \to \infty$ and the sequence $\{\lambda^n C_{\varphi_n}\}$ is bounded, i.e. $\lambda C_\varphi$ is power bounded. In the elliptic automorphism case without loss of generality we can assume $\varphi(0) = 0$. For $f \in \mathcal{B}$ with $||f|| = 1$ we have:

$$||\lambda^n C_{\varphi_n} f|| = |\lambda^n|||f o \varphi_n|| = |\lambda^n|||f|| \leq 1,$$

so $||\lambda^n C_{\varphi_n}|| \leq 1$ and $\lambda C_\varphi$ will be power bounded. Now, if $\varphi(a) = a$ where $a \in \mathbb{D}$ and $a \neq 0$, define $\phi(z) = \varphi_a^{-1}\, o\, \varphi\, o\, \varphi_a$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. Easy calculation shows, $\phi(0) = 0$, so by previous cases $\lambda C_\phi$ is power bounded. Since $\lambda^n C_{\phi_n} = C_{\varphi_a^{-1}}\ o\ (\lambda^n C_{\varphi_n})\ o\ C_{\varphi_a}$, $\lambda C_\varphi$ is also power bounded. In the case $|\lambda| = 1$, $||\lambda^n C_{\varphi_n}|| = ||C_{\varphi_n}||$, so $\lambda C_\varphi$ is power bounded if and only if $C_\varphi$ is power bounded. In [5] the authors have shown that $C_\varphi$ is power bounded if and only if $\varphi$ has interior fixed point. □

## References


1. R. Allen and F. Colonna, *Weighted composition operators on the Bloch space of bounded homogenous domain*, Oper. Theory, Adv. Appl. **202** (2010) 11–37.
2. W. Arendt, I. Chalendar and M. Kumar, *Powers of composition operators: asymptotic behaviour on Bergman, Dirichlet and Bloch spaces*, J. Australian Math. So. **108** (3) (2020) 289–320.
3. M. J. Beltrán Meneu, M. Carmen Gómez Collado and E. Jordá, *Mean ergodicity of weighted composition operators on spaces of holomorphic functions*, J. Math. Anal. Appl. **444** (2) (2016) 1–12.
4. C. Cowen and B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, 1995.
5. E. Jordá and A. Rodríguez-Arenas, *Ergodic properties of composition operators on Banach spaces of analytic functions*, J. Math. Anal. Appl. **486** (1) (2020) 123891.
6. Y. Katznelson and L. Tzafriri, *On power bounded operators*, J. Func. Anal. **68** (3) (1986) 313–328.
7. E. Wolf, *Power bounded weighted composition operators*, New York J. Math. **18** (2012) 201–212.

8. K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Graduate Texts in Mathematics, Vol. 226, Springer, New York, 2005.

E-mail: zkamali@shirazu.ac.ir

# Some Variants of Young Type Inequalities

Gholamreza Karamali*
Faculty of Bsic Sciences, Shahid Sattari Aeronautical University of Science and Technology, South Mehrabad, Tehran, Iran
and Hamid Reza Moradi
Faculty of Bsic Sciences, Shahid Sattari Aeronautical University of Science and Technology, South Mehrabad, Tehran, Iran

Abstract. The simple inequality

$$\sqrt{ab} \leq \frac{a+b}{2}, \quad a,b>0$$

is known in the literature as the arithmetic-geometric mean (AM-GM) inequality. Though simple, this inequality has received a considerable attention due to its applications in mathematical inequalities. This article presents a new treatment of the arithmetic-geometric mean inequality and its sibling, the Young inequality.
**Keywords:** Operator inequality, Young inequality, Arithmetic-geometric mean inequality, Positive operator.
**AMS Mathematical Subject Classification [2010]:** 47A63, 47A60.

## 1. Introduction

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the $C^*$-algebra of all bounded linear operators on $\mathcal{H}$ with the operator norm $\|\cdot\|$ and the identity $I_{\mathcal{H}}$. For an operator $A \in \mathcal{B}(\mathcal{H})$, we write $A \geq 0$ if $A$ is positive, and $A > 0$ if $A$ is positive and invertible. For $A, B \in \mathcal{B}(\mathcal{H})$, we say $A \geq B$ if $A - B \geq 0$.

Let $A, B \in \mathcal{B}(\mathcal{H})$ be two positive operators and $\nu \in [0,1]$. $\nu$-weighted arithmetic mean of $A$ and $B$, denoted by $A\nabla_\nu B$, is defined as

$$A\nabla_\nu B = (1-\nu)A + \nu B.$$

If $A$ is invertible, $\nu$-geometric mean of $A$ and $B$, denoted by $A\sharp_\nu B$, is defined by

$$A\sharp_\nu B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\nu A^{\frac{1}{2}}.$$

When $\nu = \frac{1}{2}$, we write $A\nabla B$ and $A\sharp B$ for brevity, respectively. We also use the same notations for scalars.

The well-known Young's inequality is a classical result attributed to the English mathematician William Henry Young (1863-1942) stating that

$$a^\nu b^{1-\nu} \leq \nu a + (1-\nu)b, \tag{1}$$

where $a$ and $b$ are distinct positive real numbers and $\nu \in [0,1]$.

*Presenter

The inequality (1) was refined by Kittaneh and Mansarah [1] in the following form

(2) $$\left(a^{\nu}b^{1-\nu}\right)^2 + r^2(a-b)^2 \le (\nu a + (1-\nu)\,b)^2,$$

where $r = \min\{\nu, 1-\nu\}$.

The authors of [4] obtained another refinement of the Young inequality as follows:

(3) $$\left(a^{\nu}b^{1-\nu}\right)^2 + r(a-b)^2 \le \nu a^2 + (1-\nu)\,b^2.$$

More recently, Hu in [2] gave the following Young type inequalities:

(4) $$\begin{cases} \left((\nu a)^{\nu}b^{1-\nu}\right)^2 + \nu^2(a-b)^2 \le \nu^2a^2 + (1-\nu)^2b^2, \ 0 \le \nu \le \frac{1}{2}, \\ \left\{(a^{\nu}\,(1-\nu)\,b)^{1-\nu}\right\}^2 + (1-\nu)^2(a-b)^2 \le \nu^2a^2 + (1-\nu)^2b^2, \ \frac{1}{2} \le \nu \le 1. \end{cases}$$

When comparing inequalities (4) with the inequalities (2) and (3), it is easy to observe that both the left-hand and the right-hand sides of inequalities (4) are greater than or equal to the corresponding sides in (2) and (3), respectively. It should be noticed that neither inequalities (4) nor (2) and (3) is uniformly better than the other.

The primary objective of this paper is to present new inequalities of Young's-type. We first propose a refinement of the inequalities in (4). Furthermore, a new refinement and a reverse for the arithmetic-geometric mean inequality are proved. Finally, we use these inequalities to obtain corresponding operator inequalities.

It should be mentioned here that this talk is based on the papers [3] by the authors and [6] by the second author.

## 2. Main Results

**2.1. Scalar Inequalities.** Before starting the first result, we recall the following notations from [5]:

$$S_N\,(\nu; a, b) = \sum_{j=1}^{N} s_j\,(\nu)\left(\sqrt[2^j]{b^{2^{j-1}-k_j(\nu)}a^{k_j(\nu)}} - \sqrt[2^j]{a^{k_j(\nu)+1}b^{2^{j-1}-k_j(\nu)-1}}\right)^2,$$

with $k_j\,(\nu) = \left[2^{j-1}\nu\right]$, $r_j\,(\nu) = \left[2^j\nu\right]$ and
$s_j\,(\nu) = (-1)^{r_j(\nu)}2^{j-1}\,(\nu) + (-1)^{r_j(\nu)+1}\left[\frac{r_j(\nu)+1}{2}\right]$, for $N \in \mathbb{N}$ and $j = 1, 2, \ldots, N$.
Notice that $[x]$ is the greatest integer less than or equal to $x$.

Recall that, in [5], it has been shown that:

$$a^{\nu}b^{1-\nu} + S_N\,(\nu; a, b) \le \nu a + (1-\nu)\,b.$$

Now, we start with some numerical results.

Theorem 2.1. *Let $a, b > 0$ and $N \in \mathbb{N}$.*

(i) *If $0 \le \nu \le \frac{1}{2}$, then*

$$bS_N\,(2\nu; \nu a, b) + \left((\nu a)^{\nu}b^{1-\nu}\right)^2 + \nu^2(a-b)^2 \le \nu^2a^2 + (1-\nu)^2b^2.$$

(ii) *If $\frac{1}{2} \le \nu \le 1$, then*

$$aS_N\,(2\nu-1; a, (1-\nu)\,b) + \left\{a^{\nu}((1-\nu)\,b)^{1-\nu}\right\}^2 + (1-\nu)^2(a-b)^2 \le \nu^2a^2 + (1-\nu)^2b^2.$$

Remark 2.2. Since $bS_N(2\nu;\nu a,b)$, $aS_N(2\nu-1;a,(1-\nu)b)\geq 0$, so Theorem 2.1 improves the inequalities in (4).

As a direct consequence of Theorem 2.1, we have:

Corollary 2.3. *Let* $a,b>0$ *and* $N\in\mathbb{N}$.

(i) *If* $0\leq\nu\leq\frac{1}{2}$, *then*

$$\sqrt{b}S_N\left(2\nu;\nu\sqrt{a},\sqrt{b}\right)+\nu^{2\nu}\left(a^{\nu}b^{1-\nu}\right)+\nu^2\left(\sqrt{a}-\sqrt{b}\right)^2\leq\nu^2a+(1-\nu)^2b.$$

(ii) *If* $\frac{1}{2}\leq\nu\leq 1$, *then*

$$\sqrt{a}S_N\left(2\nu-1;\sqrt{a},(1-\nu)\sqrt{b}\right)+(1-\nu)^{2(1-\nu)}\left(a^{\nu}b^{1-\nu}\right)+(1-\nu)^2\left(\sqrt{a}-\sqrt{b}\right)^2$$
$$\leq\nu^2a+(1-\nu)^2b.$$

Corollary 2.4. *Assume that* $a,b\geq 1$.

(i) *If* $0\leq\nu\leq\frac{1}{2}$, *then*

$$S_N(2\nu;\nu a,b)+\left((\nu a)^{\nu}b^{1-\nu}\right)^2+\nu^2(a-b)^2\leq\nu^2a^2+(1-\nu)^2b^2.$$

(ii) *If* $\frac{1}{2}\leq\nu\leq 1$, *then*

$$S_N(2\nu-1;a,(1-\nu)b)+\left(a^{\nu}((1-\nu)b)^{1-\nu}\right)^2+(1-\nu)^2(a-b)^2\leq\nu^2a^2+(1-\nu)^2b^2.$$

Next, we present new non trivial refinement and reverse of the simple inequality $\sqrt{ab}\leq\frac{a+b}{2}$, or $a\sharp b\leq a\nabla b$. It is worth noting that this inequality has not been refined or reversed in the literature, although the Young inequality (1) has been extensively studied.

Theorem 2.5. *Let* $a,b>0$.

1) *If* $0\leq p\leq\frac{1}{2}$, *then*

$$\sqrt{ab}+2\left(\frac{|a^p-b^p|}{2}\right)^{\frac{1}{p}}\leq\frac{a+b}{2}.$$

2) *If* $\frac{1}{2}\leq p\leq 1$, *then*

$$\sqrt{ab}+2\left(\frac{|a^p-b^p|}{2}\right)^{\frac{1}{p}}\geq\frac{a+b}{2}.$$

*The equality in* (1) *and* (2) *holds if and only if* $p=1/2$ *or* $a=b$.

The case $p=1/4$ in Theorem 2.5 reduces to the following inequality

$$\sqrt{ab}+\left[F_{1/4}(a,b)-H_{1/4}(a,b)\right]\leq\frac{a+b}{2},$$

where $H_v(a,b)=\frac{a^{1-v}b^v+a^vb^{1-v}}{2}$ and $F_v(a,b)=(1-v)\sqrt{ab}+v\frac{a+b}{2}$ are the Heinz mean and the Heron mean, respectively.

**2.2. Operator Versions.** Here, the operator versions of the inequalities proved in the previous section are established.

THEOREM 2.6. *Let* $A, B \in \mathcal{B}(\mathcal{H})$ *be two positive and invertible operators, and* $N \in \mathbb{N}$. *If* $0 \le \nu \le \frac{1}{2}$ *and* $\alpha_j(2\nu) = \frac{k_j(2\nu)}{2^j}$, *then*

$$\begin{aligned}
&\sum_{j=1}^{N} s_j(2\nu)\Big(\nu^{2\alpha_j(2\nu)} A\sharp_{\alpha_j(2\nu)}B + \nu^{2\alpha_j(2\nu)+2^{1-j}} A\sharp_{\alpha_j(2\nu)+2^{-j}}B \\
&\quad -2\nu^{2\alpha_j(2\nu)+2^{-j}} A\sharp_{\alpha_j(2\nu)+2^{-(j+1)}}B\Big) \\
&\quad + \nu^{2\nu} A\sharp_\nu B + 2\nu^2 (A\nabla B - A\sharp B) \\
&\quad \le ((1-\nu)A)\nabla_\nu(\nu B).
\end{aligned}$$

*On the other hand, if* $\frac{1}{2} \le \nu \le 1$ *and* $\beta_j(2\nu-1) = \frac{k_j(2\nu-1)}{2^j}$, *then*

$$\begin{aligned}
&\sum_{j=1}^{N} s_j(2\nu-1)\Big((1-\nu)^{1-2\beta_j(2\nu-1)} B\sharp_{\frac{1}{2}-\beta_j(2\nu-1)}A \\
&\quad + (1-\nu)^{1-2\beta_j(2\nu-1)-2^{1-j}} B\sharp_{\frac{1}{2}-\beta_j(2\nu-1)-2^{-j}}A \\
&\quad -2(1-\nu)^{1-2\beta_j(2\nu-1)-2^{-j}} B\sharp_{\frac{1}{2}-\beta_j(2\nu-1)-2^{-(j+1)}}A\Big) \\
&\quad + (1-\nu)^{2(1-\nu)} A\sharp_\nu B + 2(1-\nu)^2 (A\nabla B - A\sharp B) \\
&\quad \le ((1-\nu)A)\nabla_\nu(\nu B).
\end{aligned}$$

As for the operator inequalities for Theorem 2.5, we have the following.

THEOREM 2.7. *Let* $A, B \in \mathcal{B}(\mathcal{H})$ *be positive operators such that* $A > B$.

(1) *If* $0 \le p \le \frac{1}{2}$, *then*

$$A\sharp B + 2^{1-\frac{1}{p}} A\sharp_{\frac{1}{p}}(A - A\sharp_p B) \le A\nabla B.$$

(2) *If* $\frac{1}{2} \le p \le 1$, *then*

$$A\sharp B + 2^{1-\frac{1}{p}} A\sharp_{\frac{1}{p}}(A - A\sharp_p B) \ge A\nabla B.$$

## References


1. O. Hirzallah and F. Kittaneh, *Matrix Young inequalities for the Hilbert-Schmidt norm*, Linear Algebra Appl. **308** (2000) 77–84.
2. X. Hu, *Young type inequalities for matrices*, J. East China Norm. Univ. Natur. Sci. **4** (2012) 12–17.
3. G. Karamali, P. K. Harikrishnan and H. R. Moradi, *On refined Young's inequality*, Ital. J. Pure Appl. Math. to appear.
4. F. Kittaneh and Y. Manasrah, *Improved Young and Heinz inequalities for matrices*, J. Math. Anal. Appl. **361** (1) (2010) 262–269.
5. M. Sababheh and D. Choi, *A complete refinement of Young's inequality*, J. Math. Anal. Appl. **440** (1) (2016) 379–393.
6. M. Sababheh, S. Furuichi, Z. Heydarbeygi and H. R. Moradi, *On the arithmetic-geometric mean inequality*, J. Math. Inequal. to appear.



E-mail: rezakaramali918@gmail.com

E-mail: hrmoradi@mshdiau.ac.ir

# General Additive Functional Equations in k-Ary Banach Algebras

Vahid Keshavarz*
Department of Mathematics, Shiraz University of Technology, Shiraz, Iran
and Sedigheh Jahedi
Department of Mathematics, Shiraz University of Technology, Shiraz, Iran

Abstract. In this paper, we introduce the concept of k-ary hom-derivation. We investigate on the relation between the generalized additive functional equations and $\mathbb{C}-$linearity. We also, prove the Hyers-Ulam stability of these equations in k-ary Banach algebras.
**Keywords:** k-Ary hom-derivation, k-Ary Banach algebras, Hyers-Ulam stability.
**AMS Mathematical Subject Classification [2010]:** 17A40, 39B52, 17B40, 47B47.

## 1. Introduction

Let $(X, [\,])$ be a k-ary Banach algebra, i.e. a linear space over $F$ endowed with a k-linear associative composition law. It has been shown that many familiar notions from the theory of binary algebras can quite naturally generalized to k-linear case. Let $A$ and $B$ be Banach spaces. The function $f : A \to B$ is called additive if satisfies the functional equation

$$f(a + b) = f(a) + f(b),$$

for all $a, b \in A$.

A number of authors investigated the stability problem of additive functional equation, [4, 5, 6, 7]. The stability problem of functional equations has been first raised by Ulam [9] which asks whether or not there is a true solution of the functional equation

$$\mathcal{E}_1(f) = \mathcal{E}_2(f),$$

in some sense, near to its approximate solution. Hyers [3] gave a first affirmative answer to the question of Ulam for Banach spaces. The generalizations of this result have been published by Aoki [1] and Rassias [8] for additive mappings and linear mappings, respectively. In this presentation, we extend the definition of hom-derivation [7] to the sense of k-ary Banach algebras, and we investigate Hyers-Ulam stability of the generalized additive functional equation by using the fixed point method.

Theorem 1.1. [2] *Let $(X, d)$ be a complete generalized metric space and let $F : X \to X$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$.*

*Presenter

*Then for each given element $x \in X$, either*

$$d(F^n(x), F^{n+1}(x)) = \infty,$$

*for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that*
(1) $d(F^n(x), F^{n+1}(x)) < \infty, \ \forall n \geq n_0$,
(2) *the sequence $\{F^n(x)\}$ converges to a unique fixed point $y^*$ of $F$ in the set* $Y = \{y \in X \mid d(F^{n_0}x, y) < \infty\}$,
(3) $d(y, y^*) \leq \frac{1}{1-L} d(y, F(y))$ *for all $y \in Y$.*

## 2. Results

Throughout the paper, $(X, [\ ])$ is a k-ary Banach algebra over $\mathbb{C}$ and $\mathbb{T}^1 := \{\zeta \in \mathbb{C} : \ |\zeta| = 1\}$.
A $\mathbb{C}$-linear mapping $h : X \to X$ is called a k-ary homomorphism if

$$h\Big([x_1, x_2, \ldots, x_k]\Big) = [h(x_1), h(x_2), \ldots, h(x_k)],$$

for all $x_i \in X$, $1 \leq i \leq k$.

Definition 2.1. Let $h : X \to X$ be a k-ary homomorphism. A $\mathbb{C}$-linear mapping $D : X \to X$ is called a k-ary hom-derivation if

$$\begin{aligned} D\Big([x_1, x_2, \ldots, x_k]\Big) &= [D(x_1), h(x_2), \ldots, h(x_k)], \\ &+ [h(x_1), D(x_2), \ldots, h(x_k)] + \cdots + [h(x_1), h(x_2), \ldots, D(x_k)], \end{aligned}$$

for all $x_i \in X$, $1 \leq i \leq k$.

Let $f : X \to X$ be a function satisfying the functional equation

$$f\Big(\sum_{i=1}^{k} \lambda x_i\Big) = \sum_{i=1}^{k} \lambda f(x_i) + \sum_{i=1}^{k} \lambda f(x_i - x_{i-1}), \tag{1}$$

where $x_0 = x_k$, and $x_i \in X$, $1 \leq i \leq k$.

Clearly by considering $x_0 = x_k$, $x_i = 0$, $1 \leq i \leq k$, we obtain $f(0) = 0$. For $\lambda = 1$, if we consider $x_0 = x_2$, $1 \leq k \leq 2$, then for any function $f : X \to X$ which satisfies (1), $f$ is additive. Also, any additive function is a solution of (1). In general, if $f : X \to X$ is a $\mathbb{C}-$linear mapping then $f$ satisfies in functional Eq. (1).

For the converse we have the following result.

Proposition 2.2. *Let $\delta : X^k \to [0, +\infty)$, be a function such that*

$$\Delta(x_1, \ldots, x_k) = \sum_{j=0}^{\infty} 2^{-j} \delta(2^j x_1, \ldots, 2^j x_k) < \infty.$$

*Let $f : X \to X$ be a mapping satisfying the functional Eq. (1) and*

$$\Big\| f\Big(\sum_{i=1}^{k} \lambda x_i\Big) - \sum_{i=1}^{k} \lambda f(x_i)\Big\| \leq \delta(x_1, \ldots, x_k),$$

*for all $x_1, x_2 \ldots, x_n \in X$ and $\lambda \in \mathbb{T}^1$. Then $f$ is a $\mathbb{C}-$linear mapping.*

In the following Theorem, let $\varphi_i$, $i = 1, 2$, be functions from $X^k$ into $[0, \infty)$, for which there exists a $0 < L < 1$ such that

$$\varphi_1(\frac{x_1}{2}, \frac{x_2}{2}, \ldots, \frac{x_k}{2}) \leq \frac{L}{2}\varphi_1(x_1, x_2 \ldots, x_k), \tag{2}$$

$$\varphi_2(\frac{x_1}{2}, \frac{x_2}{2}, \ldots, \frac{x_k}{2}) \leq \frac{L}{2^k}\varphi_2(x_1, x_2 \ldots, x_k). \tag{3}$$

Therefore

$$\lim_{n\to\infty} 2^n \varphi_1(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \ldots, \frac{x_k}{2^n}) = 0,$$

$$\lim_{n\to\infty} 2^{nk} \varphi_2(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \ldots, \frac{x_k}{2^n}) = 0,$$

for all $x_1, x_2 \ldots, x_k \in X$. Hence $\varphi_1(0, \ldots, 0) = 0$ and $\varphi_2(0, \ldots, 0) = 0$.

Theorem 2.3. *Suppose $f, g : X \to X$ are two functions satisfy in*

$$\Big\|f\Big(\sum_{i=1}^{k} \lambda x_i\Big) - \sum_{i=1}^{n} \lambda f(x_i) - \sum_{i=1}^{k} \lambda f(x_i - x_{i-1})\Big\| \leq \varphi_1(x_1, x_2, \ldots x_k),$$

$$\Big\|g\Big(\sum_{i=1}^{k} \lambda x_i\Big) - \sum_{i=1}^{k} \lambda g(x_i) - \sum_{i=1}^{k} \lambda g(x_i - x_{i-1})\Big\| \leq \varphi_1(x_1, x_2, \ldots x_k),$$

$$\|f([x_1, x_2, \ldots, x_k]) - [f(x_1), f(x_2), \ldots, f(x_k)]\| \leq \varphi_2(x_1, x_2, \ldots, x_k),$$

$$\Big\|g([x_1, x_2, \ldots, x_k]) - [g(x_1), f(x_2), \ldots, f(x_k)] - [f(x_1), g(x_2), f(x_3), \ldots, f(x_k)] - \cdots - [f(x_1), f(x_2), \ldots, f(x_{k-1}), g(x_k)]\Big\| \leq \varphi_2(x_1, x_2, \ldots, x_k),$$

*where $\varphi_i$, $i = 1, 2$, is a function fulfill (2) and (3). Then there exists a unique $k$-ary homomorphism $h : X \to X$ and a unique $k$-ary hom-derivation $D : X \to X$ such that*

$$\|f(x) - h(x)\| \leq \frac{L}{2(1-L)}\varphi_1(x, x, 0, \ldots, 0),$$

$$\|g(x) - D(x)\| \leq \frac{L}{2(1-L)}\varphi_1(x, x, 0, \ldots, 0),$$

*for all $x \in X$.*

Corollary 2.4. *Let $p > 1$ be a positive real number and $\theta \geq 0$ be a real number. If $f, g : X \to X$ are mappings satisfying $f(0) = g(0) = 0$ and*

$$\Big\|f\Big(\sum_{i=1}^{k} \lambda x_i\Big) - \sum_{i=1}^{k} \lambda f(x_i) - \sum_{i=1}^{k} \lambda f(x_i - x_{i-1})\Big\| \leq \theta \sum_{i=1}^{k} \|x_i\|^p,$$

$$\Big\|g\Big(\sum_{i=1}^{k} \lambda x_i\Big) - \sum_{i=1}^{k} \lambda g(x_i) - \sum_{i=1}^{k} \lambda g(x_i - x_{i-1})\Big\| \leq \theta \sum_{i=1}^{k} \|x_i\|^p,$$

$$\|f([x_1, x_2, \ldots, x_n]) - [f(x_1), f(x_2), \ldots, f(x_k)]\| \leq \theta \prod_{i=1}^{k} \|x_i\|^p,$$

$$\Big\| g([x_1, x_2, \ldots, x_k]) - [g(x_1), f(x_2), \ldots, f(x_k)] - [f(x_1), g(x_2), f(x_3), \ldots, f(x_k)] - \cdots - [f(x_1), f(x_2), \ldots, f(x_{k-1}), g(x_k)] \Big\| \leq \theta \prod_{i=1}^{k} \|x_i\|^p,$$

*for all* $x_1, x_2, \ldots, x_n \in X$*, then there exists a unique* $k$*-ary homomorphism* $h : X \to X$ *and a unique* $k$*-ary hom-derivation* $D : X \to X$ *such that*

$$\|f(x) - h(x)\| \leq \frac{3\theta}{|2 - 2^p|} \|x\|^p,$$

$$\|g(x) - D(x)\| \leq \frac{3\theta}{|2 - 2^p|} \|x\|^p,$$

*for all* $x \in X$*.*

## References


1. T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950) 64–66.
2. J. B. Diaz and B. Margolis, *A fixed point theorem of the alternative for contractions on the generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968) 305–309.
3. D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U. S. A. **27** (1941) 222–224.
4. V. Keshavarz and S. Jahedi, *Hyperstability of k-ary Jordan derivations*, The 49th Annual Iranian Mathematics Conference, Iran University of Science and Technology, Tehran, Iran, 2018.
5. P. Nakmahachalasint, *On the Hyers-Ulam-Rassias stability of an n-dimensional additive functional equation*, Thai J. Math. **5** (2007) 81–86.
6. C. Park, *Homomorphisms between Poisson* $JC^*$*-algebras*, Bull. Braz. Math. Soc. **36** (2005) 79-97.
7. C. Park, J. R. Lee and X. Zhang, *Additive s-functional inequality and hom-derivations in Banach algebras*, J. Fixed Point Theory Appl. **21** (2019) 1661–7738.
8. Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978) 297–300.
9. S. M. Ulam, *Problems in Modern Mathematics*, Wiley, New York, 1940.


E-mail: v.keshvarz68@yahoo.com
E-mail: jahedi@sutech.ac.ir

# Hyers-Ulam Stabilities for 3D Cauchy-Jensen $\rho$-Functional

Vahid Keshvarz*
University of Applied Science and Technology, Center of Sambol Shimi (Tandis), Qazvin, Iran
and Zohreh Kefayati
Department of Pure Mathematics, Imam Khomeini International University, Qazvin, Iran

Abstract. In this paper, we introduce and solve the following $3D$ Cauchy-Jensen $\rho-$functional

$$\begin{aligned}&J\Big(\frac{ta+tb}{2}+tc\Big)+J\Big(\frac{ta+tc}{2}+tb\Big)+J\Big(\frac{tb+tc}{2}+ta\Big)\\&-2tJ(a)-2tJ(b)-2tJ(c)\\&=\rho\Big(J(ta+tb+tc)-tJ(a)-tJ(b)-tJ(c)\Big),\end{aligned}$$

where $\rho\neq 0,\pm 1$ is a real number. We investigate the Hyers-Ulam stability of ternary Jordan derivation in ternary algebras for 3D Cauchy-Jensen $\rho-$functional equation.

**Keywords:** Ternary Jordan derivation, Ternary algebras, $3D$ Cauchy-Jensen.
**AMS Mathematical Subject Classification [2010]:** 39B52, 39B82, 22D25.

## 1. Introduction

The study of stability problem functional equations which had been raised by Ulam [8] have been done by several authors on different functional equations (See [1, 3, 6]). In 1941 [5], Hyers solved the approximately additive mappings on the setting of Banach spaces. Hyers Theorem was generalized by Th. M. Rassias [7]. A generalization of the theorem of Th. M. Rassias was obtained by Găvruta [4] by replacing a general control function $\varphi : X\times X\longrightarrow[0,\infty)$.
A ternary Banach algebra $\mathfrak{A}$ is a complex linear space, endowed with a ternary product $(u_1,u_2,u_3)\to[u_1,u_2,u_3]$ from $\mathfrak{A}^3$ into $\mathfrak{A}$ such that

$$[[u_1,u_2,u_3],v_1,v_2]=[u_1,[u_2,u_3,v_1],v_2]=[u_1,u_2,[u_3,v_1,v_2]],$$

and satisfy $\|[[u_1,u_2,u_3]\|\leq\|u_1\|.\|u_2\|.\|u_3\|$, $\|[u,u,u]\|=\|u\|^3$ (See [9]).

Definition 1.1. Let $\mathfrak{D}$ be a $\mathbb{C}$-linear mapping from $\mathfrak{A}$ into $\mathfrak{A}$ satisfies

$$\mathfrak{D}\Big([a,b,c]\Big)=[\mathfrak{D}(a),b,c]+[a,\mathfrak{D}(b),c]+[a,b,\mathfrak{D}(c)],\qquad\forall\ a,b,c\in\mathfrak{A},$$

then equation above is called a ternary derivation, if it satisfying

$$\mathfrak{D}\Big([a,a,a]\Big)=[\mathfrak{D}(a),a,a]+[a,\mathfrak{D}(a),a]+[a,a,\mathfrak{D}(a)],\qquad\forall\ a\in\mathfrak{A},$$

*Presenter

then equation above is called a ternary Jordan derivation.

## 2. Main Results

Throughout the paper, suppose that $\mathfrak{A}$ is a ternary algebra and $t \in \mathbb{T}^1_{1/n0}$ the set of all complex numbers $e^{i\theta}$, where $0 \le \theta \le \frac{2\pi}{n_0}$ and $\rho \ne 0, \ \pm 1$ is a real number.

Lemma 2.1. *Let $J$ be an odd mapping from $\mathfrak{A}$ into $\mathfrak{A}$ for all $a, b, c \in \mathfrak{A}$ satisfies*

$$\begin{aligned} J\Big(\frac{a+b}{2}+c\Big) &+ J\Big(\frac{a+c}{2}+b\Big) + J\Big(\frac{b+c}{2}+a\Big) - 2J(a) - 2J(b) - 2J(c) \\ &= \rho\Big(J(a+b+c) - J(a) - J(b) - J(c)\Big), \end{aligned} \tag{1}$$

*then $J$ is additive.*

Proof. First of all, let $a = b = c = 0$ in (1), we get $J(0) = 0$. Let $a = b = 0$ in (1), we have

$$J(\frac{c}{2}) = \frac{1}{2}J(c),$$

for all $c \in \mathfrak{A}$. Putting $c = -b$ in (1), we get

$$J(\frac{a-b}{2}) + J(\frac{a+b}{2}) - J(a) = 0, \tag{2}$$

for all $a, b \in \mathfrak{A}$. Again put $a = a + b$ and $b = a - b$ in (2), we have

$$J(a+b) = J(a) + J(b).$$

The proof is complete. □

Lemma 2.2. [2] *Let $J$ be an linear mapping from $\mathfrak{A}$ into $\mathfrak{A}$. Then the following assertions are equivalent:*

$$\begin{aligned} J([a,a,a]) &= [J(a),a,a] + [a,J(a),a] + [a,a,J(a)], \\ J([a,b,c]+[b,c,a]+[c,a,b]) &= [J(a),b,c] + [a,J(b),c] + [a,b,J(c)] \\ &+ [J(b),c,a] + [b,J(c),a] + [b,c,J(a)] \\ &+ [J(c),a,b] + [c,J(a),b] + [c,a,J(b)]. \end{aligned}$$

Theorem 2.3. *Suppose $\psi$ be a function from $\mathfrak{A}^3$ into $[0,\infty)$ such that*

$$\widetilde{\psi}(a,b,c) := \sum_{n=0}^{\infty} \frac{1}{2^n}\psi(2^n a, 2^n b, 2^n c) < \infty,$$

*for all $a, b, c \in \mathfrak{A}$. Let $J$ is an odd mapping from $\mathfrak{A}$ into $\mathfrak{A}$ satisfying*

$$\begin{aligned} \|J(\frac{ta+tb}{2}+tc) + J(\frac{ta+tc}{2}+tb) + J(\frac{tb+tc}{2}+ta) - 2tJ(a) - 2tJ(b) - 2tJ(c) \\ - \rho(J(ta+tb+tc) - tJ(a) - tJ(b) - tJ(c))\| \le \psi(a,b,c), \end{aligned}$$

*and*

$$\begin{aligned} \|J([a,b,c]+[b,c,a]+[c,a,b]) - [J(a),b,c] - [a,J(b),c] - [a,b,J(c)] \\ - [J(b),c,a] - [b,J(c),a] - [b,c,J(a)] \\ - [J(c),a,b] - [c,J(a),b] - [c,a,J(b)].\| \le \psi(a,b,c), \end{aligned}$$

*for all* $t \in \mathbb{T}^1_{\frac{1}{n_0}}$ *and all* $a, b, c \in \mathfrak{A}$. *Then there exists a unique ternary Jordan derivation* $\mathfrak{D} : \mathfrak{A} \to \mathfrak{A}$ *such that*

$$\|J(a) - \mathfrak{D}(a)\| \leq \frac{1}{6}\widetilde{\psi}(a, 0, 0),$$

*for all* $a \in \mathfrak{A}$.

COROLLARY 2.4. *Let* $\theta, p_i, q_i,\ i = 1, 2, 3$ *are positive real such that* $p_i < 1$ *and* $q_i < 3$ . *Suppose that* $J$ *is an odd mapping from* $\mathfrak{A}$ *into* $\mathfrak{A}$ *such that*

$$\begin{aligned}\|J(\frac{ta+tb}{2} + tc) + J(\frac{ta+tc}{2} + tb) + J(\frac{tb+tc}{2} + ta) - 2tJ(a) - 2tJ(b) - 2tJ(c)\\ - \rho(J(ta+tb+tc) - tJ(a) - tJ(b) - tJ(c))\|\\ \leq \theta(\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3}),\end{aligned}$$

$$\begin{aligned}\|J([a,b,c] + [b,c,a] + [c,a,b]) - [J(a),b,c] - [a,J(b),c] - [a,b,J(c)]\\ - [J(b),c,a] - [b,J(c),a] - [b,c,J(a)]\\ - [J(c),a,b] - [c,J(a),b] - [c,a,J(b)].\|\\ \leq \theta(\|a\|^{q_1} + \|b\|^{q_2} + \|c\|^{q_3}),\end{aligned}$$

*for all*$a, b, c \in \mathfrak{A}$ *and* $t \in \mathbb{T}^1_{\frac{1}{n_0}}$. *Then is a unique ternary Jordan derivation* $\mathfrak{D}$ *from* $\mathfrak{A}$ *into* $\mathfrak{A}$ *such that*

$$\|J(a) - \mathfrak{D}(a)\| \leq \frac{\theta}{3}\{\frac{1}{2-2^{p_1}}\|a\|^{p_1} + \frac{1}{2-2^{p_2}}\|a\|^{p_2} + \frac{1}{2-2^{p_3}}\|a\|^{p_3}\},$$

*for all* $a \in A$.

## Acknowledgement

The first author would like to acknowledge of University of Applied Science and Technology Center of Sambol Shimi(Tandis) for their support and contribution to this study.

## References

1. I. I. El-Fassi, S. Kabbaj and A. Charifi, *Hyperstability of Cauchy-Jensen functional equations*, Indagat. Math. **27** (3) (2016) 855–867.
2. M. Eshaghi Gordji, S. Bazeghi, C. Park and S. Y. Jang, *Ternary Jordan ring derivations on Banach ternary algebras: A fixed point approach*, J. Comput. Anal. Appl. **21** (5) (2016) 829–834.
3. M. E. Gordji, V. Keshavarz, C. Park and J. R. Lee, *Approximate ternary Jordan bi-derivations on Banach Lie triple systems*, J. Comput. Anal. Appl. **22** (1) (2017) 45–51.
4. P. Găvruţa, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (3) (1994) 431–436.
5. D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (4) (1941) 222–224.
6. L. Naranjani, M. Hassani and M. Mirzavaziri, *Local higher derivations on $C^*$-algebras are higher derivations*, Int. J. Nonlinear Anal. Appl. **9** (1) (2018) 111–115.
7. T. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (2) (1978) 297–300.

8. S. M. Ulam, *Problems in Modern Mathematics*, Science Editions John Wiley & Sons, Inc., New York, 1964.
9. H. Zettl, *A characterization of ternary rings of operators*, Adv. in Math. **48** (2) (1983) 117–143.

E-mail: v.keshavarz68@yahoo.com
E-mail: zohrekefayati68@gmail.com

# Some New Inequality for Operator Means and the Hadamard Product

Somayeh Malekinejad*
Department of Mathematics, Payame Noor University, Tehran, Iran

Abstract. The paper contains some new theorems for Hadamard product. Some inequalities for Heinz and Heron means has been proved using operator means.

**Keywords:** Hadamard product, Heinz means, Heron means, Mean adjoint, Positive operator.
**AMS Mathematical Subject Classification [2010]:** 47A63, 15A42, 15A45.

## 1. Introduction

Throughout the paper, let $B(H)$ be the set of all bounded linear operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. For $A, B \in B(H)$, $A^*$ denotes conjugate operator of $A$. An operator $A \in B(H)$ is positive, and we write $A \geq 0$, if $\langle Ax, x\rangle \geq 0$ for every vector $x \in H$. If $A$ and $B$ are self-adjoint operators, then order relation $A \geq B$ means, as usual, that $A - B$ is a positive operator. The set of all positive invertible operators is denoted by $B(H)_{++}$.

Let $A, B \in B(H)$ be two positive operator and $v \in [0, 1]$. The $v$-weighted arithmetic mean of $A$ and $B$ denoted by $A\nabla_v B$, is defined as $A\nabla_v B = (1 - v)A + vB$. If $A$ is invertible, then $v$-geometric mean of $A$ and $B$ denoted by $A\sharp_v B$ is defined as $A\sharp_v B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^v A^{\frac{1}{2}}$. In addition if both $A$ and $B$ are invertible. $v$-harmonic mean of $A$ and $B$, denoted by $A!_v B$, is defined as $A!_v B = ((1 - v)A^{-1} + vB^{-1})^{-1}$ for more detail, see [1]. When $v = \frac{1}{2}$, we write $A\nabla B$, $A\sharp B$, $A!B$ for brevity, respectively. The operator version of the Heinz means, denoted by

$$H_v(A, B) = \frac{A\nabla_v B + A\nabla_{1-v} B}{2},$$

where $A, B \in B(H)_{++}$, and $v \in [0, 1]$. The operator version of the Heron means, denoted by

$$F_\alpha(A, B) = (1 - \alpha)(A\sharp B) + \alpha(A\nabla B),$$

for $0 \leq \alpha \leq 1$.

It is well known that if $A$ and $B$ are positive invertible operators, then

$$A\nabla_v B \geq A\sharp_v B \geq A!_v B,$$

for $0 < v < 1$.

The usual arithmetic, geometric and harmonic means correspond to $\nu = \frac{1}{2}$. The following identity holds. See [4]

$$(A\sigma B)\sharp(B\sigma^{\perp} A) = A\sharp B.$$

*Presenter

THEOREM 1.1. [3, Theorem 5.7] *Every operator mean $\sigma$ is subadditive:*

$$A\sigma C + B\sigma D \leq (A+B)\sigma(C+D),$$

*and jointly concave:*

$$\lambda(A\sigma C) + (1-\lambda)(B\sigma D) \leq (\lambda A + (1-\lambda)B)\sigma(\lambda C + (1-\lambda)D),$$

*for* $0 \leq \lambda \leq 1$.

Mond et al. [3] studied inequalities for the mixed operator and the mixed matrix means in 1996-1997. A simple inequalities of this type are:

$$\begin{aligned} &A\sharp_\mu(A\nabla_\lambda) \geq A\nabla_\lambda(A\sharp B), \\ &A!_\lambda(A\sharp_\mu B) \geq A\sharp_\mu(A!_\lambda B), \\ &A!_\mu(A\nabla_\lambda B) \geq A\nabla_\lambda(A!_\mu B), \end{aligned}$$

where $A, B \in B_{++}(H)$ are invertible and $\lambda, \mu \in (0,1)$.

Also, the following important inequalities were obtained by Moslehian and Bakherad [2],

$$(\sum_{i=1}^{k}(A_i\sigma B_i)) \circ (\sum_{i=1}^{k}(A_i\sigma^{\perp} B_i)) \geq (\sum_{i=1}^{k}(A_i\sharp B_i)),$$

$$(\sum_{i=1}^{k} A_i) \circ (\sum_{i=1}^{k} B_i) \geq (\sum_{i=1}^{k}(A_i\sharp B_i)) \circ (\sum_{i=1}^{k}(A_i\sharp B_i)).$$

## 2. Main Results

THEOREM 2.1. *Let $A, B \in B(H)^{++}$ and $v, \lambda, \mu \in (0,1)$. Then*

i) $H_v(A, A!_\lambda B) \leq A!_\lambda H_v(A,B)$,

ii) $F_\mu(A, A!_\lambda B) \leq A!_\lambda F_\mu(A,B)$,

iii) $H_\mu(A, A!\lambda B) \leq A^{1-\lambda}F^{\lambda}_{(2\mu-1)^2}(A,B)$.

PROOF. i)

$$\begin{aligned} H_v(A, A!_\lambda B) &= \frac{A\sharp_v(A!_\lambda B) + A\sharp_{1-v}(A!_\lambda B)}{2} \\ &\leq \frac{A!_\lambda(A\nabla_v B) + A!_\lambda(A\sharp_{1-v} B)}{2} \\ &\leq \frac{(A+A)!_\lambda((A\sharp_v B) + (A\sharp_{1-v} B))}{2} \\ &= \frac{2A!_\lambda(2H_v(A,B))}{2} = A!_\lambda H_v(A,B), \end{aligned}$$

ii)

$$\begin{aligned}F_\mu(A, A!_\lambda B) &= (1-\mu)(A\sharp(A!_\lambda B)) + \mu(A\nabla(A!_\lambda B))\\ &\le \mu A!_\lambda(A\nabla B) + (1-\mu)(A!_\lambda(A\sharp B))\\ &= A!_\lambda(\mu A\nabla B + (1-\mu)A\sharp B)\\ &= A!(F_\mu(A,B)),\end{aligned}$$

iii)

$$\begin{aligned}H_\mu(A, A!_\lambda B) &\le A!_\lambda F_{(2\mu-1)^2}(A,B)\\ &= ((1-\lambda)A^{-1} + \lambda F^{-1}_{(2\mu-1)^2}(A,B))^{-1}\\ &= \frac{1}{(1-\lambda)A^{-1} + \lambda F^{-1}_{(2\mu-1)^2}(A,B)}\\ &\le \frac{1}{(A^{-1})^{1-\lambda} F^{-\lambda}_{(2\mu-1)^2}(A,B)}\\ &= A^{1-\lambda} F^{\lambda}_{(2\mu-1)^2}(A,B).\end{aligned}$$

□

THEOREM 2.2. *Let $A, B \in B(H)^{++}$. Then*

$$(A\sigma^* B)\sharp(B\sigma A) = A\sharp B.$$

COROLLARY 2.3. *Let $A, B \in B(H)^{++}$. Then*

$$\left(\frac{A^{-1}+B^{-1}}{2}\right)\sharp\left(\frac{A+B}{2}\right)^{-1} = (A\sharp B)^{-1}.$$

THEOREM 2.4. *Let $A, B \in B(H)_{++}$. Then*

$$(\sum_{i=1}^{k} A_i^{-1}) \circ (\sum_{i=1}^{k} B_i^{-1}) \ge (\sum_{i=1}^{k}(A_i\sharp B_i)^{-1}) \circ (\sum_{i=1}^{k}(A_i\sharp B_i)^{-1}).$$

THEOREM 2.5. *Let $A, B \in B(H)_{++}$. Then*

$$(\sum_{i=1}^{k}(A_i\sharp B_i)^{-1}) \circ (\sum_{i=1}^{k}(A_i\sharp B_i)^{-1}) \ge \sum_{i=1}^{k}(A_i\sharp B_i)^{-1}.$$

## References


1. F. Kubo and T. Ando, *Means of positive linear operators*, Math. Ann. **246** (1979/80) 205–224.
2. M. S. Moslehian and M. Bakherad, *Chebyshev type inequalities for Hilbert space operators*, J. Math. Anal. Appl. **420** (2014) 737–749.
3. J. Pecaric, T. Furuta, J. Micic Hot and Y. Seo, *Mond-Pecaric Method in Operator Inequalities*, 1. ELEMENT, Zagreb, 2005.
4. M. Singh, J. S. Aujla and H. L. Vasudeva, *Inequalities for hadamard product and unitarily invariant norms of matrices*, Linear Multilinear Algebra **48** (2001) 247–262.


E-mail: maleki60313@pnu.ac.ir

# Bounds for Heron Mean by Heinz Mean and other Means

Somayeh Malekinejad*
Department of Mathematics, Payame Noor University, Tehran, Iran

Abstract. In this paper, some bound for Heron mean by Heinz mean and other means are peresented. we give some new inequality for scalars and we use them to establish new inequality for operators.
**Keywords:** Heinz operator means, Heron operator means, Positive operator.
**AMS Mathematical Subject Classification [2010]:** 15A45, 47A63.

## 1. Introduction

Let $a, b \geq 0$ and $0 \leq \nu \leq 1$. The Heinz means are defined as follows:

$$H_\nu(a,b) = \frac{a^\nu b^{1-\nu} + a^{1-\nu} b^\nu}{2},$$

and Heron means are defined as follows:

$$F_{\alpha(\nu)}(A,B) = (1-\alpha)(A \sharp B) + \alpha(A \nabla B).$$

We have

$$\sqrt{ab} \leq F_{\alpha(\nu)}(a,b) \leq \frac{a+b}{2}.$$

Heinz means interpolate between the geometric mean and arithmetic mean:

$$\sqrt{ab} \leq H_\nu(a,b) \leq \frac{a+b}{2}. \tag{1}$$

The second inequality of (1) is know as Heinz inequality for nonnegative real numbers.

R. Bhatia [1] proved that the Heinz and Heron means satisfy the following inequality

$$H_\nu(a,b) \leq F_{\alpha(\nu)}(a,b),$$

for $\nu \in [0,1]$, where $\alpha(\nu) = 1 - 4(\nu - \nu^2)$.

The following important inequalities were obtained by Cartwright and Field [2],

$$a^\nu b^{1-\nu} + \frac{\nu(1-\nu)}{2m}(a-b)^2 \geq \nu a + (1-\nu)b,$$

where $a > 0$, $b > 0$, $m = \min\{a,b\}$, $M = \max\{a,b\}$ and $0 \leq \nu \leq 1$. To reach inequalities for bounded self-adjoint operators on Hilbert space, we shall use the following monotonicity property for operator functions:

*Presenter

If $X \in B_h(H)$ with a spectrum $Sp(X)$ and $f, g$ are continuous real-valued functions on $Sp(X)$, then

(2) $$f(t) \geq g(t),\ t \in Sp(X) \Rightarrow\ f(X) \geq g(X).$$

For more details about this property, the reader is referred to [3].

Yang and Ren [4] proved that

THEOREM 1.1. *If $A$ and $B$ be two positive and invertible operators, $I$ be the identity operator, and $\nu \in [0,1]$, then we have*

$$\nu(1-\nu)(A\nabla B - A\sharp B) + A\sharp B \leq F_\nu(A, B),$$

*and*

$$F_\nu(A, B) \leq A\nabla B - \nu(1-\nu)(A\nabla B - A\sharp B).$$

Recently Zuo and Jiang in [5] obtained the other inequalities:

THEOREM 1.2. *The Heinz and Heron means satisfy*

$$F_\alpha(a, b) \geq H_\nu(a, b) + 4\nu(1-\nu)(a\nabla b - a\sharp b),$$

*for $a, b \geq 0$, $\nu \in [0,1]$, $\alpha = 1 - 4(\nu - \nu^2)$ and $(\frac{b}{a})^{\nu-\frac{1}{2}} + (\frac{b}{a})^{\frac{1}{2}-\nu} \geq 4$.*

THEOREM 1.3. *Let $a, b \geq 0$ and $\nu \in [0,1]$, then we can have*

$$(a+b)^2 \geq 4(H_\nu(a,b))^2 + 8\nu(1-\nu)(a^2\nabla b^2 - a^2\sharp b^2),$$

*for $(2\nu - 1)(b^2 - a^2) \geq 0$.*

## 2. Main Results

THEOREM 2.1. *Let $a, b \geq 0$ and $\frac{1}{2} \leq \nu \leq 1$ then*

(3) $$F_{2\nu-1}(a, b) \leq H_\nu(a, b) + (2\nu - 1)(a\nabla b - a\sharp b).$$

PROOF. Inequality (3), in expanded forms, says

$$(2\nu-1)(\frac{a+b}{2}) + 2(1-\nu)\sqrt{ab} \leq \frac{a^\nu b^{1-\nu} + a^{1-\nu}b^\nu}{2} + (2\nu-1)(\frac{a+b}{2} - \sqrt{ab}).$$

Put $a = 1$, $b = t$,

$$(2\nu-1)(\frac{1+t}{2}) + 2(1-\nu)\sqrt{t} \leq \frac{t^\nu + t^{1-\nu}}{2} + (2\nu-1)(\frac{1+t}{2} - \sqrt{t}).$$

Let

$$f(t) = \frac{t^\nu + t^{1-\nu}}{2} + (2\nu-1)(\frac{1+t}{2} - \sqrt{t}) - (2\nu-1)(\frac{1+t}{2}) - 2(1-\nu)\sqrt{t}.$$

Then

$$f^{'}(t) = \frac{\nu t^{\nu-1} + (1-\nu)t^{-\nu}}{2} + (2\nu-1)(\frac{1}{2} - \frac{1}{2\sqrt{t}}) - \frac{(2\nu-1)}{2} - (1-\nu)\frac{1}{\sqrt{t}},$$

and

$$f^{''}(t) = \frac{\nu(\nu-1)t^{\nu-2} - \nu(1-\nu)t^{-\nu-1}}{2} + (2\nu-1)(\frac{1}{4}t^{-\frac{3}{2}}) + (1-\nu)(\frac{1}{2}t^{-\frac{3}{2}}).$$

Finally

$$f^{''}(t) = \frac{\nu(\nu-1)[t^{\nu-2} + t^{-\nu-1}]}{2} + (\frac{1}{4}t^{-\frac{3}{2}}),$$

we have $f^{''}(t) \geq 0$. Then $f(1) = f'(1) = 0$ which means that $f(t)$ is decreasing on $(0,1]$ and increasing on $(1,\infty)$, respectively. Consequently, $f(t) \leq 0$ for $\nu \in [0,1]$. This proved that

$$F_{2\nu-1}(a,b) \leq H_\nu(a,b) + (2\nu-1)(a\nabla b - a\sharp b),$$

holds for $a, b > 0$, $\nu \in [0,1]$. □

Theorem 2.2. *If $A$ and $B$ be two positive and invertible operators then*

$$F_{2\nu-1}(A,B) \leq H_\nu(A,B) + (2\nu-1)(A\nabla B - A\sharp B),$$

*for* $\nu \in [\frac{1}{2}, 1]$.

Proof. If $\nu \in [0, \frac{1}{2}]$, the inequality (3) for $a = 1, b > 0$, becomes

$$(2\nu-1)(\frac{1+b}{2}) + (2-2\nu)\sqrt{b} \leq \frac{b^\nu + b^{1-\nu}}{2} + (2\nu-1)(\frac{1+b}{2} - \sqrt{b}).$$

Since the operator $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ has a positive spectrum. According to rule (2), we can insert $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in above inequality, i.e., we have

$$(2\nu-1)\left(\frac{1+A^{-\frac{1}{2}}BA^{-\frac{1}{2}}}{2}\right) + (2-2\nu)(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}} \leq \frac{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\nu + (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-\nu}}{2}$$
$$+ (2\nu-1)\left(\frac{1+A^{-\frac{1}{2}}BA^{-\frac{1}{2}}}{2} - (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}\right).$$

Finally, if we multiply inequality by $A^{\frac{1}{2}}$ on the left and right, we get

$$\frac{A\sharp_\nu B + A\sharp_{1-\nu}B}{2} + (2\nu-1)(\frac{A+B}{2} - A\sharp B) \geq (2\nu-1)(\frac{A+B}{2}) + 2(1-\nu)(A\sharp B).$$

□

## References


1. R. Bhati, *Interpolating the arithmetic-geometric mean inequality and its operator version*, Linear Algebra Appl. **413** (2006) 355–363.
2. D. I. Cartwright and M. J. Field, *A refinement of the arithmetic mean-geometric mean inequality*, Proc. Amer. Math. Soc. **71** (1) (1978) 36–38.
3. J. E. Pecaric, T. Furuta, J. M. Hot and Y. Seo, *Mond-Pecaric Method in Operator Inequalities*, Element, Zagreb, 2005.
4. C. Yang and Y. Ren, *Some results of Heron mean and Youngs inequalities*, J. Inequal. Appl. **2018** (1) (2018) 172.
5. H. Zuo and F. Jiang, *More accurate Young, Heinz-Heron mean and Heinz inequalities for scalar and matrix*, Ital. J. Pure Appl. Math. **44** (2020) 567–575.


E-mail: maleki60313@pnu.ac.ir

# Generalized $T_F$-contractive Mappings and Solving Some Polynomials

Sirous Moradi*
Department of Mathematics, Faculty of Sciences, Lorestan University, Khorramabad, 68151-4-4316, Iran

ABSTRACT. In this paper, by considering generalized $T_F$-contractive mapping and the concept of sequentially convergent, we give the existence and uniqueness of a fixed point. These conditions are analogous to Ćirić conditions. Also, we show that the concept of sequentially convergent is a special case of the concept of graph closed. Finally, by using the main theorem, we present an application to solving some polynomials.
**Keywords:** Contractive mapping, Generalized $T_F$-contractive mapping, Graph closed.
**AMS Mathematical Subject Classification [2010]:** 46J10, 46J15, 47H10.

## 1. Introduction

A new class of contractive mappings were introduced by Moradi and Beiranvand [5] in 2010. By this new definition, they extended the Branciari's theorem as follows:

THEOREM 1.1. *Let $(X,d)$ be a complete metric space, $\alpha \in [0,1)$, $T,g : X \longrightarrow X$ be two mappings such that $T$ is one-to-one and graph closed and $g$ is a $T_F$-contraction; that is:*

$$F(d(Tgx,Tgy)) \leq \alpha F(d(Tx,Ty)),$$

*for all $x,y \in X$, where $F : [0,+\infty) \longrightarrow [0,+\infty)$ is nondecreasing and continuous with $F^{-1}(0) = \{0\}$; then $g$ has a unique fixed point $a \in X$. Also for every $x \in X$, the sequence of iterates $\{Tg^n x\}$ converges to $Ta$.*

In 2015, Mehmet Kir and Hukmi Kiziltunc [3], extended Kannan fixed point theorem by using $T_F$-contraction mappings. After that in 2017, Dubey et al. [2], proved some fixed point results for $T_F$ type of contractive mappings in the framework of complete metric spaces. Some of their results, as follows:

THEOREM 1.2. [2] *Let $T,g : X \longrightarrow X$ be tow mappings on complete metric space $(X,d)$ and such that $T$ is continuous, one-to-one and subsequentially convergent. If $a,b \in [0,1)$ and $x,y \in X$*

$$F(d(Tgx,Tgy)) \leq a[F(d(Tx,Ty))] + b[F(d(Tx,Tgx)) + F(d(Tx,Tgy))],$$

*where $F : [0,+\infty) \longrightarrow [0,+\infty)$ is nondecreasing and continuous from the right with $F^{-1}(0) = \{0\}$. Then $g$ has a unique fixed point $a \in X$. Also by considering*

*Presenter

*the sequentially convergent of $T$, we conclude that, for every $x_0 \in X$ the sequence $\{g^n x_0\}$ converges to the fixed point of $g$.*

Theorem 1.3. [2] *Let $T, g : X \longrightarrow X$ be tow mappings on complete metric space $(X, d)$ and such that $T$ is continuous, one-to-one and subsequentially convergent. If $a, b, c \in [0, 1)$ and $x, y \in X$*

$$\begin{aligned} F(d(Tgx, Tgy)) &\leq a[F(d(Tx, Ty))] + b[F(d(Tx, Tgy)) + F(d(Ty, Tgx))] \\ &+ c[F(d(Tx, Tgx)) + F(d(Tx, Tgy))], \end{aligned}$$

*where $F : [0, +\infty) \longrightarrow [0, +\infty)$ is continuous from the right and is nondecreasing with $F^{-1}(0) = \{0\}$. Then $g$ has a unique fixed point $a \in X$. Also by considering the sequentially convergent of $T$ we conclude that, for every $x_0 \in X$ the sequence $\{g^n x_0\}$ converges to the fixed point of $g$.*

Theorem 1.4. [2] *Let $T, g : X \longrightarrow X$ be tow mappings on complete metric space $(X, d)$ and such that $T$ is continuous, one-to-one and subsequentially convergent. For all $x, y \in X$*

$$\begin{aligned} F(d(Tgx, Tgy)) &\leq a(x, y)[F(d(Tx, Ty))] \\ &+ b(x, y)[F(d(Tx, Tgy)) + F(d(Ty, Tgx))] \\ &+ c(x, y)[F(d(Tx, Tgx)) + F(d(Tx, Tgy))], \end{aligned}$$

*where $a(x, y), b(x, y), c(x, y) \geq 0$ and*

$$\sup[a(x, y) + 2b(x, y) + 2c(x, y)] \leq \lambda < 1,$$

*and where $F : [0, +\infty) \longrightarrow [0, +\infty)$ is nondecreasing and continuous from the right with $F^{-1}(0) = \{0\}$. Then $g$ has a unique fixed point $a \in X$. Also by considering the sequentially convergent of $T$ we conclude that, for every $x_0 \in X$ the sequence $\{g^n x_0\}$ converges to the fixed point of $g$.*

Remark 1.5. In the proof of above theorems, the boundedness of the sequence $\{Tg^n x_0\}$ is used by the authores, but not proved. Also the authores just considered $a, b, c \in [0, 1)$ for Theorem 1.2 and Theorem 1.3. In the following, we give counterexamples for these tow theorems. In the main resualts of this paper, we extend and correct the above theorems.

Example 1.6. Let $X = \{1, 2\}$ endowed with the Euclidean metric and let $g : X \longrightarrow X$ defined by $g(1) = 2, g(2) = 1$. Suppose that $a = b = c = \frac{2}{3}$ and $T(x) = F(x) = x$ for all $x \in X$. It can be easily verified that, the condition of Theorems 1.2 and 1.3 are hold. But $g$ has not fixed point.

In all part of this paper, $(X, d)$ denotes a complete metric space.

Definition 1.7. [5] A mapping $T : X \longrightarrow X$ is said to be graph closed if for every sequence $\{x_n\}$ with $\lim_{n \to \infty} Tx_n = a$ we can find some $b \in X$ such that $Tb = a$.

Definition 1.8. [4] A mapping $T : X \longrightarrow X$ is said to be sequentially convergent if we have, for every sequence $\{y_n\}$ , if $\{Ty_n\}$ is convergent then $\{y_n\}$ also is convergent. $T$ is said to be subsequentially convergent if we have, for every sequence $\{y_n\}$ , if $\{Ty_n\}$ is convergent then $\{y_n\}$ has a convergent subsequence.

Let $\Psi$ denotes the class of all nondecreasing and continuous maps $F : [0, +\infty) \to [0, +\infty)$ with $F^{-1}\{0\} = \{0\}$.

DEFINITION 1.9. A mapping $g : X \longrightarrow X$ is said to be generalized $T_F$-contractive, if there exists $F \in \Psi$ and one-to-one and graph closed mapping $T : X \longrightarrow X$ such that

$$F(d(Tgx, Tgy)) \leq \alpha F(N(x, y)),$$

for all $x, y \in X$ and some $\alpha \in [0, 1)$, where

$$N(x, y) = \max\{d(Tx, Ty), d(Tx, Tgx), d(Ty, Tgy), \frac{d(Tx, Tgy) + d(Ty, Tgx)}{2}\}.$$

For our main results, at first, we prove the following useful lemma.

LEMMA 1.10. *Let $T : X \longrightarrow X$ be a mapping on the complete metric space $(X, d)$ , such that $T$ is continuous and subsequentially convergent. Then $T$ is a graph closed map.*

PROOF. Suppose that $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Tx_n = a$. Since $T$ is subsequentially convergent, then there exists a subsequence $\{x_{n(k)}\}$ such that $\lim_{k\to\infty} x_{n(k)} = b$. Since $T$ is continuous and $\lim_{n\to\infty} Tx_n = a$, then we conclude that $Tb = a$. This completes the proof. □

REMARK 1.11. In 2012 Aydi et al. [1] proved that the main results of some papers; that consider the sequentially convergent; are particular results of previous existing theorems in the literature. We can not conclude that, every graph closed map is subsequentially convergent. For example, suppose that $X = \mathbb{R}$ endowed with the Euclidean metric and $T : X \longrightarrow X$ defined by, $Tx := \sin x$. Obviousley, $T$ is continuous and graph closed, but $T$ is not subsequentially convergent. Because the sequence $\{\sin(2n\pi)\}$ is convergent, but the sequence $\{2n\pi\}$ has not any convergent subsequence. In this paper we consider the graph closed mappins for the main results.

## 2. Main Results

Our main result of this paper is the following theorem.

THEOREM 2.1. *Let $g : X \longrightarrow X$ be a mapping on complete metric space $(X, d)$ such that satisfies the following condition:*

$$F(d(Tgx, Tgy)) \leq \alpha F(N(x, y)), \tag{1}$$

*for all $x, y \in X$ and some $\alpha \in [0, 1)$ (i.e., generalized $T_F$-contractive) where $F \in \Psi$ and $T : X \longrightarrow X$ is a one-to-one and graph closed map. Then $g$ has a unique fixed point $b \in X$ and for every $x \in X$ the sequence of iterates $\{Tg^n x\}$ converges to $Tb$. Also by considering the sequentially convergent of $T$ we conclude that, for every $x \in X$ the sequence $\{g^n x_0\}$ converges to to $b$ (the fixed point of $g$).*

PROOF. Unicity of the fixed point follows from (1).
Since $F \in \Psi$, for every $\varepsilon > 0$

$$F(\varepsilon) > 0.$$

From (1) if $x \neq y$ then,

$$d(Tgx, Tgy) < N(x, y).$$

Let $x \in X$. Define $x_n = Tg^n x$.
We break the argument into four steps.

**Step 1**. $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

**Step 2**. $\{x_n\}$ is a bounded sequence.

**Step 3**. $\{x_n\}$ is a Cauchy sequence.

**Step 4**. $g$ has a fixed point. □

REMARK 2.2. By considering $Tx = x$ and $F(t) = \int_0^t \phi(s)ds$ in the above theorem, we conclude that, Theorem 2.1 is a real generalization of the Rhoades theorem (See the following example).

EXAMPLE 2.3. Let $S : [1, +\infty) \longrightarrow [1, +\infty)$ defined by $Sx = 4\sqrt{x}$ on the Euclidean metric space $X = [1, +\infty)$. Obviously $S$ has a unique fixed point $b = 16$. By define $T : X \longrightarrow X$ by $Tx = \ln(e.x)$. Obviously $T$ is one-to-one and graph closed. By taking $F(t) = t$, all conditions of Theorem 2.1 are hold and therefore $S$ has a unique fixed point.

The following corollary, is a new extension of Theorem 1.4.

COROLLARY 2.4. *Let $g : X \longrightarrow X$ be a mapping on complete metric space $(X, d)$ such that satisfies the following condition:*

$$\begin{aligned} F(d(Tgx, Tgy)) &\leq a(x,y)[F(d(Tx,Ty))] \\ &+ b(x,y)[F(d(Tx,Tgy)) + F(d(Ty,Tgx))] \\ &+ c(x,y)[F(d(Tx,Tgx)) + F(d(Tx,Tgy))], \end{aligned}$$

*where $a(x,y), b(x,y), c(x,y) \geq 0$ for all $x, y \in X$ and*

$$\sup_{x,y\in X} [a(x,y) + 2b(x,y) + 2c(x,y)] \leq \lambda < 1,$$

*for some $\lambda \in [0,1)$, and where $F \in \Psi$ and $T : X \longrightarrow X$ is a one-to-one and graph closed map. Then $g$ has a unique fixed point $b \in X$ and for every $x \in X$ the sequence of iterates $\{Tg^n x\}$ converges to $Tb$. Also by considering the sequentially convergent of $T$ we conclude that, for every $x \in X$ the sequence $\{g^n x_0\}$ converges to to $b$ (the fixed point of $g$).*

PROOF. One can esealy shows that

$$\begin{aligned} a(x,y)[F(d(Tx,Ty))] + b(x,y)[F(d(Tx,Tgy)) + F(d(Ty,Tgx))] \\ +c(x,y)[F(d(Tx,Tgx)) + F(d(Tx,Tgy))] \leq \lambda F(N(x,y)), \end{aligned}$$

for all $x, y \in X$. Now by using Theorem 2.1, the result is obtained. □

## 3. Application to Solving Polynomials

As an application of the main theorem of this paper we conclude the existence of solution of some polynomials.

THEOREM 3.1. *Let $b, c > 0$ and $n > 1$. Then the equation*

$$y^n = by + c, \tag{2}$$

*has a unique solution on* $[\sqrt[n]{c}, +\infty)$.

PROOF. Let $0 < \varepsilon < b\sqrt[n]{c}$ be arbitrary. Put $\alpha = c + \varepsilon$. It is enough to show that the problem (2) has a unique solution on $[\sqrt[n]{\alpha}, +\infty)$.
There exists $\beta > 0$ such that $\ln(\alpha - c) + \beta \geq \alpha$. Suppose $g : [\alpha, +\infty) \longrightarrow [\alpha, +\infty)$ defined by $gx = b\sqrt[n]{x} + c$ and $T : [\alpha, +\infty) \longrightarrow [\alpha, +\infty)$ defined by $Tx = \ln(x - c) + \beta$. For all $x, y \in [\alpha, +\infty)$ with $x > y$ we have

$$|Tgx - Tgy| = \ln(\frac{\sqrt[n]{x}}{\sqrt[n]{y}}) = \frac{1}{n}\ln(\frac{x}{y}) < \frac{1}{n}\ln(\frac{x-c}{y-c}) = \frac{1}{n}|Tx - Ty| \leq \frac{1}{n}N(x, y).$$

Hence $g$ is generalized $T_F$-contractive. So $g$ has a unique fixed point $z$ on $[\alpha, +\infty)$ and the sequence of iterates $\{Tg^n(c+1)\}$ converges to $Tz$ and therefore, the sequence of iterates $\{g^n(c+1)\}$ converges to $z$. Therefore the equation $x = b\sqrt[n]{x}+c$ has a unique solution on $[\alpha, +\infty)$. Also there exists a unique $y > 0$ such that $y^n = z$. Obviously $y \in [\sqrt[n]{\alpha}, +\infty)$. Hence from $z = b\sqrt[n]{z} + c$ we have $y^n = by + c$ and this completes the proof. □

## Acknowledgement

The author is grateful to the referees for his or her suggestions to the improvement of the paper.

## References

1. H. Aydi, E. Karapınar and B. Samet, *Remarks on some recent fixed point theorems*, Fixed Point Theory and Appl. **2012** (2012) 76.
2. A. K. Dubey, U. Mishra and R. Pal, *New fixed point theorems for $t_f$ type contractive conditions*, Global J. Pure & Appl. Math. **13** (11) (2017) 529–539.
3. M. Kir and H. Kiziltunc, *Some generalized fixed point theorems in the context of ordered metric spaces*, J. Nonlinear Sci. Appl. **8** (2015) 7955–7962.
4. S. Moradi and D. Alimohammadi, *New extensions of Kannan fixed-point theorem on complete metric and generalized metric spaces*, Int. J. Math. Analysis **5** (47) (2011) 2313–2320.
5. S. Moradi and A. Beiranvand, *Fixed Point of $T_F$−contractive Single-valued Mappings*, Iranian J. Math. Sci. Info. **5** (2) (2010) 25–32.

E-mail: moradi.s@lu.ac.ir; sirousmoradi@gmail.com

# A Perturbation of Controlled Generalized Frames

Kamran Musazadeh*
Department of Mathematics, College of Science, Mahabad Branch, Islamic Azad University, Mahabad, Iran

Abstract. We define a new perturbation of controlled g-frames by appropriate bounded invertible operators to obtain new g-frames from a given one with optimal g-frame bounds. Also we generalize an identity to the controlled g-frames.
**Keywords:** g-Frames, Controlled g-frames, Perturbation.
**AMS Mathematical Subject Classification [2010]:** 42C15, 68M10, 46C05.

## 1. Introduction

In 2006, a new generalization of the frame named g-frame was introduced by Sun [5] in a complex Hilbert space. G-frames are natural generalizations of frames which cover the above generalizations of frames. Controlled frames for spherical wavelets were introduced in [1] to get a numerically more efficient approximation algorithm and the related theory for general frames were developed in [3]. Controlled g-frames with two controller operators were studied in [4]. To get a large class of controlled g-frames it is important to use of two controlling operators.

Throughout this paper $H$, $K$ are separable Hilbert spaces, $\mathcal{L}(H,K)$ denotes the space of all bounded linear operators from $H$ to $K$ and $GL(H)$ denotes the set of all bounded linear operators which have bounded inverses. Let $\{K_i : i \in I\}$ be a sequence of closed subspaces of a Hilbert space $K$ (for example $K = (\bigoplus_{i\in I} K_i)_{\ell_2} = \{\{f_i\}_{i\in I} : f_i \in K_i, \forall i \in I, \sum_{i\in I} \|f_i\|^2 < \infty\}$).

Definition 1.1. Let $T, U \in GL(H)$ and $\Lambda = \{\Lambda_i \in \mathcal{L}(H, K_i) : i \in I\}$ be a sequence of bounded linear operators. We say that $\Lambda$ is a $(T, U)$-controlled generalized frame, or simply a $(T, U)$-CGF, for $H$ with respect to $\{K_i : i \in I\}$ if there exist two positive constants $0 < C_{TU} \leq D_{TU} < \infty$ such that

$$C_{TU}\|f\|^2 \leq \sum_{i\in I} < \Lambda_i T f, \Lambda_i U f > \leq D_{TU}\|f\|^2, \qquad \forall f \in H.$$

We call $C_{TU}$ and $D_{TU}$ the lower and upper CGF bounds, respectively.
We call $\Lambda$ a $C_{TU}$-tight CGF (TCGF) if $C_{TU} = D_{TU}$ and we call it a Parseval CGF (PCGF) if $C_{TU} = D_{TU} = 1$. If only the second inequality holds, then we call it a $(T, U)$-controlled G-Bessel sequence, or simply a $(T, U)$-CGBS.

Let $\Lambda$ be a G-Bessel sequence for a Hilbert space $H$ and $T \in GL(H)$. Then we define the *Analysis operator* $\theta_{\Lambda T} : H \to (\bigoplus_{i\in I} K_i)_{\ell_2}$ for $\Lambda$ as follows:

$$\theta_{\Lambda T} f = \{\Lambda_i T f\}_{i\in I}, \qquad \forall f \in H.$$

*Presenter

So its adjoint $\theta^*_{\Lambda T} : (\bigoplus_{i\in I} K_i)_{\ell_2} \to H$ Which is called the *Synthesis operator* for $\Lambda$ is defined as follows:

$$\theta^*_{\Lambda T}\Big(\{f_i\}_{i\in I}\Big) = \sum_{i\in I} T^*\Lambda_i^* f_i, \qquad \forall \{f_i\}_{i\in I} \in (\bigoplus_{i\in I} K_i)_{\ell_2}.$$

Therefore, The controlled g-frame operator $S_{TU} : H \to H$ with respect to a $(T,U)$-CGF $\Lambda$ can be defined as follows:

$$S_{TU} f = \theta^*_{\Lambda U}\theta_{\Lambda T} f = \sum_{i\in I} U^*\Lambda_i^*\Lambda_i T f = U^* S_\Lambda T f, \qquad \forall f \in H,$$

where $S_\Lambda f = \sum_{i\in I}\Lambda_i^*\Lambda_i f$. Furthermore, $C_{TU} Id_H \le S_{TU} \le D_{TU} Id_H$ . So $S_{TU}$ is a well-defined bounded linear operator which is also positive and invertible.

Proposition 1.2. *Let $T, U \in GL(H)$ and $\Lambda = \{\Lambda_i \in \mathcal{L}(H, K_i) : i \in I\}$ be a sequence of bounded linear operators. Then the following statements hold:*

i) *If $\Lambda$ is a $(T,U)$-CGF for $H$. Then $\Lambda$ is a g-frame for $H$.*
ii) *If $\Lambda$ is a g-frame for $H$ and $U^*S_\Lambda T$ is a positive operator, then $\Lambda$ is a $(T,U)$-CGF for $H$.*

Proof. It is straight forward. □

## 2. Main Results

We have the following identity which is a generalization of a result in [2].

Proposition 2.1. *Let $\Lambda$ be a g-frame for $H$ and $T, U$ be operators for which $\Lambda$ is a $(T,U)$-PCGF for $H$. For any subset $K \subset I$ let $\Lambda_K = \{\Lambda_i\}_{i\in K}$. Then the following statements hold for all $f \in H$.*

i) $< \theta_{\Lambda_K T} f, \theta_{\Lambda_K U} f > - \|\theta^*_{\Lambda_K U}\theta_{\Lambda_K T} f\|^2 = < \theta_{\Lambda_{I-K} T} f, \theta_{\Lambda_{I-K} U} f > - \|\theta^*_{\Lambda_{I-K} U}\theta_{\Lambda_{I-K} T} f\|^2$.
ii) $< \theta_{\Lambda_K T} f, \theta_{\Lambda_K U} f > + \|\theta^*_{\Lambda_{I-K} U}\theta_{\Lambda_{I-K} T} f\|^2 \ge \frac{3}{4}\|f\|^2$.

Proof.

i) Define $S_K f := \theta^*_{\Lambda_K U}\theta_{\Lambda_K T} f = \sum_{i\in K} U^*\Lambda_i^*\Lambda_i T f$ for each $f \in H$. Since $\Lambda$ is a $(T,U)$-PCGF for $H$, then $S_K + S_{I-K} = S_{TU} = Id_H$. Therefore,

$$S_K - S_K^2 = S_{I-K} - S_{I-K}^2.$$

So for each $f \in H$ we have

$$< S_K f, f > - < S_K f, S_K f > = < S_{I-K} f, f > - < S_{I-K} f, S_{I-K} f > .$$

Hence,

$$\sum_{i\in K} < \Lambda_i T f, \Lambda_i U f > - \|\sum_{i\in K} U^*\Lambda_i^*\Lambda_i T f\|^2 = \sum_{i\in I-K} < \Lambda_i T f, \Lambda_i U f > - \|\sum_{i\in I-K} U^*\Lambda_i^*\Lambda_i T f\|^2.$$

ii) It is similar to the proof of the in [2, Theorem 2.1].

□

Definition 2.2. Let $\Lambda = \{\Lambda_i \in \mathcal{L}(H, K_i) : i \in I\}$ and $\Gamma = \{\Gamma_i \in \mathcal{L}(H, K_i) : i \in I\}$ be two sequences of bounded linear operators. Let $T, U \in GL(H)$ and

$0 \leq \lambda_1, \lambda_2 < 1$ be real numbers. We say that $\Gamma$ is a $(\lambda_1, \lambda_2, T, U)$-perturbation of $\Lambda$ if for all $f \in H$,

$$\| (\theta_{\Gamma U} - \theta_{\Lambda T})f\|_2 \leq \lambda_1 \| \theta_{\Gamma U} f\|_2 + \lambda_2 \| \theta_{\Lambda T} f\|_2.$$

We have the following important result.

Proposition 2.3. *Let $T, U \in GL(H)$ and $\Lambda$ be a $(T,T)$-CGF for $H$ with frame bounds $C, D$. Let $\Gamma$ be a $(\lambda_1, \lambda_2, T, U)$-perturbation of $\Lambda$. Then $\Gamma$ is also a $(U,U)$-CGF for $H$ with frame bounds*

$$\left(\frac{(1-\lambda_2)\sqrt{C}}{1+\lambda_1}\right)^2, \qquad \left(\frac{(1+\lambda_2)\sqrt{D}}{1-\lambda_1}\right)^2.$$

Proof. Let $f \in H$. Then by triangular inequality we have

$$\|\theta_{\Gamma U} f\|_2 = \|(\theta_{\Gamma U} - \theta_{\Lambda T})f + \theta_{\Lambda T} f\|_2 \leq \|(\theta_{\Gamma U} - \theta_{\Lambda T})f)f\|_2 + \|\theta_{\Lambda T} f\|_2$$
$$\leq \lambda_1 \| \theta_{\Gamma U} f\|_2 + \lambda_2 \| \theta_{\Lambda T} f\|_2 + \|\theta_{\Lambda T} f\|_2.$$

So

$$(1-\lambda_1)\|\theta_{\Gamma U} f\|_2 \leq (1+\lambda_2)\|\theta_{\Lambda T} f\|_2 \leq (1+\lambda_2)\sqrt{D}\|f\|.$$

Therefore,

$$\|\theta_{\Gamma U} f\|_2 \leq \frac{(1+\lambda_2)\sqrt{D}}{1-\lambda_1}\|f\|.$$

On the other hand,

$$\begin{aligned}\|\theta_{\Gamma U} f\|_2 &= \|\theta_{\Lambda T} f - (\theta_{\Lambda T} - \theta_{\Gamma U})f\|_2 \\ &\geq \|\theta_{\Lambda T} f\|_2 - \|(\theta_{\Gamma U} - \theta_{\Lambda T})f\|_2 \\ &\geq \|\theta_{\Lambda T} f\|_2 - \lambda_1\|\theta_{\Gamma U} f\|_2 - \lambda_2\|\theta_{\Lambda T} f\|_2.\end{aligned}$$

So

$$(1+\lambda_1)\|\theta_{\Gamma U} f\|_2 \geq (1-\lambda_2)\|\theta_{\Lambda T} f\|_2 \geq (1-\lambda_2)\sqrt{C}\|f\|.$$

Therefore,

$$\frac{(1-\lambda_2)\sqrt{C}}{1+\lambda_1}\|f\| \leq \|\theta_{\Gamma U} f\|_2.$$

Now the result follows. □

Proposition 2.4. *Let $\Lambda = \{\Lambda_i \in \mathcal{L}(H, K_i) : i \in I\}$ and $\Gamma = \{\Gamma_i \in \mathcal{L}(H, K_i) : i \in I\}$ be two $(T,U)$-CGBS. Suppose that there exists $0 < \varepsilon < 1$ such that*

$$\|f - \theta^*_{\Gamma U}\theta_{\Lambda T} f\| \leq \varepsilon\|f\|, \qquad \forall f \in H.$$

*Then $\Lambda$ and $\Gamma$ are $(T,T)$-controlled and $(U,U)$-controlled g-frames for $H$, respectively.*

Proof. For each $f \in H$ we have

$$\|\theta^*_{\Gamma U}\theta_{\Lambda T} f\| \geq \|f\| - \|f - \theta^*_{\Gamma U}\theta_{\Lambda T} f\| \geq (1-\varepsilon)\|f\|.$$

Therefore, we have

$$\begin{aligned}(1-\varepsilon)\|f\| \leq \|\theta^*_{\Gamma U}\theta_{\Lambda T}f\| &= \sup_{g\in H,\|g\|=1}\Big| < \theta^*_{\Gamma U}\theta_{\Lambda T}f, g > \Big| \\ &= \sup_{g\in H,\|g\|=1}\Big| < \theta_{\Lambda T}f, \theta_{\Gamma U}g > \Big| \\ &= \sup_{g\in H,\|g\|=1}\Big| \sum_{i\in I} < \Lambda_i Tf, \Gamma_i Ug > \Big| \\ &\leq \sup_{g\in H,\|g\|=1}\Big(\sum_{i\in I}\|\Lambda_i Tf\|^2\Big)^{\frac{1}{2}}\Big(\sum_{i\in I}\|\Gamma_i Ug\|^2\Big)^{\frac{1}{2}} \\ &\leq \sqrt{D_\Gamma}\Big(\sum_{i\in I}\|\Lambda_i Tf\|^2\Big)^{\frac{1}{2}},\end{aligned}$$

where $D_\Gamma$ is a controlled Bessel bound for $\Gamma$. Hence,

$$\frac{(1-\varepsilon)^2}{D_\Gamma}\|f\|^2 \leq \sum_{i\in I}\|\Lambda_i Tf\|^2.$$

Therefore, $\Lambda$ is a $(T,T)$-controlled g-frame for $H$. Similarly, we can show that $\Gamma$ is also a $(U,U)$-controlled g-frames for $H$. □

**Acknowledgement**

The author express his gratitude to the referee for carefully reading and useful comments, which improved the manuscript.

## References

1. I. Bogdanova, P. Vandergheynst, J. P. Antoine, L. Jacques and M. Morvidone, *Stereographic wavelet frames on the sphere*, Appl. Comput. Harmon. Anal. **19** (2005) 223–252.
2. P. Gavruţa, *On some identities and inequalities for frames in Hilbert spaces*, J. Math. Anal. Appl. **321** (2006) 469–478.
3. K. Musazadeh and H. Khandani, *Some results on controlled frames in Hilbert spaces*, Acta Math. Sci. **36B** (3) (2016) 655–665.
4. A. Rahimi and A. Fereydooni, *Controlled G-Frames and Their G-Multipliers in Hilbert spaces*, An. Şt. Univ. Ovidius Constanţa **21** (2) (2013) 223–236.
5. W. Sun, *G-frames and g-Riesz bases*, J. Math. Anal. Appl. **322** (1) (2006) 437–452.

E-mail: kamran2007mg2@gmail.com

# On $\varphi$-Connes Amenability of Dual Banach Algebras and $\varphi$-Splitting

Ebrahim Tamimi*
Department of Mathematics, University of Semnan, Semnan, Iran

Abstract. Let $\varphi$ and $\psi$ be $\omega^*$-continuous homomorphisms from dual Banach algebras to $\mathbb{C}$. We present a characterization of $\varphi$-Connes amenability of a dual Banach algebra $\mathcal{A}$ with predual $\mathcal{A}_*$ in terms of so-called $\varphi$-splitting of the short exact sequences. Also, we investigate the relation between $\varphi$-splitting of the certain short exact sequence and $\varphi$-$\sigma wc$ virtual diagonal of a Banach algebra. The relation between $\varphi$-splitting and $\psi$-splitting with $\varphi\otimes\psi$-splitting of the certain short exact sequence is obtained. Other results in this direction are also obtained.
**Keywords:** $\varphi$-$\sigma wc$ Virtual diagonal, $\varphi$-Connes amenability, $\varphi$-Splitting, Dual Banach algebra.
**AMS Mathematical Subject Classification [2010]:** 46J10, 43A22, 16D40.

## 1. Introduction

Connes amenability of certain Banach algebras in terms of normal virtual diagonals is characterized by Effros in [2]. Ghaffari and Javadi in [3], investigated $\varphi$-Connes amenability for dual Banach algebras and semigroup algebras, where $\varphi$ was an homomorphism from a Banach algebra on $\mathbb{C}$. In [5], Runde proved that the measure algebra $M(G)$ for a locally compact group $G$ is Connes amenable if and only if it has a normal virtual diagonal if and only if $G$ is amenable. Also in [4], Ghaffari et al. investigated $\phi$-Connes module amenability of dual Banach algebras that $\phi$ is a $\omega^*$-continuous bounded module homomorphism from a Banach algebra on itself.

In [1, Proposition 4. 4], Daws proved that a Banach algebra is Connes amenable if and only if the short exact sequence splits.

What is the relation between $\varphi$-splitting and $\varphi$-Connes amenability, where $\varphi$ is $\omega^*$-continuous homomorphism from Banach algebra onto $\mathbb{C}$?

Motivated by above question and [6], to study $\varphi$-Connes amenability and $\varphi$-splitting. In fact, we obtain a characterization for $\varphi$-Connes amenability of a dual Banach algebra $\mathcal{A} = (\mathcal{A}_*)^*$ in terms of so-called $\varphi$-splitting of the short exact sequences. We investigate the relation between $\varphi$-splitting and $\varphi$-$\sigma wc$ virtual diagonals of Banach algebras in Theorem 2.7. Also, the relation between two short exact sequences $\Sigma_\varphi$ and $\Sigma_\psi$ with $\Sigma_{\varphi\otimes\psi}$ that are $\varphi, \psi$ and $\varphi\otimes\psi$-splitting, respectively is investigated in Theorem 2.8. The equivalence relation between $\varphi\otimes\psi$-$\sigma wc$ virtual diagonals and $\varphi\otimes\psi$-Connes amenability of projection tensor product of Banach algebras is obtained in Corollary 2.9. The biflat of a Banach algebra under some natural conditions, is investigated in Lemma 2.10. For a

*Presenter

certain Banach algebra, we fund the relationship between the identity of kernel of $\varphi$ and the $\varphi$-splitting of the short exact sequence Theorem 2.11. In finally, we obtain a condition for dual Banach algebra under which, the short exact sequence $\varphi$-splits Corollary 2.13.

We recall that for Banach algebra $\mathcal{A}$, the projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule in the canonical way. Now, we define the map $\mathcal{A}$-bimodule homomorphism $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}$ by $\pi(a \otimes b) = ab$. A Banach $\mathcal{A}$-bimodule $E$ is dual if there is a closed submodule $E_* \subseteq E^*$, predual of $E$, such that $E = (E_*)^*$. A dual Banach $\mathcal{A}$-bimodule $E$ is normal if the module actions of $\mathcal{A}$ on E are $\omega^*$-continuous. A Banach algebra is dual if it is dual as a Banach $\mathcal{A}$-bimodule. We write $\mathcal{A} = (\mathcal{A}_*)^*$ if we wish to stress that $\mathcal{A}$ is a dual Banach algebra with predual $\mathcal{A}_*$.

Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra and let $E$ be a Banach $\mathcal{A}$-bimodule. Then $\sigma wc(E)$, a closed submodule of $E$, stands for the set of all elements $x \in E$ such that the following maps are $\omega^*$-$\omega$ continuous

$$\mathcal{A} \longrightarrow E, \quad a \longmapsto a.x, \quad a \longmapsto x.a.$$

The Banach $\mathcal{A}$-bimodules $E$ that are relevant to us are those the left action is of the form $a.x = \varphi(a)x$. For the brevity's sake, such $E$ will occasionally be called a Banach $\varphi$-bimodule.

Throughout the paper, $\Delta(\mathcal{A})$ and $\Delta_{\omega^*}(\mathcal{A})$ will denote the sets of all homomorphisms and $\omega^*$-continuous homomorphisms from the Banach algebra $\mathcal{A}$ onto $\mathbb{C}$, respectively.

## 2. Main Results

Let $\mathcal{A}$ be a Banach algebra, and let $E$ be a Banach A-bimodule. A derivation from $\mathcal{A}$ to $E$ is a bounded, linear map $D : \mathcal{A} \to E$ satisfying $D(ab) = a.D(b) + D(a).b$ $(a, b \in \mathcal{A})$. A derivation $D : \mathcal{A} \to E$ is called inner if there is $x \in E$ such that $Da = a.x - x.a$ $(a \in \mathcal{A})$.

Definition 2.1. Let $\mathcal{A}$ be a dual Banach algebra and $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$. $\mathcal{A}$ is $\varphi$-Connes amenable if for every normal $\varphi$-bimodule $E$, every bounded $\omega^*$-continuous derivation $D : \mathcal{A} \to E$ is inner.

Definition 2.2. Let $\mathcal{A}$ be a Banach algebra, and let $3 \leq n \in \mathbb{N}$. A sequence

$$\mathcal{A}_1 \overset{\varphi_1}{\to} \mathcal{A}_2 \overset{\varphi_2}{\to} \cdots \overset{\varphi_{n-1}}{\longrightarrow} \mathcal{A}_n,$$

of $\mathcal{A}$-bimodules $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ and $\mathcal{A}$-bimodule homomorphisms $\varphi_i : \mathcal{A}_i \to \mathcal{A}_{i+1}$ for $i \in \{2, \ldots, n-1\}$ is called exact at position $i = 2, \ldots, n-1$ if $\varphi_{i-1} = ker\varphi_i$. It is called exact if it is exact at every position $i \in \{2, \ldots, n-1\}$.

We restrict ourselves to exact sequences with few bimodules, and a few bimodules (short exact sequences) respectively. Therefore, an exact sequence of the following form

$$0 \to \mathcal{A}_1 \overset{\varphi}{\to} \mathcal{A}_2 \overset{\psi}{\to} \mathcal{A}_3 \to 0,$$

is called a short exact sequence.

In the following we define the admissible and splitting short exact sequences.

DEFINITION 2.3. Let $\mathcal{A}$ be a Banach algebra. A short exact sequence

$$\Theta : 0 \to \mathcal{A}_1 \xrightarrow{\varphi_1} \mathcal{A}_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{n-1}} \mathcal{A}_n \to 0,$$

of Banach $\mathcal{A}$-bimodules $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ and $\mathcal{A}$-bimodule homomorphisms $\varphi_i : \mathcal{A}_i \to \mathcal{A}_{i+1}$ for $i = 1, 2, \dots, n-1$ is admissible, if there exists a bounded linear maps $\rho : \mathcal{A}_{i+1} \to \mathcal{A}_i$ such that $\rho o \varphi_i$ on $\mathcal{A}_i$ for $i = 1, 2, \dots, n-1$ is the identity map on $\mathcal{A}_{i+1}$. Further, $\Theta$ splits if we may choose $\rho$ to be an $\mathcal{A}$-bimodule homomorphism.

Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a unital dual Banach algebra by unit of $e$. Then the short exact sequence

$$\sum : 0 \longrightarrow \mathcal{A}_* \xrightarrow{\pi^*} \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*) \longrightarrow \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)/\pi^*(\mathcal{A}_*) \longrightarrow 0,$$

of A-bimodules is admissible (indeed, the map $\rho : \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*) \to (\mathcal{A}_*)$ defined by $\rho(T) = T(e)$ is a bounded left inverse to $\pi^*|_{\mathcal{A}_*}$ ). We restrict ourselves to the case where $\varphi \in \Delta_{\omega^*}(\mathcal{A}) \cap \mathcal{A}_*$. Because we are interested in the splitting of the short exact sequence $\sum$. Then our result would be comparable to the Daws's theorem; $\mathcal{A}$ is Connes-amenable if and only if $\sum$ splits [1, Proposition 4.4].

DEFINITION 2.4. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a unital dual Banach algebra, and let $\varphi \in \Delta_{\omega^*}(\mathcal{A}) \cap \mathcal{A}_*$ We say that $\sum$ $\varphi$-splits if there exists a bounded linear map $\rho : \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*) \to \mathcal{A}_*$ such that $\rho o \pi^*(\varphi) = \varphi$ and $\rho(T.a) = \varphi(a)\rho(T)$, for all $a \in \mathcal{A}$ and $T \in \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)$.

In this note, if we denote the bounded linear operators from $\mathcal{A}^*$ to $(\mathcal{A}\widehat{\otimes}\mathcal{A})^*$ by $\mathcal{L}(\mathcal{A}^*, (\mathcal{A}\widehat{\otimes}\mathcal{A})^*)$, then $\mathcal{H}_{\mathcal{A}}(\mathcal{A}^*, (\mathcal{A}\widehat{\otimes}\mathcal{A})^*)$, denotes the $\mathcal{A}$-bimodule homomorphisms in $\mathcal{L}(\mathcal{A}^*, (\mathcal{A}\widehat{\otimes}\mathcal{A})^*)$.

DEFINITION 2.5. Let $\mathcal{A}$ be a dual Banach algebra, and let $\varphi \in \Delta_{\omega^*}(\mathcal{A}) \cap \mathcal{A}_*$. An element $M \in \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^*$ is a $\varphi$-$\sigma wc$ virtual diagonal for $\mathcal{A}$ if

i) $a.M = \varphi(a)M, \quad (a \in \mathcal{A});$
ii) $\langle \varphi \otimes \varphi, M \rangle = 1.$

In throughout this paper, let $\otimes_\omega$ stand for the injective tensor product of Banach algebras.
We consider the following short exact sequences, which have three non-zero terms:

$$\sum_{\varphi} : 0 \to \mathcal{A}_* \xrightarrow{\pi^*_{\mathcal{A}}} \sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^* \to \sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^*/\pi^*_{\mathcal{A}}(\mathcal{A}_*) \to 0,$$

$$\sum_{\psi} : 0 \to \mathcal{B}_* \xrightarrow{\pi^*_{\mathcal{B}}} \sigma wc(\mathcal{B}\widehat{\otimes}\mathcal{B})^* \to \sigma wc(\mathcal{B}\widehat{\otimes}\mathcal{B})^*/\pi^*_{\mathcal{B}}(\mathcal{B}_*) \to 0,$$

and

$$\sum_{\varphi\otimes\psi} : 0 \to \mathcal{A}_* \otimes_\omega \mathcal{B}_* \xrightarrow{\pi^*_{\mathcal{A}\widehat{\otimes}\mathcal{B}}} \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{B})\widehat{\otimes}(\mathcal{A}\widehat{\otimes}\mathcal{B}))^* \to \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{B})\widehat{\otimes}(\mathcal{A}\widehat{\otimes}\mathcal{B}))^*/\pi^*_{\mathcal{A}\widehat{\otimes}\mathcal{B}}(\mathcal{A}_*\otimes_\omega\mathcal{B}_*) \to 0.$$

DEFINITION 2.6. Let $\mathcal{A}$ be a Banach algebra and $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$ is the projection induced product map. Then $\mathcal{A}$ is biflat if $\pi^* : \mathcal{A}^* \to (\mathcal{A}\widehat{\otimes}\mathcal{A})^*$ has a bounded left inverse which is an $\mathcal{A}$-bimodule homomorphism.

In following we extend Daws's theorem under certain condition on a Banach algebra [1].

Theorem 2.7. *Let* $\mathcal{A} = (\mathcal{A}_*)^*$ *be a unital biflat dual Banach algebra, and let* $\varphi \in \Delta_{\omega^*}(\mathcal{A}) \bigcap \mathcal{A}_*$. *Then the following are equivalent:*

i) *The short exact sequence* $\sum_{\varphi}$ $\varphi$*-splits.*
ii) *There is a* $\varphi$*-*$\sigma wc$ *virtual diagonal for* $\mathcal{A}$.

Theorem 2.8. *Suppose that* $\mathcal{A} = (\mathcal{A}_*)^*$, $\mathcal{B} = (\mathcal{B}_*)^*$ *and* $\mathcal{A}\widehat{\otimes}\mathcal{B} = (\mathcal{A}_*\otimes_\omega\mathcal{B}_*)^*$ *be unital dual Banach algebras, and let* $\varphi \in \Delta_{\omega^*}(\mathcal{A}) \cap \mathcal{A}_*$ *and* $\psi \in \Delta_{\omega^*}(\mathcal{B}) \cap \mathcal{B}_*$. *Then the short exact sequence* $\sum_{\varphi\otimes\psi}$ $\varphi\otimes\psi$*-splits if and only if the short exact sequence* $\sum_{\varphi}$ $\varphi$*-splits and the short exact sequence* $\sum_{\psi}$ $\psi$*-splits.*

Corollary 2.9. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be dual Banach algebras, let* $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$ *and* $\psi \in \Delta(\mathcal{B}) \cap \mathcal{B}_*$. *Then the following are equivalent:*

i) $\mathcal{A}\widehat{\otimes}\mathcal{B}$ *is* $\varphi\otimes\psi$*-Connes amenable.*
ii) *There is a* $\varphi\otimes\psi$*-*$\sigma wc$ *virtual diagonal for* $\mathcal{A}\widehat{\otimes}\mathcal{B}$.
iii) *The short exact sequence* $\sum_{\varphi\otimes\psi}$ $\varphi\otimes\psi$*-splits.*

Lemma 2.10. *A Banach algebra* $\mathcal{A}$ *is biflat if and only if there is an* $\mathcal{A}$*-bimodule homomorphism* $\rho : \mathcal{A} \to (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ *such that* $\pi^{**}o\rho$ *is the canonical embedding of* $\mathcal{A}$ *into* $\mathcal{A}^{**}$.

Let $\mathcal{A}$ be a Banach algebra and $(e_\alpha)$ be a bounded approximate identity for $\mathcal{A}$. Let $\mathcal{A}$ is bifiat, thus $\pi^*_{\mathcal{A}}$ has a left inverse, say $\rho \in \mathcal{H}_{\mathcal{A}}((\mathcal{A}\widehat{\otimes}\mathcal{A})^*, \mathcal{A}^*)$. We may suppose that $\rho^*(e_\alpha)$ converges in the $\omega^*$-topology on $(\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$, say to M. Hence for each $a \in \mathcal{A}$ and $\Lambda \in (\mathcal{A}\widehat{\otimes}\mathcal{A})^*$, we have

$$\begin{aligned}\langle a.M, \Lambda\rangle = \langle M, \Lambda.a\rangle &= \lim_\alpha \langle \rho^*(e_\alpha), \Lambda.a\rangle = \lim_\alpha \langle e_\alpha, \rho(\Lambda.a)\rangle \\ &= \lim_\alpha \langle a.e_\alpha, \rho(\Lambda)\rangle = \langle a, \rho(\Lambda)\rangle.\end{aligned}$$

Similarly as, we obtain $\langle M.a, \Lambda\rangle = \langle a, \rho(\Lambda)\rangle$.

Theorem 2.11. *Let* $\mathcal{A} = (\mathcal{A}_*)^*$ *be a unital dual Banach algebra, and let* $\varphi \in \Delta_{\omega^*}(\mathcal{A}) \bigcap \mathcal{A}_*$. *Then the short exact sequence* $\sum_{\varphi}$ $\varphi$*-splits if and only if* $ker\varphi$ *has a left identity.*

Definition 2.12. Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$ and $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$. A linear functional $m \in \mathcal{A}^{**}$ is called a mean if $m(\varphi) = 1$. A $m$ is $\varphi$-invariant mean if $m(a.f) = \varphi(a)m(f)$ for all $a \in \mathcal{A}$ and $f \in \mathcal{A}_*$.

Corollary 2.13. *Let* $\mathcal{A} = (\mathcal{A}_*)^*$ *be a dual Banach algebra, and let* $\varphi \in \Delta_{\omega^*}(\mathcal{A}) \bigcap \mathcal{A}_*$. *If* $\mathcal{A}^{**}$ *has a* $\varphi$*-invariant mean, then the short exact sequence* $\sum_{\varphi}$ $\varphi$*-splits.*

## References

1. M. Daws, *Connes-amenability of bidual and weighted semigroup algebras*, Math. Scand. **99** (2) (2006) 217–246.
2. E. G. Effros, *Amenability and virtual diagonals for von Neumann algebras*, J. Funct. Anal. **78** (1) (1988) 137–153.

3. A. Ghaffari and S. Javadi, *$\phi$-Connes amenability of dual Banach algebras*, Bull. Iranian Math. Soc. **43** (1) (2017) 25–39.
4. A. Ghaffari, S. Javadi and E. Tammi, *$\varphi$-Connes module amenability of dual Banach algebras*, J. Algebr. Syst. **8** (1) (2020) 69–82 (in Persian).
5. V. Runde, *Connes amenability and normal, virtual diagonals for measure algebras, II*, Bull. Austral. Math. Soc. **68** (2) (2003) 325–328.
6. V. Runde, *Lectures on Amenability*, Lecture Notes in Mathematics 1774, Springer Verlag, Berlin, 2002.

E-mail: tamimi_ebrahim@semnan.ac.ir

# Contributed Posters

## Geometry and Topology

# An Expansion for the Prime Counting Function

Mehdi Hassani*
Department of Mathematics, University of Zanjan, Zanjan, Iran

Abstract. In this article, we obtain a connection between the function $\omega(n) = \sum_{\substack{p|n \\ p \text{ is prime}}} 1$ and the prime counting function $\pi(x)$. This connection implies an elementary formula for $\pi(x)$ in terms of the Möbius function $\mu(n)$. Also, we obtain a conditional asymptotic expansion for the fractional part sum $\sum_{p\leqslant x}\{\frac{x}{p}\}$.
**Keywords:** Arithmetic function, Prime number.
**AMS Mathematical Subject Classification [2010]:** 11A25, 11A41.

## 1. Introduction and Summary of the Results

Let $\omega(n)$ be the number of distinct prime divisors of the positive integer $n$. Also, let $\pi(x)$ be the number of primes not exceeding $x$. Hardy and Ramanujan [3] proved the following average relation

$$\frac{1}{x}\sum_{n\leqslant x}\omega(n) = \log\log x + M + R(x),$$

where $R(x) = O(\frac{1}{\log x})$ and

$$M = \gamma + \sum_{p}\left(\log\left(1 - p^{-1}\right) + p^{-1}\right),$$

known as the Meissel–Mertens constant [2]. The later sum runs over all primes, and $\gamma$ denotes the Euler constant. In this article, we obtain a connection between the sum $\sum_{n\leqslant x}\omega(n)$ and the prime counting function $\pi(x)$. The prime number theorem asserts that

$$\pi(x) \sim \frac{x}{\log x},$$

as $x \to \infty$. The study of the function $\pi(x)$ is very closely related to the study of the zeros of the Riemann zeta function which is defined, for $\Re(s) > 1$, by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

and extended by analytic continuation to the complex plan with one simple pole at $s = 1$ with residue 1. The Riemann hypothesis states that non-trivial zeros of the Riemann zeta function all lie on the line $\Re(s) = \frac{1}{2}$.

*Presenter

THEOREM 1.1. *For each real $x \geqslant 2$, we have*

$$\sum_{n \leqslant x} \omega(n) = \sum_{n \leqslant \frac{x}{2}} \pi\Big(\frac{x}{n}\Big). \tag{1}$$

As an immediate consequence, by using the Möbius inversion [6], we obtain the following expansion for the prime counting function $\pi(x)$.

COROLLARY 1.2. *For each real $x \geqslant 2$, we have*

$$\pi(x) = \sum_{n \leqslant \frac{x}{2}} \mu(n) \sum_{k \leqslant \frac{x}{n}} \omega(k). \tag{2}$$

As another application of (1), we obtain a conditional asymptotic expansion for the fractional part sum

$$\mathcal{S}(x) := \sum_{p \leqslant x} \left\{\frac{x}{p}\right\}.$$

Note that $\{x\} = x - [x]$, where $[x]$ denotes the integer part of $x$. The fractional part sum $\mathcal{S}(x)$ has been studied by de la Vallée Poussin [7] who showed, by elementary methods, that

$$\mathcal{S}(x) \sim (1-\gamma)\frac{x}{\log x},$$

as $x \to \infty$. In this note, we study $\mathcal{S}(x)$ under assuming that the Riemann hypothesis is true.

COROLLARY 1.3. *Let $m \geqslant 2$ be a fixed integer. Assume that the Riemann hypothesis is true. Then, as $x \to \infty$, we have*

$$\mathcal{S}(x) = (1-\gamma)\frac{x}{\log x} - x\sum_{j=2}^{m} \frac{a_j}{\log^j x} + O\Big(\frac{x}{\log^{m+1} x}\Big), \tag{3}$$

*where the coefficients $a_j$ are computable constants given by the following improper convergent integral.*

$$a_j = -\int_1^\infty \frac{\{t\}}{t^2}(\log t)^{j-1}\mathrm{d}t. \tag{4}$$

**Notations 1.4.** *We will use $p$ for a prime number, and $m$, $n$, $k$ for integers. Also, we will use $x$ for a real number.*

## 2. Proofs

PROOF OF THEOREM 1.1. We have

$$\sum_{n \leqslant x} \omega(n) = \sum_{n \leqslant x} \sum_{p | n} 1 = \sum_{p \leqslant x} \sum_{\substack{n \leqslant x \\ p | n}} 1 = \sum_{p \leqslant x} \left[\frac{x}{p}\right]. \tag{5}$$

Note that $\frac{x}{n+1} < p \leqslant \frac{x}{n}$ holds if and only if $n \leqslant \frac{x}{p} < n+1$. Hence,

$$\begin{aligned}\sum_{p\leqslant x}\left[\frac{x}{p}\right] &= \sum_{1\leqslant n\leqslant \frac{x}{2}}\ \sum_{\frac{x}{n+1}<p\leqslant \frac{x}{n}}\left[\frac{x}{p}\right]\\ &= \sum_{1\leqslant n\leqslant \frac{x}{2}}\ \sum_{\frac{x}{n+1}<p\leqslant \frac{x}{n}} n\\ &= \sum_{1\leqslant n\leqslant \frac{x}{2}} n\left(\pi\Big(\frac{x}{n}\Big)-\pi\Big(\frac{x}{n+1}\Big)\right)\\ &= \sum_{1\leqslant n\leqslant \frac{x}{2}}\left((n-1)\pi\Big(\frac{x}{n}\Big)-n\pi\Big(\frac{x}{n+1}\Big)\right)+\sum_{1\leqslant n\leqslant \frac{x}{2}}\pi\Big(\frac{x}{n}\Big).\end{aligned}$$

As $\frac{x}{[\frac{x}{2}]+1} < 2$, we get

$$\sum_{1\leqslant n\leqslant \frac{x}{2}}\left((n-1)\pi\Big(\frac{x}{n}\Big)-n\pi\Big(\frac{x}{n+1}\Big)\right) = -\left[\frac{x}{2}\right]\pi\Big(\frac{x}{[\frac{x}{2}]+1}\Big) = 0.$$

Thus, we obtain (1), and this completes the proof of Theorem 1.1. □

Proof of Corollary 1.2. Note that $\frac{x}{2} < n \leqslant x$ is equivalent to $1 \leqslant \frac{x}{n} < 2$. This allows us to rewrite (1) as follows.

$$\sum_{n\leqslant x}\omega(n) = \sum_{n\leqslant x}\pi\Big(\frac{x}{n}\Big).$$

Applying the Möbius inversion, implies

$$\pi(x) = \sum_{n\leqslant x}\mu(n)\sum_{k\leqslant \frac{x}{n}}\omega(k) = \sum_{n\leqslant \frac{x}{2}}\mu(n)\sum_{k\leqslant \frac{x}{n}}\omega(k) + \sum_{\frac{x}{2}<n\leqslant x}\mu(n)\sum_{k\leqslant \frac{x}{n}}\omega(k).$$

As we mentioned, for $n$ with $\frac{x}{2} < n \leqslant x$, we have $1 \leqslant \frac{x}{n} < 2$ and so $\sum_{k\leqslant \frac{x}{n}}\omega(k) = 0$. Hence we get (2). This completes the proof of Corollary 1.2. □

Proof of Corollary 1.3. For a given integer $m \geqslant 1$, Saffari [4] used Dirichlet's hyperbola method to prove a more general result implies that

$$R(x) = \sum_{j=1}^{m}\frac{a_j}{\log^j x} + O\Big(\frac{1}{\log^{m+1} x}\Big), \tag{6}$$

where the coefficients $a_j$ are given by (4). More precisely, by [2], it is known that

$$a_1 = \gamma - 1.$$

Later, Diaconis reproved (6) by applying Perron's formula [6] on the Dirichlet series $\sum_{n=1}^{\infty}\omega(n)n^{-s}$ and complex integration methods (See [1]). Considering (5) and (1), we get

$$\mathcal{S}(x) = x\mathcal{A}(x) - \sum_{n\leqslant x}\omega(n) = x\mathcal{A}(x) - \sum_{n\leqslant \frac{x}{2}}\pi\Big(\frac{x}{n}\Big),$$

where

$$\mathcal{A}(x) := \sum_{p\leqslant x}\frac{1}{p}.$$

By [5, Corollary 2], if the Riemann hypothesis is true, then we have

$$|\mathcal{A}(x) - \log\log x - M| < \frac{3\log x + 4}{8\pi\sqrt{x}}, \tag{7}$$

for each $x \geqslant 13.5$. Now, assume that the Riemann hypothesis is true and apply the conditional bound (7) to obtain

$$\mathcal{S}(x) = -xR(x) + O\left(\sqrt{x}\log x\right).$$

Hence, by using the asymptotic expansion (6), for each fixed $m \geqslant 2$, we deduce the validity of the conditional expansion (3), under the assumption of the Riemann hypothesis, where the coefficients $a_j$ are as in (6). □

**Acknowledgement**

The author wishes to express his thanks to the referee for several helpful comments.

## References

1. P. Diaconis, *Asymptotic expansions for the mean and variance of the number of prime factors of a number n*, Technical Report No. 96, Department of Statistics, Stanford University, California 1976.
2. S. R. Finch, *Mathematical Constants*, Encyclopedia of Mathematics and its Applications, Vol. 94, Cambridge University Press, Cambridge 2003.
3. G. H. Hardy and S. Ramanujan, *The normal number of prime factors of a number n*, Quart. J. Math. **48** (1917) 76–92.
4. B. Saffari, *Sur quelques applications de la "méthode de l'hyperbole" de Dirichlet a la théorie des nombres premiers*, Enseignement Math. **14** (1970) 205–224.
5. L. Schoenfeld, *Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$, II*, Math. Comp. **30** (134) (1976) 337–360.
6. G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, 3rd ed., Amer. Math. Soc. Providence 2015.
7. Ch. de la Vallée Poussin, *Sur les valeurs moyennes de certaines fonctions arithmétiques*, Ann. Soc. Sci. Bruxelles **22** (1898) 84–90.

E-mail: mehdi.hassani@znu.ac.ir

# Some Results on Generalized Harmonic Maps with Potential

Keyvan Salehi*
Central of Theoretical Physic and Chemistry (CTPC), Massey University, Auckland, Newzealand
Seyed Mehdi Kazemi Torbaghan
Faculty of Basic Sciences, Univesity of Bojnord, Bojnord, Iran
and Salman Babayi
Department of mathematics, Faculty of Science, Urmia University, Urmia, Iran

Abstract. In this paper, the second variation formula for exponential harmonic maps with potential is obtained. As an application, instability and nonexistence theorems for exponential harmonic maps with potential are given.
**Keywords:** Exponential harmonic maps, Stability, Riemannian manifolds, Calculus of variations.
**AMS Mathematical Subject Classification [2010]:** 53C43, 58E20.

## 1. Introduction

Eells and Sampson introduced harmonic maps between Riemannian manifolds in 1964. They proved that any smooth map from a compact Riemannian manifold into a Riemannian manifold with non-positive sectional curvature can be deformed into a harmonic map. This theorem is known as fundamental existence thorem for harmonic maps. This topic is extensively applied in physics and engineering, [5].

Harmonic maps with potential, was first introduced by Ratto in [6] and lately developed by many scholars : V. Branding [1], Y. Chu [2], A. Fardoun and all [4] and other. Let $(M, g)$ and $(N, h)$ be Riemannian manifolds, $\phi : (M, g) \longrightarrow (N, h)$ be a smooth map and $H \in C^\infty(N)$ The $H-$energy function of $\phi$ is denoted by $E_H(\phi)$ and defined by

$$E_H(\phi) = \int_M [e(\phi) - H(\phi)]\nu_g,$$

where $e(\phi) := \frac{1}{2} \mid d\phi \mid^2$. The critical points of $E_H$ is called is called harmonic maps with potential $H$.

The notion of harmonic maps to exponential harmonic maps is extended by Eells and Lemaire, [3]. They studied the stability of these maps under the curvature conditions on the target manifold. They defined the exponential energy functional of $\phi : (M, g) \longrightarrow (N, h)$ as follows:

$$E_e(\phi) = \int_M \exp(\frac{\mid d\phi \mid^2}{2})\nu_g.$$

*Presenter

The critical points of exponential energy functional is said to be exponential harmonic maps. In terms of the Euler-Lagrange equation, $\phi$ is exponential harmonic if $\phi$ satisfies the following equation

$$\tau_e(\phi) = \tau(\phi) + d\phi(grad\ \exp(e(\phi))) = 0.$$

The section $\tau_e(\phi) \in \Gamma(\phi^{-1}TN)$ is called exponential tension field of $\phi$, [3].

In this paper, first, we derive the first and second variation formulas for exponential harmonic maps with potential. Then, the stability of exponential harmonic maps with potential from a compact Riemannian manifold to the unit sphere equipped with induced metric is studied.

## 2. Main Results

Now, we compute the first and second variation formulas of exponential energy functional with potential $H$. Then instability and nonexistence theorems for exponential harmonic maps with potential are given.

Suppose that $\phi : M \longrightarrow N$ is a $C^3$ map. Throughout this paper, we denote the Levi-Civita connection of $M, N$ and $\phi^{-1}TN$ by ${}^M\nabla, {}^N\nabla$ and $\hat{\nabla}$. Noting that the induced connection $\hat{\nabla}$ on $\phi^{-1}TN$ defined by $\hat{\nabla}_Y Z = {}^N\nabla_{d\phi(Y)}Z$, where $Y \in \chi(M)$ and $Z \in \Gamma(\phi^{-1}TN)$.

Definition 2.1. Let $\phi : (M, g) \longrightarrow (N, h)$ and $H \in C^\infty(N)$. The exponential energy functional of $\phi$ with potential $H$is denoted by $E_{e,H}(\phi)$ and defined by

$$E_{e,H}(\phi) = \int_M [e(\phi) - H(\phi)]\nu_g,$$

The critical points of $E_{e,H}$ is called exponential harmonic with potential $H$.

By choosing a local orthonormal frame field $\{e_i\}$ on $M$, The exponential tension field of $\phi$ with potential $H$, $\tau_{e,H}(\phi)$, is defined by

$$\tau_{e,H}(\phi) = \exp(\frac{\mid d\phi \mid^2}{2})\tau(\phi) + d\phi(grad\ \exp(\frac{\mid d\phi \mid^2}{2})) + {}^N\nabla H \circ \phi,$$

here $\tau(\phi) = \sum_{i=1}^m \{\hat{\nabla}_{e_i} d\phi(e_i) - d\phi({}^M\nabla_{e_i} e_i)\}$.

According to the above notations we get

Lemma 2.2. *Let $\phi : (M, g) \longrightarrow (N, h)$ be a smooth map. Then*

$$\frac{d}{dt} E_{e,H}(\phi_t) \mid_{t=0} = -\int_M h(\tau_{e,H}(\phi), V)\nu_g,$$

*where $V = \frac{d\phi_t}{dt} \mid_{t=0}$ .*

Definition 2.3. A map $\phi$ is called exponential harmonic with potential $H$ if $\tau_{e,H}(\phi) = 0$.

Definition 2.4. Let $\phi : (M, g) \longrightarrow (N, h)$ be an exponential harmonic map with potential $H$, and let $\phi_t : M \longrightarrow N$ $(-\epsilon < t < \epsilon)$ be a smooth variation of $\phi$ and $V = \frac{\partial \phi_t}{\partial t} \mid_{t=0}$. Setting

$$I(V) = \frac{d^2}{dt^2} E_{e,H}(\phi_t) \mid_{t=0} .$$

The map $\phi$ is called stable if $I(V) \geq 0$ for any compactly supported vector field $V$ along $\phi$.

Let $W$ and $Z$ be vector fields on $M$ such that

$$g(W,X) = \exp(\frac{|\ d\phi\ |^2}{2})<\hat{\nabla}V, d\phi>.h(d\phi(X), V),$$

$$g(Z,X) = \exp(\frac{|\ d\phi\ |^2}{2})h(\hat{\nabla}_X V, V),$$

for any vector fields $X$ on $M$, respectively. By (1) and considering the divergence of $W$ and $Z$, and Green's Theorem, $I(V)$ can be obtained as follows.

THEOREM 2.5. *Let $\phi : (M,g) \longrightarrow (N,h)$ be an exponential harmonic map with potential H, and let $\phi_t : M \longrightarrow N$ $(-\epsilon < t < \epsilon)$ be a compactly supported variation such that $\phi_0 = \phi$. Then*

$$\begin{aligned} I(V) = &\int_M \exp(\frac{|\ d\phi\ |^2}{2})\langle\hat{\nabla}V, d\phi\rangle^2 dv_g \\ &+ \int_M \exp(\frac{|\ d\phi\ |^2}{2})\Big\{\langle|\ \hat{\nabla}V\ |^2 - h(trace_g\, {}^N R(V, d\phi)d\phi \\ &- (\nabla^N_V grad^N H)\circ\phi, V)\Big\}\nu_g, \end{aligned}$$

*where $V = \frac{\partial\phi_t}{\partial t}\ |_{t=0}$.*

THEOREM 2.6. *Suppose that $\phi : (\mathbb{S}^n, g) \longrightarrow (N,h)$ is a stable exponential harmonic map with potential $H$ from $\mathbb{S}^n$ $(n > 2)$ to a Riemannian manifold $(N,h)$, and let $trace_g h(\nabla d\phi(\cdot, grad^{\mathbb{S}^n} \exp(e(\phi))), d\phi(\cdot)) \neq 0$. Then $\phi$ is constant.*

## References


1. V. Branding, *The heat flow for the full bosonic string*, Ann. Global Anal. Geom. **50** (2016) 347–365.
2. Y. Chu and X. Liu, *Regularity of harmonic maps with the potential*, Sci. China Ser. A. **49** (2006) 599–610.
3. J. Eells and L. Lemaire, *Some properties of exponentially harmonic maps*, in: Partial Differential Equations, Part 1, 2 (Warsaw, 1990), 129136, Banach Center Pulb. 27, Polish Acad. Sci, Warsaw (1992)
4. A. Fardoun, A. Ratto and R. Regbaoui, *On the heat flow for harmonic maps with potential*, Ann. Global Anal. Geom. **18** (2000) 555–567.
5. Z. Pirbodaghi, M. Mirmohammad Rezaii and S. M. Kazemi Torbaghan, *Stability of λ-Harmonic Maps*, J. Math. **6** (2018) 94. DOI:0.3390/math6060094.
6. A. Ratto, *Harmonic maps with potential*, Proceedings of the Workshop on Differential Geometry and Topology (Palermo, 1996). Rend. Circ. Mat. Palermo (2) Suppl. **49** (1997) 229–242.



E-mail: m.kazemi@ub.ac.ir

E-mail: K.salehi@massey.ac.nz

E-mail: S.babayi@urmia.ac.ir

# A New Generalization of Orbifolds Using of Generalized Groups

Hassan Maleki*
Faculty of Mathematical Sciences, Malayer University, Malayer, Iran
and Mohammad Reza Molaei
Department of Pure Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran

Abstract. Our ultimate goal in this paper is to introduce a special type of topological spaces including manifolds and also, orbifolds. Because of using of generalized groups, we call them *GG-spaces*. We will study their properties, and then we will introduce a special *GG*-space that is not manifold and orbifold. Finally we obtain conditions that cause a *GG*-space to become manifold.
**Keywords:** Generalized group,T-Space, Quotient space, Orbifold.
**AMS Mathematical Subject Classification [2010]:** 22A20, 22A99, 16W22.

## 1. Introduction

One of the interesting problems in geometry is to extend our definitions in order to add more objects to a certain category. We know geometric objects like torus and spheres are manifolds, but cones aren't. Extending the notion of manifolds one can define a new structure called orbifold to include cones and some other objects as well. Intuitively, a manifold is a topological space locally modeled on Euclidean space $\mathbb{R}^n$. Manifolds have origins in Carl Friedrich Gauss's works and Bernhard Riemann's lecture in Gottingen in 1854 laid the foundations of higher-dimensional differential geometry. As an extension of manifolds, an orbifold is a topological space locally modeled on a quotient of $\mathbb{R}^n$ by the action of a finite group. The simplest examples of orbifolds are cones, lens spaces and $\mathbb{Z}_p$-teardrops. Orbifolds lie at the intersection of many different areas of mathematics, including algebraic and differential geometry, topology, algebra and string theory [10]. *GG-spaces* are a fascinating extension of orbifolds and manifolds. We can be roughly described a *GG*-space as a topological space that is locally modeled on a quotient of $\mathbb{R}^n$ by the *generalized action* of a *topological generalized group*. *GG*-spaces will yield a geometrical and algebraic device useful for showing the existence of structures that are not a manifold or an orbifold such as Example 3.5.

Let us recall the definition of orbifolds. They were first introduced into topology and differential geometry by Satake [9], who called them *V-manifolds*. Satake described them as topological spaces generalizing smooth manifolds and generalized concepts such as de Rham cohomology and the Gauss-Bonnet theorem to orbifolds. The late 1970s, orbifolds were used by Thurston in his work on three-manifolds [10]. The name *V-manifold* was replaced by the word *orbifold* by

*Presenter

Thurston. An orbifold $\mathbb{O}$, consists of a paracompact, Hausdorff topological space $\mathbb{X}_{\mathbb{O}}$ called the *underlying space*, such that for each $x \in \mathbb{X}_{\mathbb{O}}$ and neighborhood $U$ of $x$, there exists a neighborhood $U_x \subseteq U$, an open set $\tilde{U}_x \cong \mathbb{R}^n$, a finite group $G_x$ acting continuously and effectively on $\tilde{U}_x$ which fixes $0 \in \tilde{U}_x$, and a homeomorphism $\phi_x : \tilde{U}_x \big/ G_x \to U_x$ with $\phi_x(0) = x$ [2].

## 2. Preliminaries

Generalized groups or completely simple semi-groups [1] are an extension of groups. This notion has been studied first in 1999 [4, 5, 7]. Topological generalized groups have been applied in geometry, dynamical systems and also genetic [6]. The notion of generalized action [4] is an extension of the notion of group actions. Furthermore, the notion of $T$-spaces have been introduced and studies as an extentionof the notion of $G$-spaces using of topological generalized groups [3]. We refer to [3, 7, 8] for more details. We start by recalling the notions of topological generalized groups and their generalized action on a topological space.

Definition 2.1. [5] A *topological generalized group* is a Hausdorff topological space $T$ which is endowed with a semigroup structure such that the following conditions hold:

- For each $t \in T$, there is a unique $e(t) \in T$ such that $t \cdot e(t) = e(t) \cdot t = t$.
- For each $t \in T$, there is $s \in T$ such that $s \cdot t = t \cdot s = e(t)$.
- For each $s, t \in T$, $e(s \cdot t) = e(s) \cdot e(t)$.
- The generalized group operations $m_1 : T \to T$ defined by $m_1(t) = t^{-1}$ and $m_2 : T \times T \to T$ defined by $m_2((s,t)) = s \cdot t$ are continuous maps, where $t^{-1} \in T$ with $t \cdot t^{-1} = t^{-1} \cdot t = e(t)$.

Example 2.2. Let $T$ be the topological space $\mathbb{R} \setminus \{0\}$. We can see that $T$ with the multiplication $x \cdot y = x|y|$ is a topological generalized group. The identity set $e(T)$ is $\{-1, 1\}$.

Example 2.3. If $T$ is the topological space

$$\mathbb{R}^2 - \{(0,0)\} = \{re^{i\theta} | \quad r > 0 \quad \text{and} \quad 0 \leqslant \theta < 2\pi\},$$

with the Euclidean metric, then $T$ with the multiplication

$$(r_1 e^{i\theta_1}) \cdot (r_2 e^{i\theta_2}) = r_1 r_2 e^{i\theta_2},$$

is a topological generalized group. We have $e(re^{i\theta}) = e^{i\theta}$ and $(re^{i\theta})^{-1} = \frac{1}{r} e^{i\theta}$. So we can see the identity set $e(T)$ is the unit circle $S^1$. However, $T$ is not a topological group.

Definition 2.4. Let $X$ be a topological space and let $T$ be a topological generalized group. A *generalized action* of $T$ on $X$ is a continuous map $\lambda : T \times X \longrightarrow X$ such that the following conditions hold:

- $\lambda(s, \lambda(t, x)) = \lambda(s \cdot t, x)$, for $s, t \in T$ and $x \in X$;
- If $x \in X$, then is $e(t) \in T$ such that $\lambda(e(t), x) = x$.

## 3. Main Results

DEFINITION 3.1. For each $x \in T$, we define
$$T_x = \{t \in T| \quad tx = x\},$$
called the *stabilizer* of $x$ in $T$. A generalized action $\lambda$ of $T$ on $X$ is called *perfect* if $e(T) \subseteq T_x$ for each $x \in X$. Moreover, $\lambda$ is called *super perfect* if for each $x \in X$, $e(T) = T_x$.

Now we are ready to define *GG*-spaces. A *GG*-space is a topological space that is locally homeomorphic to a quotient of $\mathbb{R}^n$ by the generalized action of a topological generalized group. First, we need to define charts.

DEFINITION 3.2. Let $X$ be a topological space. Then a *chart* for $X$ is a $(U, \widetilde{U}, \varphi, T)$ where $U$ is an open subset of $X$, $\widetilde{U}$ is an open subset of $\mathbb{R}^n$, $T$ is a topological generalized group that acts continuously on $\widetilde{U}$ by a generalized action $\lambda$ and $\varphi : \widetilde{U} \longrightarrow U$ is a continuous map inducing a homeomorphism between $\widetilde{U}/T$ and $U$.

DEFINITION 3.3. The collection $\{(U_i, \widetilde{U}_i, \varphi_i, T_i) : i \in I\}$ of charts of $X$ is said to be an *atlas* for $X$ if the following properties are satisfied:

- $\{U_i : i \in I\}$ is a cover of $X$ that closed under finite intersection;
- whenever $U_i \subset U_j$, there is an injective generalized group homomorphism
$$f_{ij} : T_i \hookrightarrow T_j,$$
and an embedding
$$\psi_{ij} : \widetilde{U}_i \hookrightarrow \widetilde{U}_j,$$
such that for $t \in T_i$,
$$\psi_{ij}(tx) = f_{ij}(t)\psi_{ij}(x),$$
and also
$$\varphi_j \circ \psi_{ij} = \varphi_i.$$

DEFINITION 3.4. An *GG-space* is a pair $(X, \mathcal{A})$ where $X$ is a topological space and $\mathcal{A}$ is an atlas for $X$.

In the following example, the distinction between the geometrical structure of GG-spaces and classical geometrical structures such as Manifolds and orbifolds is well illustrated. In the Manifold theory, no center is considered for the unit circle, but in the concept of GG-spaces we are able to consider the unit circle with its center as a connected geometric structure.

EXAMPLE 3.5. Let $Y = \mathbb{R}^2$ and $T$ be the generalized group of Example 2.3 which acts on $Y$ by
$$(r_1 e^{i\theta_1}).(r_2 e^{i\theta_2}) = r_1 r_2 e^{i\theta_2}.$$
We can see that $T_x = e(T)$, for each $x \in Y$, so the action of $T$ is super perfect. For $x = r_1 e^{i\theta_1}$ and $y = r_2 e^{i\theta_2}$, $[x] = [y]$ if and only if $\theta_1 = \theta_2$. Now suppose $X := Y/T$. We can see that $(X, Y, \pi, T)$ is a chart for $X$ where $\pi : Y \to X$ is the projection map. Moreover, $X$ is homeomorphic to $S^1 \bigcup \{(0, 0)\}$ (See Figure 1). Note that $X$ is a connected space with the quotient topology.

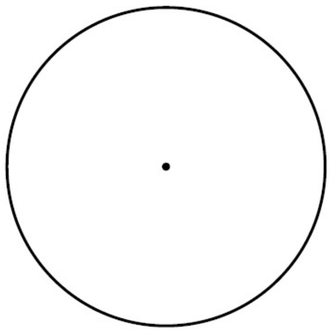


FIGURE 1. The $GG$-space which is not an orbifold.

THEOREM 3.6. *The GG-space $(X, \mathcal{A})$ is an orbifold if every topological generalized group $T_i$ is a finite group. Moreover, $(X, \mathcal{A})$ is a manifold if every topological generalized group $T_i$ is trivial.*

PROOF. Using the definition of an orbifold [10] and a manifold , we can proof this theorem. □

**Note**. There are $GG$-spaces that are not a orbifold (See Example 3.5).

THEOREM 3.7. *For any open connected $T$-space $(X, T, \lambda)$ that $X \subseteq \mathbb{R}^n$, the quotient space $X/_T$ is a GG-space.*

THEOREM 3.8. *Let $(X, \mathcal{A})$ be a GG-space. If every topological generalized group $T_i$ is finite and its generalized action is super perfect, then $X$ is a manifold.*

PROOF. We know that for each $x \in X$ there is a chart $(U, \widetilde{U}, \varphi, T)$ such that $x \in U$ and $\widetilde{U} \subseteq \mathbb{R}^n$ and a continuous map $\varphi : \widetilde{U} \to U$ induces a homeomorphic between $\widetilde{U}/_T$ and $U$. We claim that $\widetilde{U}/_T$ is locally Euclidean ,i.e. $U$ is locally Euclidean and then $X$ is a manifold.

Since the generalized action of $T$ on $\widetilde{U}$ is super perfect, $tz \neq z$ for each $t \notin e(T)$ and for each $z \in \widetilde{U}$. Moreover, $T$ is finite, so we can say that for each $z \in \widetilde{U}$ there is a neighborhood $\widetilde{V} \subseteq \widetilde{U}$ of $z$ such that

$$t\widetilde{V} \bigcap \widetilde{V} = \varnothing, \tag{1}$$

where $t \notin e(T)$.

Now we consider the projection map $\pi : \widetilde{U} \to \widetilde{U}/_T$. We will show that $\pi(\widetilde{V})$ is an open subset of $\widetilde{U}/_T$ that is homeomorphic to the open subset $\widetilde{V}$ of $\mathbb{R}^n$. This implies that $\widetilde{U}/_T$ and also $U$ are locally Euclidean.

We can see that $\pi^{-1}(\pi(\widetilde{V})) = \bigcup t\widetilde{V}$, where $t \in T$. Since the action of $T$ on $\widetilde{U}$ is perfect, so every $\lambda_t : X \to X$ defined by $\lambda_t(x) = tx$, is a homeomorphism and so is an open map. So $t\widetilde{V} = \lambda_t(\widetilde{V})$ is an open subset of $\widetilde{U}$. So $\pi^{-1}(\pi(\widetilde{V}))$ is open in $\widetilde{U}$. According to the quotient topology, $\pi(\widetilde{V})$ is open in $\widetilde{U}/_T$. Moreover, we knew that $\pi_{|_{\widetilde{V}}} : \widetilde{V} \to \pi(\widetilde{V})$ is an open surjective continuous map. Also using

(1), it is injective. So $\pi(\widetilde{V})$ is homeomorphic to $\widetilde{V}$ and $\widetilde{U}/T$ is locally Euclidean. Therefore $U$ is locally Euclidean. □

## References


1. J. Araujo and J. Konieczny, *Molaei's generalized groups are completely simple semigroups*, Buletinul Institului Polithnic Din Iasi **48** (52) (2004) 1–5.
2. J. E. Borzellino and V. Brunsden, *Determination of the topological structure of an orbifold by its group of orbifold diffeomorphisms*, J. Lie Theory **13** (2003) 311–327.
3. H. Maleki and M. R. Molaei, *T-Spaces*, Turk J. Math. **39** (2015) 851–863.
4. M. R. Molaei, *Generalized actions*, Proceedings of the First International Conference on Geometry, Integrability and Quantization, Coral Press Scientific Publishing, September (1999) 175–180.
5. M. R. Molaei, *Topological generalized groups*, Int. J. Pure Appl. Math. **2** (9) (2000) 1055–1060.
6. M. R. Molaei, *Top spaces*, J. Interdiscip. Math. **7** (2) (2004) 173–181.
7. M. R. Molaei, *Mathematical Structures Based on Completely Simple Semigroups*, Hadronic press, Incorporated, 2005.
8. M. R. Molaei, G. S. Khadekar and M. R. Farhangdoost, *On top spaces*, Balkan J. Geom. Appl. **11** (1) (2009) 101–106.
9. I. Satake, *On a generalization of the notion of manifold*, Proc. Nat. Acad. Sci. USA **42** (6) (1956) 359–363.
10. W. Thurston, *The Geometry and Topology of Three-Manifolds*, Princeton University Press, New Jersey, 1997.


E-mail: Hassanmaleki62@gmail.com
E-mail: Mrmolaei@uk.ac.ir

# Notes on Maximal Subrings of Rings of Continuous Functions

Mehdi Parsinia*
Faculty of Mathematical Sciences and Informatics, Shahid Chamran University of Ahvaz, Ahvaz, Iran

ABSTRACT. In this paper, by using the notion of singly generated subrings and subalgebras, and realcompactifications generated by subsets of $C(X)$, we investigate some new observations on maximal subrings of rings of continuous functions form which some new proofs to some results of [3] follow.
**Keywords:** Maximal subing, Intermediate ring, Realcompactification, Singly generated subalgebra.
**AMS Mathematical Subject Classification [2010]:** 54C30, 46E25.

## 1. Introduction

All topological spaces are assumed to be completely regular and Hausdorff throughout this note. As usual, for a given topological space $X$, $C(X)$ denotes the algebra of all real-valued continuous functions on $X$ and $C^*(X)$ denotes the subalgebra of $C(X)$ consisting of all bounded elements. Also, $\beta X$ and $\upsilon X$ denote the Stone-Cech compactification and the Hewitt-realcompactification of $X$, respectively. The reader is referred to [6] for undefined notations and terminologies concerning $C(X)$. It is manifest that every $f \in C(X)$ could be considered as a continuous function from $X$ into the one-point compactification $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$ of the real line $\mathbb{R}$ and thus it has the Stone-extension $f^* : \beta X \to \mathbb{R}^*$, and we have $f^* = f^\beta$ whenever $f$ is bounded. The set of all points in $\beta X$ where $f^*$ takes real values is denoted by $\upsilon_f X$, i.e., $\upsilon_f X = \{p \in \beta X : f^*(p) \neq \infty\}$. For each $A(X) \subseteq C(X)$, we set $\upsilon_A X = \bigcap_{f \in A(X)} \upsilon_f X$; i.e., $\upsilon_A X = \{p \in \beta X : f^*(p) < \infty, \forall f \in A(X)\}$. It follows that $\upsilon_C X = \upsilon X$ and $\upsilon_{C^*} X = \beta X$. Also, $\upsilon X \subseteq \upsilon_A X$. It is easy to see that $\upsilon_A X$ is a realcompactification of $X$ and every realcompactification of $X$ contained in $\beta X$ is of the form $\upsilon_A X$ for some $A(X) \subseteq C(X)$, see [6, 8B (2)]. Note that, by a realcompactification of $X$, we mean a realcompact space containing $X$ as a dense subspace, see [1] and [8] for more details about the spaces $\upsilon_A X$.

A subring $A(X)$ of $C(X)$ is called a $C$-ring if it is isomorphic to $C(Y)$ for some Tychonoff space $X$. Also, $A(X)$ is called an intermediate ring if contains $C^*(X)$. Intermediate $C$-rings are intermediate rings which are also $C$-rings, see [7] for more details about intermediate $C$-rings. We should emphasize that by a subring of $C(X)$ we mean a non-unital subring, unless otherwise, we explicitly assert. We denote by $[f]$, the subalgebra of $C(X)$ generated by $f$ which is the set $\{\Sigma_{i=1}^n c_i f^i : i \in \mathbb{N}, c_i \in \mathbb{R}\}$. Moreover, for a subalgebra $A$ and $f \in C(X)$, the singly generated subalgebra over $A$ by $f$ is denoted by $A[f]$ which the smallest subalgebra

*Presenter

of $C(X)$ containing both $A$ and $f$. It is easy to see that $A[f] = A + [f] + A.[f]$; i.e., $A[f] = \{\Sigma_{i=1}^{n} c_i f^i + \Sigma_{j=0}^{m} g_j f^j : n \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}, c_i \in \mathbb{R}, g_j \in A\}$. It is easy to see that whenever $A$ is a unital subalgebra, then $A[f] = \{\Sigma_{i=0}^{n} g_i f^i : n \in \mathbb{N} \cup \{0\}, g_i \in A\}$. Moreover, the subring of $C(X)$ generated by a subring $A$ and an element $f$ equals to $A[f] = \{\Sigma_{i=1}^{n} n_i f^i + \Sigma_{j=1}^{m} g_j f^j : n \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}, n_i \in \mathbb{Z}, g_j \in A\}$ and whenever $A$ is unital, $A[f] = \{\Sigma_{i=1}^{n} g_i f^i : n \in \mathbb{N} \cup \{0\}, g_i \in A\}$ whenever $A(X)$ is an intermediate ring and $f \in C(X)$, $A(X)[f]$ is said to be the singly generated intermediate ring over $A(X)$ by $f$. As by [4, Proposition 2.1], intermediate rings are exactly absolutely convex subrings of $C(X)$, we can conclude that an intermediate ring contains a given function $g \in C(X)$ if and only if it contains $|g| + c$ for any $c \in \mathbb{R}$. Hence, $A(X)[g] = A(X)[|g| + c]$ which means that every intermediate ring that is singly generated over $A(X)$ is $A(X)[f]$ for some $f \geq c$.

A proper subring $S$ of a commutative ring $R$ is called a maximal subring of $R$, if $S = Q$ or $Q = R$ whenever $Q$ is another subring of $R$ containing $S$. Maximal subrings of $C(X)$ have been first studied by E.M. Vechtomov in [10]. Maximal subrings of commutative rings have been extensively studied recently by A. Azarang et al, see for example [2]. We aim in this paper to investigate some properties of maximal subrings of rings of continuous functions by using the notion of realcompactifications generated by subrings of $C(X)$ and singly generated subalgebras of $C(X)$. From these properties, new approaches to some results of [3] concerning maximal subrings of $C(X)$ follows.

## 2. Main Results

In the following statement, we show that no intermediate ring of $C(X)$ could be a maximal subring of $C(X)$.

THEOREM 2.1. *No maximal subring of $C(X)$ is an intermediate ring of $C(X)$.*

PROOF. Let $A(X)$ be a proper intermediate ring of $C(X)$ different from $C^*(X)$. Hence, there exists $f \in C(X) \setminus A(X)$ and $p \in \beta X \setminus \upsilon_A X$. As $p \in \beta X \setminus \upsilon_A X$, there exists $g \in A(X)$ such that $g^*(p) = \infty$. It follows that $f^2 + g^2$ does not belong to $A(X)$ and $(f^2 + g^2)^*(p) = \infty$. Now, set $h = 1 + f^2 + g^2$. It follows that $|h| \geq 1$ and $\frac{1}{h} \in C^*(X) \subseteq A(X)$. From these, we can infer that $A(X)[h] = \{kh^n : k \in A(X), n \in \mathbb{N} \cup \{0\}\}$, since, $\Sigma_{i=0}^{n} k_i h^i = (\frac{k_0}{h^n} + \frac{k_1}{h^{n-1}} + \cdots + k_n)h^n$ for each $n \in \mathbb{N}$. We claim that $e^h \notin A(X)[h]$. Assume on the contrary that $e^h \in A(X)[h]$. Hence, there exists $k_0 \in A(X)$ and $n \in \mathbb{N}$ such that $e^h = k_0 h^n$. This implies $k_0 = \frac{e^h}{h^n}$ and thus $k_0^*(p) = \infty$. This contradiction proves our claim. Therefore, we have $A(X) \subset A(X)[h] \subset C(X)$; i.e., $A(X)$ is not a maximal subring of $C(X)$. □

It is inferred from [4, Proposition 2.1] and Theorem 2.1 that no maximal subring of $C(X)$ could be an absolutely convex subring.

REMARK 2.2. From Theorem 2.1, it easily follows that $(M^p)^u + \mathbb{R}$ could not be a maximal subring of $C(X)$ for any $p \in \beta X$. Indeed, $(M^p)^u + \mathbb{R} = M^p + C^*(X)$ for each $p \in \beta X \setminus \upsilon X$ which is an intermediate ring, and $(M^p)^p + \mathbb{R} = M^p + \mathbb{R} = C(X)$ for each $p \in \upsilon X$, see the notes preceding [1, Corollary 3.3] and [9, Remark 2.14]. This fact establishes a simple short proof to [3, Theorem 3.6 and Corollary 3.7].

Following [5], an intermediate ring $A(X)$ is said to closed under finite composition if whenever $g \in C(\mathbb{R}^n)$ and $f_1, \dots, f_n \in A(X)$, the composition $go(f_1, \dots, f_n)$ is in $A(X)$, for any $n \in \mathbb{N}$. It is easy to prove that $A(X)$ is closed under finite composition if and only if it is closed under composition with elements of $C(\mathbb{R})$. It is obvious that every intermediate $C$-ring is closed under finite composition, however, the converse of this fact does not hold, in general. For example, let $X$ be a $C^*$-embedded but not $C$-embedded closed subspace of a realcompact space $Y$. Then the image of $C(Y)$, under the restriction morphism from $C(Y)$ to $C(X)$ is not an intermediate $C$-ring, but, is closed under finite composition. The next statement shows that the realcompactification generated by a maximal subalgebra of an intermediate ring $A(X)$ of $C(X)$ which is closed under finite composition is the same as the realcompactification generated by $A(X)$.

THEOREM 2.3. *Let $A(X)$ be a subring of $C(X)$ which is closed under composition with elements of $C(\mathbb{R})$. Then if $R$ is a maximal subring of $A(X)$, then $\upsilon_R X = \upsilon_A X$.*

PROOF. If $p \notin \upsilon_A X$, then there exists $f \in A(X)$ such that $f^*(p) = \infty$. Now, as $R \subseteq R[f]$, we must have $R[f] = R$ or $R[f] = A(X)$. If $R[f] = R$, then $f \in R$ which implies that $p \notin \upsilon_R X$. If $R[f] = A(X)$, then, by the hypothesis, $e^f \in A(X)$ and, hence, there exists $g_i \in R$ for $0 \le i \le m$ such that $\Sigma_{i=1}^n n_i f^i + \Sigma_{j=1}^m g_j h_j = e^f$ in which $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ and $n_i \in \mathbb{Z}$ for $0 \le i \le n$. It follows that $1 = \Sigma_{i=0}^n n_i \frac{f^i}{e^f} + \Sigma_{j=0}^m \frac{g_j f^j}{e^f}$ and hence, $g_i^*(p) = \infty$ for some $0 \le i \le m$. Indeed, if we consider $(x_\lambda)$ as a net in $X$ converging to $p$, then, as $f^*(p) = \infty$, we have $f(x_\lambda) \to \infty$ and thus $\frac{f^n(x_\lambda)}{e^{f(x_\lambda)}} \to 0$ for each $n \in \mathbb{N}$ which implies that there must exists some $g_i \in R$ such that $g_i(x_\lambda) \to \infty$. Therefore, $g_i^*(p) = \infty$ which means $p \notin \upsilon_R X$; i.e., $\upsilon_R X \subseteq \upsilon_A X$. The reverse inclusion is evident. □

COROLLARY 2.4. *Let $R$ be a maximal subring of $C(X)$. Then $\upsilon_R X = \upsilon X$.*

REMARK 2.5. The converse of Theorem 2.3 does not necessarily hold, in general, since, for $p \in \beta X \setminus \upsilon X$, let $A_p = M^p + C^*(X)$ and $A(X) = A_p[f]$ in which $f \in C(X)$ with $f^*(p) = \infty$. It follows that $\upsilon_A X = \upsilon_{A_p} X \cap \upsilon_f X = \upsilon X$. However, by Theorem 2.1, $A(X)$ could not be a maximal subring of $C(X)$. Note that, in [1], it is shown that $\upsilon_{I+\mathbb{R}} X = \upsilon_{I^u+\mathbb{R}} X = \upsilon X \cup \theta(I)$ for each ideal $I$ in $C(X)$ where $\theta(I)$ denotes the set $\bigcap_{f \in I} \mathrm{cl}_{\beta X} Z(f)$.

In [10], Vechtomov has shown that for any two elements $x, y$ of a Tychonoff space $X$, subrings of the form $A_{x,y} = \{f \in C(X) : f(x) = f(y)\}$ is a maximal subalgebra of $C(X)$. It is clear that $A_{x,y} = (M_x \cap M_y) + \mathbb{R}$ which means that the class of subrings of $C(X)$ of the form $A_{x,y}$ is a subclass of subrings of the form $I + \mathbb{R}$ where $I$ is an ideal in $C(X)$. The next statement establishes a new proof to [3, Theorem 3.3 (b)].

THEOREM 2.6. *Let $I$ be an ideal of $C(X)$. Then $I + \mathbb{R}$ is a maximal subring of $C(X)$ if and only if $I = M^p \cap M^q$ for two distinct elements $p, q$ of $\upsilon X$.*

PROOF. We only prove the necessity. It is inferred from Theorem 2.3 that if a subring of the form $I + \mathbb{R}$ or $I^u + \mathbb{R}$, for some ideal $I$ in $C(X)$, is a maximal subring of $C(X)$, then $\upsilon_{I+\mathbb{R}} X = \upsilon_{I^u+\mathbb{R}} X = \upsilon X \cup \theta(I) = \upsilon X$, which implies that

$\theta(I) \subseteq \upsilon X$. Also, whenever $\theta(I) \subseteq \theta(J) \subseteq \upsilon X$, then $J+\mathbb{R} \subseteq I+\mathbb{R}$. Therefore, as by [3, Proposition 1.1], $M^p+\mathbb{R} = C(X)$ for each $p \in \upsilon X$, $\theta(I)$ must exactly has two distinct elements of $\upsilon X$. □

Corollary 2.7. *A topological space $X$ is pseudocompact if and only if for every maximal subring $R$ of $C(X)$ we have $\upsilon_R X = \beta X$.*

Proof. ⇒) This is clear, since, for each subring $R$ of $C^*(X)$, we have $\upsilon_R X = \beta X$.

⇐) Assume on the contrary that $X$ is not pseudocompact. Thus, there exits some $p \in \beta X$ such that $f^*(p) = \infty$. Now, let $q_1, q_2$ be two distinct elements of $\upsilon X$ and $f^*(q_1) = r_1, f^*(q_2) = r_2$ where $r_1, r_2 \in \mathbb{R}$. It follows that $g = (f-r_1)(f-r_2) \in I = M^{q_1} \cap M^{q_2}$. Also, by the above statement, $I+\mathbb{R}$ is a maximal subring of $C(X)$. Therefore, $g \in I+\mathbb{R}$ and $\upsilon_{I+\mathbb{R}} \neq \beta X$, since, $g \in I+\mathbb{R}$ and $g^*(p) = \infty$ which means $p \notin \upsilon_{I+\mathbb{R}} X$. □

Question 2.8. It follows from these facts that whenever $R$ is a maximal subalgebra of $C(X)$ which is of the form $I+\mathbb{R}$ or $M^p$, then $R$ is closed under the uniform topology on $C(X)$. Could the same fact could be said for all maximal subrings of $C(X)$?

## References


1. A. R. Aliabad and M. Parsinia, *Remarks on subrings of $C(X)$ of the form $I+C^*(X)$*, Quaest. Math. **40** (1) (2017) 63–73.
2. A. Azarang and O. A. S. Karamzadeh, *On maximal subrings of commutative rings*, Algebra Colloq. **19** (2012) 1125–1138.
3. F. Azarpanah and A. R. Olfati, *On subrings of the form $I+\mathbb{R}$ of $C(X)$*, J. Commut. Algebra **11** (2019) 479–509.
4. F. Azarpanah and M. Parsinia, *On the sum of z-ideals in subrings of C(X)*, J. Commut. Algebra **12** (2020) 459–466.
5. J. M. Dominguez, J. Gomez-Perez and M. A. Mulero, *Intermediate algebras between $C^*(X)$ and $C(X)$ as ring of fractions of $C^*(X)$*, Topology Appl. **77** (1997) 115–130.
6. L. Gillman and M. Jerison, *Rings of Continuous Functions*, Springer-Verlag, New York, 1978.
7. M. Parsinia, *Remarks on intermediate $C$-rings of $C(X)$*, Quaest. Math. **41** (5) (2018) 675–682.
8. M. Parsinia, *Mappings to realcompactifications*, Categ. General Alg. Struct. Appl. **10** (1) (2019) 107–116.
9. D. Rudd, *On the structure spaces of ideals in rings of continuous functions*, Trans. Amer. Math. Soc. **190** (1974) 393–403.
10. E. M. Vechtomov, *The lattice of subalgebras of ring of continuous functions and Hewitt spaces*, Math. Notes. **62** (5) (1997) 575–580.


E-mail: parsiniamehdi@gmail.com

# Contributed Posters

## Graphs and Combinatorics

# Global Accurate Dominating Set of Trees

Morteza Alishahi*
Department of mathematics, Nazarabad center, Karaj branch, Islamic Azad University, Karaj, Iran

Abstract. A dominating set $D$ of a graph $G = (V, E)$ is an accurate dominating set, if $V - D$ has no dominating set of cardinality $|D|$. An accurate dominating set $D$ of a graph $G$ is a global accurate dominating set, if $D$ is also an accurate dominating set of $\overline{G}$. The global accurate domination number $\gamma_{ga}(G)$ is the minimum cardinality of a global accurate dominating set. In this paper we study the global accurate dominating sets of trees and characterize the trees by their global accurate domination numbers.
**Keywords:** Global accurate dominating set, Tree.
**AMS Mathematical Subject Classification [2010]:** 05C65.

## 1. Introduction

The usual graph theory notions not herein, refer to [5]. The *open neighborhood* of vertex $u$ is denoted by $N(u) = \{v \in V(G) : uv \in E(G)\}$. A set $B \subseteq V(G)$ is an *independent set* of $G$ if for every edge $ab \in E(G)$ , $a \notin B$ or $b \notin B$. The *diameter* of connected graph $G$ is defined as $diam(G) = \max\{d(u, v) : u, v \in V(G)\}$. For a vertex $u \in V(G)$, the *eccentricity* of $u$, defined as $\epsilon(u) = \max\{d(u, v) : v \in V(G)\}$. The *radius* of a graph $G$ defined as $R(G) = \min\{\epsilon(u) : u \in V(G)\}$. The *center* of a graph $G$ is defined as $C(G) = \{u \in V(G) : \epsilon(u) = R(G)\}$. The number of vertices of a graph $G$ is denoted by $n(G)$ and the degree of vertex $u$ is denoted by $d(u)$ and $\Delta(G) = \max\{d(u) : u \in V(G)\}$. A path with $k$ vertices denoted by $P_k$.

A set $D \subseteq V(G)$ is a *dominating set* (D.S) of $G$ if every vertex of $V(G) - D$ is adjacent to at least one vertex of $D$. The cardinality of the smallest D.S. of $G$, denoted by $\gamma(G)$, is called the *domination number* of $G$. A D.S of cardinality $\gamma(G)$ is called a $\gamma$-set of $G$ [3]. A set $S \subseteq V(G)$ is a *global dominating set* (G.D.S) of $G$ if $S$ is a dominating set of $G$ and $\overline{G}$. The cardinality of the smallest G.D.S of $G$, denoted by $\gamma_g(G)$, is called the *global domination number* of $G$ [1, 2]. A set $S \subseteq V(G)$ is an *accurate dominating set* (A.D.S) of $G$ if $S$ is a dominating set of $G$ and $V(G) - S$ has no dominating set of cardinality $|S|$. The cardinality of the smallest A.D.S of $G$, denoted by $\gamma_a(G)$, is called the *accurate domination number* of $G$. A set $S \subseteq V(G)$ is a *global accurate dominating set* (G.A.D.S) of $G$ if $S$ is an A.D.S of $G$ and $\overline{G}$. The cardinality of the smallest G.A.D.S of $G$, denoted by $\gamma_{ga}(G)$, is called the *global accurate domination number* of $G$. A G.A.D.S of cardinality $\gamma_{ga}(G)$ is called a $\gamma_{ga}$-set of $G$.

*Presenter

We need the families $A$ and $B$ of trees which are defined as follows:

$$A = \{T : \ T \ is \ a \ tree, \ diam(T) = 3, \ C(T) = \{u, v\} \ and \ |d(u) - d(v)| \leq 1\},$$

$$B = \{T : \ T \ is \ a \ tree, \ diam(T) = 4, \ C(T) = \{u\} \ and \ d(u) = \frac{n(T)}{2}\}.$$

Kulli and Kattimani obtained some bounds for $\gamma_{ga}(G)$ and exact values of $\gamma_{ga}(G)$ for some standard graphs [4]. In this paper we characterize all the trees as their global accurate domination numbers.

## 2. Main Results

We characterize the trees with global accurate dominating sets.

LEMMA 2.1. *Let $T$ be a tree and $D$ be a G.A.D.S of $T$. If $C$ is a subset of $V(T) - D$ and $|C| = |D|$, then there exist a vertex $u \in D$ such that $u$ is adjacent to all vertices of $C$.*

PROOF. If $T = P_1$, then the result holds. Now suppose $T \neq P_1$. Since $D$ is a G.D.S of $T$, so $|D| \geq 2$. $D$ is an A.D.S of $\overline{T}$, so $C$ is not a D.S of $\overline{T}$, thus there exists a vertex $v \in V(T) - C$ such that $v$ is adjacent to all vertices of $C$ in $T$.
Claim: $v \in D$.
On the contrary suppose that $v \notin D$. Let $t_1, t_2 \in C$ and $C' = (C - \{t_1\}) \cup \{v\}$. The set $C'$ includes $|D|$ vertices of $V(T) - D$, so there exists a vertex $w \in V(T) - C'$ such that $w$ is adjacent to all vertices of $C'$. But $vwt_2v$ is a cycle, that is a contradiction, thus $v \in D$. □

THEOREM 2.2. *Let $D$ be a G.A.D.S of $T$. If $|D| \leq \frac{n(T)}{2}$, then there exists $u \in D$ such that $V(T) - D \subseteq N(u)$.*

PROOF. Let $|D| = d$. Consider the following two cases:
**Case 1)** $d = 2$.
Let $D = \{u, v\}$. On the contrary suppose there exist vertices $u', v' \in V(T) - D$ such that $u'$ is nonadjacent to $u$ and $v'$ is nonadjacent to $v$. Let $C = \{u', v'\}$. By Lemma 2.1 all the vertices of $C$ are adjacent to $u$ or $v$ that is a contradiction, therefore all the vertices of $V(T) - D$ are adjacent to $u$ or $v$.
**Case 2)** $d > 2$.
If $|V(T) - D| = d$, then by Lemma 2.1 the result holds. Now let $|V(T) - D| > d$. Let $C$ be a subset of $V(T) - D$ and $|C| = |D|$ and $t_1, t_2, t_3 \in C$. By Lemma 2.1, there exists a vertex $u \in D$ such that $u$ is adjacent to all vertices of $C$. On the contrary suppose there exists a vertex $v \in V(T) - (D \cup C)$ such that $v \notin N(u)$. Let $C' = (C - \{t_1\}) \cup \{v\}$. By Lemma 2.1, there exists a vertex $w \in D$ such that all vertices of $C'$ are adjacent to $w$, $w \neq u$. But $ut_2wt_3u$ is a cycle, that is contradiction. Consequently, $u$ is adjacent to all vertices of $V(T) - D$. □

LEMMA 2.3. *Let $G$ be a connected bipartite graph. Then $\gamma_{ga}(G) \leq \lfloor \frac{n(G)}{2} \rfloor + 1$.*

PROOF. Suppose that $G$ is a connected bipartite graph with partitions $X$ and $Y$. Let $|X| = s$ and $|Y| = k$ and $Y = \{y_1, y_2, \ldots, y_k\}$. Without lose of generality let $s \leq k$. The set $X \cup \{y_k\}$ is a G.D.S of $G$, so if $k = s$ or $k = s + 1$, then the set $X \cup \{y_k\}$ is a G.A.D.S of size $\lfloor \frac{n(G)}{2} \rfloor + 1$ but if $k \geq s + 2$, then the set $X \cup \{y_k, y_1, y_2, \ldots, y_{\lfloor \frac{n(G)}{2} \rfloor - s}\}$ is a G.A.D.S of $G$ of size $\lfloor \frac{n(G)}{2} \rfloor + 1$, so $\gamma_{ga}(G) \leq \lfloor \frac{n(G)}{2} \rfloor + 1$. □

As an immediate result we have:

Corollary 2.4. *Let $T$ be a tree. Then $\gamma_{ga}(T) \leq \lfloor \frac{n(T)}{2} \rfloor + 1$.*

Lemma 2.5. *Let $T$ be a tree. Then $\gamma_{ga}(T) \geq n(T) - \Delta(T)$ or $\gamma_{ga}(T) = \lfloor \frac{n(T)}{2} \rfloor + 1$.*

Proof. Let $D$ be a $\gamma_{ga}$-set of $T$. If $|D| \leq \frac{n(T)}{2}$, then by Theorem 2.2 there exists a vertex $u \in D$ such that $V(T) - D \subseteq N(u)$. Thus $\gamma_{ga}(T) = |D| \geq n(T) - |N(u)| = n(T) - d(u) \geq n(T) - \Delta(T)$. But if $|D| > \frac{n(T)}{2}$, then $\gamma_{ga}(T) > \frac{n(T)}{2}$. Therefore $\gamma_{ga}(T) \geq \lfloor \frac{n(T)}{2} \rfloor + 1$, and by Corollary 2.4 $\gamma_{ga}(T) = \lfloor \frac{n(T)}{2} \rfloor + 1$. □

Theorem 2.6. *Let $T$ be a tree. Then*

a) $\gamma_{ga}(T) = \lfloor \frac{n(T)}{2} \rfloor + 1$ *or* $\gamma_{ga}(T) = n(T) - \Delta(T)$ *or* $\gamma_{ga}(T) = n(T) - \Delta(T) + 1$.

b) $\gamma_{ga}(T) = \lfloor \frac{n(T)}{2} \rfloor + 1$ *if and only if* $T = P_2$ *or* $P_3$ *or* $\Delta(T) < \frac{n(T)}{2}$ *or* $T \in A \cup B$.

c) $\gamma_{ga}(T) = n(T) - \Delta(T) + 1$ *if and only if* $T = P_2$ *or* $T$ *is a star or* $diam(T) = 3$ *or* $T \in B$.

Proof. Consider all of the possible states for $T$ as follows:

**State 1)** If $\Delta(T) < \frac{n(T)}{2}$.

Let $D$ be a $\gamma_{ga}$-set of $T$. If $|D| \leq \frac{n(T)}{2}$, then by Theorem 2.2 there exists a vertex $u \in D$ such that $V(T) - D \subseteq N(u)$, so $d(u) \geq n(T) - |D| \geq n(T) - \frac{n(T)}{2} = \frac{n(T)}{2}$. Therefore $\Delta(T) \geq \frac{n(T)}{2}$, a contradiction. So $|D| > \frac{n(T)}{2}$, and $\gamma_{ga}(T) \geq \lfloor \frac{n(T)}{2} \rfloor + 1$. Now by Corollary 2.4 $\gamma_{ga}(T) = \lfloor \frac{n(T)}{2} \rfloor + 1$.

**State 2)** If $\Delta(T) \geq \frac{n(T)}{2}$.

It is obvious that $n(T) - \Delta(T) \leq \frac{n(T)}{2}$. By Lemma 2.5, $\gamma_{ga}(T) \geq n(T) - \Delta(T)$ or $\gamma_{ga}(T) = \lfloor \frac{n(T)}{2} \rfloor + 1 > \frac{n(T)}{2} \geq n(T) - \Delta(T)$. Thus if $\Delta(T) \geq \frac{n(T)}{2}$, then

$$\gamma_{ga}(T) \geq n(T) - \Delta(T). \tag{1}$$

Let $u$ be a vertex of $T$ such that $d(u) = \Delta(T)$. Let $H = N(u)$ and $D = V(T) - H$. It is clear that $H$ is an independent set. Consider two cases of $H$ as follows:

**Case 1)** None of the vertices of $H$ is adjacent to all vertices of $D$.

In this case $D$ is a G.D.S of $T$. Consider two bellow subcases:

i) If $|H| > \frac{n(T)}{2}$. Let $C$ be a subset of $H$ and $|C| = |\frac{n(T)}{2}|$. It is clear that $H - C \neq \phi$ and $C$ doesn't dominate any vertex of $H - C$, so $D$ is an A.D.S of $T$. In addition, Since $u$ is adjacent to all vertices of $C$, so $C$ does't dominate vertex $u$ in $\overline{T}$. Therefore $D$ is an A.D.S of $\overline{T}$, too. Thus $D$ is a G.A.D.S of $T$, so $\gamma_{ga}(T) \leq |D| = n(T) - d(u) = n(T) - \Delta(T)$. By (1) $\gamma_{ga}(T) = n(T) - \Delta(T)$.

ii) If $|H| = \frac{n(T)}{2}$. In this case since $u$ is adjacent to all vertices of $H$, so $H$ is not a dominating set of $\overline{T}$, therefore $D$ is an A.D.S of $\overline{T}$. Now if $H$ is not a dominating set of $T$, then $D$ is an A.D.S of $T$, too. Therefore $D$ is a G.A.D.S of $T$ and by (1), $\gamma_{ga}(T) = n(T) - \Delta(T)$. But if $H$ is a dominating set of $T$, then $D$ is not an A.D.S of $T$. Let $v$ be an arbitrary vertex of $H$. Obviously the set $D \cup \{v\}$ is a G.A.D.S of $T$, so $\gamma_{ga}(T) = \lfloor \frac{n(T)}{2} \rfloor + 1$.

In this case, since $T$ has no cycle, the subgraph $G[D]$ has no edge, therefore $D$ is an independent set. Since none of the vertices of $H$ is adjacent to all vertices of $D$, so $T \in B$.

**Case 2)** There exists a vertex in $H$ adjacent to all vertices of $D$. Let $w \in H$ is adjacent to all vertices of $D$. Since $T$ has no cycle, so $D$ and $H$ are independent sets. It is clear that $D$ is not a D.S of $\overline{T}$. Therefore $D$ is not a G.A.D.S of $T$. But the set $D' = D \cup \{w\}$ is a G.A.D.S of $T$, so $\gamma_{ga}(T) = |D| + 1 = n(T) - d(u) + 1 = n(T) - \Delta(T) + 1$.

If $|D| = |H| = 1$, then $T = P_2$, $\gamma_{ga}(T) = \lfloor \frac{n(T)}{2} \rfloor + 1$.

If $|D| = 1$ and $|H| = 2$, then $T = P_3$, $\gamma_{ga}(T) = \lfloor \frac{n(T)}{2} \rfloor + 1$.

If $|D| = 1$ and $|H| > 2$, then $T$ is a star.

If $|D| \geq 2$, then $diam(T) = 3$.

If $|D| \geq 2$ and $|H| = |D|$ or $|H| = |D| + 1$, then $|D| = \lfloor \frac{n(T)}{2} \rfloor$, therefore $\gamma_{ga}(T) = |D| + 1 = \lfloor \frac{n(T)}{2} \rfloor + 1$ and it is clear that $T \in A$. □

**Problem.** Let $G$ be a graph and $\overline{G}$ be a tree. What can say about the $\gamma_{ga}(G)$?

## References

1. R. C. Brigham and R. D. Dutton, *Factor domination in graphs,* Discrete Math. **86** (1990) 127–136.
2. R. D. Dutton and R. C. Brigham, *On global domination critical graphs*, Discrete Math. **309** (2009) 5894–5897.
3. T. Haynes, S. Hedetniemi and P. J. Slater, *Fundamentals of domination in graphs*, M. dekker, Inc., New York, 1997.
4. V. R. Kulli and M. B. Kattimani, *Global accurate domination in graphs,* Int. J. Sci. Res. Pub. **3** (2013) 1–3.
5. D. B. West, *Introduction to Graph Theory,* 2nd ed., Prentice-Hall, Upper Saddle River, NJ, 2001.

E-mail: Morteza.alishahi@gmail.com

# The Metric Dimension of the Composition Product of Some Families of Graphs

Mohsen Jannesari*
Faculty of Basic Sciences, University of Shahreza, Shahreza, Iran

Abstract. A set of vertices $W$ is a resolving set for a connected graph $G$ if every vertex is uniquely determined by its vector of distances to the vertices in $W$. The minimum cardinality of a resolving set of $G$ is the metric dimension of $G$. The composition product of graphs $G$ and $H$, $G \circ H$, is the graph with vertex set $V(G) \times V(H) := \{(u, v) \mid u \in V(G), v \in V(H)\}$, where $(a, b)$ is adjacent to $(u, v)$ whenever $a$ is adjacent to $u$, or $a = u$ and $b$ is adjacent to $v$. In this paper, the metric dimension of composition product $G \circ H$ is considered when $G$ or $H$ or both of them is in some families of graphs such as paths, cycles, bipartite graphs and Kneser graphs.
**Keywords:** Composition product, Metric dimension, Adjacency dimension.
**AMS Mathematical Subject Classification [2010]:** 05C12.

## 1. Introduction

Throughout this paper $G = (V, E)$ is a finite simple graph of order $n(G)$. We use $\overline{G}$ for the complement graph of $G$. The distance between two vertices $u$ and $v$, denoted by $d_G(u, v)$, is the length of a shortest path between $u$ and $v$ in $G$. The notations $u \sim v$ and $u \nsim v$ denote the adjacency and none-adjacency relation between $u$ and $v$, respectively. The symbols $P_n$ and $C_n$ represent a path of order $n$ and a cycle of order $n$, respectively.

The vertices of a connected graph can be represented by different ways, for example, the vectors which theirs components are the distances between the vertex and the vertices in a given subset of vertices. For an ordered set $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ and a vertex $v$ of $G$, the $k$-vector

$$r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k)),$$

is called the *(metric) representation* of $v$ with respect to $W$. The set $W$ is called a *resolving set (locating set)* for $G$ if distinct vertices have different representations in this case we say the set $W$ resolves $G$. A resolving set $W$ for $G$ with minimum cardinality is called a *basis* of $G$, and its cardinality is the *metric dimension* of $G$, denoted by $\beta(G)$.

The concept of (metric) representation is introduced by Slater [10] (See [6]). He described the usefulness of these ideas when working with U.S. sonar and Coast Guard Loran stations [10]. It was noted in [5, 9] that the problem of finding the metric dimension of a graph is $NP$-hard. For more results in this concept see [1, 2, 3, 4].

*Presenter

Caceres et al. [3] obtained the metric dimension of cartesian product of graphs $G$ and $H$, $G\Box H$, for $G,H\in\{P_n,C_n,K_n\}$. The *composition product*, $G\circ H$ of graphs $G$ and $H$ is the graph with vertex set $V(G)\times V(H):=\{(u,v)\mid u\in V(G), v\in V(H)\}$, where $(a,b)$ is adjacent to $(u,v)$ whenever $a\sim u$, or $a=u$ and $b\sim v$. It is easy to see that $G\circ H$ is a connected graph if and only if $G$ is a connected graph of order at least 2.

Jannesari and Omoomi [8] studied the metric dimension of composition product of graphs. They find $\beta(G\circ H)$ in terms of order and some other parameters of $G$ and a new parameter of $H$, called *adjacency dimension*. The definition of adjacency dimension is as following.

Definition 1.1. [8] Let $H$ be a graph and $W=\{w_1,w_2,\ldots,w_k\}$ be an ordered subset of $V(H)$. For each vertex $v\in V(H)$ the adjacency representation of $v$ with respect to $W$ is the $k$-vector $r_2(v|W):=(a_H(v,w_1),a_H(v,w_2),\ldots,a_H(v,w_k))$, where $a_H(v,w_i)=\min\{2,d_H(v,w_i)\}$. If all distinct vertices of $H$ have distinct adjacency representations, $W$ is called an adjacency resolving set for $H$. The minimum cardinality of an adjacency resolving set is called the adjacency dimension of $H$, denoted by $\beta_2(H)$. An adjacency resolving set of cardinality $\beta_2(H)$ is called an adjacency basis of $H$.

To describe results about metric dimension of $G\circ H$ in [8] the following definitions are needed.

Two distinct vertices $u,v$ are *twins* if $N(v)\backslash\{u\}=N(u)\backslash\{v\}$. It is called that $u\equiv v$ if and only if $u=v$ or $u,v$ are twins. Clearly $\equiv$ is an equivalent relation. The equivalence class of the vertex $v$ is denoted by $v^*$. Hernando et al. [7] proved that $v^*$ is a clique or an independent set in $G$. We mean by $\alpha_{_{N(G)}}$ and $\alpha_{_{K(G)}}$, the number of clique and independent classes of size at least 2, respectively. We also use $a(G)$ and $b(G)$ for the number of all vertices of $G$ which have at least an adjacent twin and a none-adjacent twin vertex in $G$, respectively. Jannesari and Omoomi considered the metric dimension of composition product of graphs through the following four theorems.

Theorem 1.2. [8] *If $H$ has two adjacency bases $B_1$ and $B_2$ such that for each $w\in V(H)$, $r_2(w|B_i)$ is not entirely $i$, $1\le i\le 2$, then $\beta(G\circ H)=\beta(G\circ\overline{H})=n\beta_2(H)$.*

Theorem 1.3. [8] *If for each adjacency basis $A$ of $H$ there exist vertices $x_{_A},y_{_A}\in V(H)$ such that for each $w\in A$, $w\sim x_{_A}$ and $w\nsim y_{_A}$, then*

$$\beta(G\circ H)=\beta(G\circ\overline{H})=n\beta_2(H)+a(G)+b(G)-\alpha_{_K}(G)-\alpha_{_N}(G).$$

Theorem 1.4. [8] *Let $H$ has an adjacency basis $W$ such that all vertices of $V(H)\backslash W$ have a neighbor in $W$. If for each adjacency basis $A$ of $H$ there exist a vertex $x_{_A}\in V(H)$ such that $x_{_A}$ is adjacent to all vertices of $A$, then $\beta(G\circ H)=n\beta_2(H)+a(G)-\alpha_{_K}(G)$.*

Theorem 1.5. [8] *Let $H$ has an adjacency basis $W$ such that all vertices of $V(H)\backslash W$ have a none-neighbor vertex in $W$. If for each adjacency basis $A$ of $H$ there exist a vertex $y_{_A}\in V(H)$ such that $y_{_A}$ is not adjacent to any vertex of $A$, then $\beta(G\circ H)=n\beta_2(H)+b(G)-\alpha_{_N}(G)$.*

Clearly to find the exact value of $\beta(G \circ H)$, we need to find $a(G), b(G), \alpha_{_K}(G)$, $\alpha_{_N}(G)$, $\beta_2(H)$, and of course the structure of adjacency bases of $H$. The aim of this paper is to find these parameter fore some families of graphs and use them to compute $\beta(G \circ H)$ for these families. To do this, the following known results are needed.

Corollary 1.6. [8] *If $G$ does not have any pair of twin vertices, then $\beta(G \circ H) = n\beta_2(H)$.*

If $H$ is a graph of order $m$, it is easy to check that $1 \leq \beta_2(H) \leq m-1$. Also, if $H$ is a connected graph with diameter 2, then $\beta(H) = \beta_2(H)$. Clearly $\beta_2(K_n) = n-1$.

Lemma 1.7. [8] *If $K_{m_1,m_2,\ldots,m_t}$ is the complete $t$-partite graph, then*

$$\beta_2(K_{m_1,m_2,\ldots,m_t}) = \beta(K_{m_1,m_2,\ldots,m_t}) = \begin{cases} m-r-1, & \text{if } r \neq t, \\ m-r, & \text{if } r = t, \end{cases}$$

*where $m_1, m_2, \ldots, m_r$ are at least 2, $m_{r+1} = \cdots = m_t = 1$, and $\sum_{i=1}^{t} m_i$.*

Against the metric dimension, adjacency dimension is also defined for disconnected graphs.

Lemma 1.8. [8] *If $H$ is a graph, then $\beta_2(H) = \beta_2(\overline{H})$.*

It is clear that $\beta_2(P_1) = \beta_2(P_2) = \beta_2(P_3) = 1$.

Lemma 1.9. [8] *If $m \geq 4$, then $\beta_2(C_m) = \beta_2(P_m) = \lfloor \frac{2m+2}{5} \rfloor$.*

## 2. Main Results

In this section we find parameters $a(G), b(G), \alpha_{_K}(G), \alpha_{_N}(G)$ and $\beta_2(H)$ fore some families of graphs and use them to compute $\beta(G \circ H)$ for these families. Let start with bipartite graphs.

Lemma 2.1. *If $G$ is a bipartite graph of order at least 3 and $H$ is an arbitrary graph, $\beta(G \circ H) = n\beta_2(H) + b(G) - \alpha_{_N}(G)$.*

The family of Keneser graphs is an important family of graphs.

Lemma 2.2. *If $G = K(k,r)$, $k \geq 2r+1$ be the Kneser graph, then $G$ does not have any pair of twin vertices.*

Note that the line graph $L(K_n)$ of $K_n$ is the complement of $K(n,2)$. Since all twin vertices of a graph are twins in its complement, by Lemma 2.2, $L(K_n)$ $(n \geq 5)$ does not have any pair of twin vertices. Also, Since the path $P_n$ $(n \geq 4)$ and the cycle $C_n$ $(n \geq 5)$ do not have any pair of twin vertices, their complements, $\overline{P}_n$ $(n \geq 4)$ and $\overline{C}_n$ $(n \geq 5)$ do not have any pair of twins. Therefore, by Corollary 1.6, we have obtained the exact value of $\beta(G \circ H)$ for $H \in \{P_m, C_m, \overline{P}_m, \overline{C}_m, K_m, \overline{K}_m, P, K_{m_1,\ldots,m_t}\}$ and the connected graph $G \in \{\overline{P}_n\ (n \geq 4), \overline{C}_n (n \geq 5), L(K_n)\ (n \geq 5), K(k,r)\}$.

By Lemma 1.9 and properties of adjacency bases of $P_n$, $C_n$ and their complements the following proposition is obtained.

Proposition 2.3. *Let $G$ be a connected graph of order $n$ and $H \in \{P_m, C_m\}$, where $m = 5k + r \notin \{2, 3\}$.*

(a) *If $r$ is even, then* $\beta(G \circ H) = \beta(G \circ \overline{H}) = n\lfloor \frac{2m+2}{5} \rfloor$.
(b) *If* $m = 6$, *then* $\beta(G \circ H) = \beta(G \circ \overline{H}) = n\lfloor \frac{2m+2}{5} \rfloor + a(G) + b(G) - \alpha_{_K}(G) - \alpha_{_N}(G)$.
(c) *If $r$ is odd and* $m \neq 6$, *then* $\beta(G \circ H) = n\lfloor \frac{2m+2}{5} \rfloor + b(G) - \alpha_{_N}(G)$ *and* $\beta(G \circ \overline{H}) = n\lfloor \frac{2m+2}{5} \rfloor + a(G) - \alpha_{_K}(G)$.

Considering properties of paths, cycles, complete graphs, and complete $t$-partite graphs, the following corollaries are obtained.

Corollary 2.4. *Let* $m = 5k + r$. *If* $H \in \{P_m, C_m\}$, *then for all* $n \geq 2$,

- $\beta(K_n \circ H) = \begin{cases} 2n-1, & \text{if } H = P_2 \text{ or } H = P_3, \\ 3n-1, & \text{if } H \in \{C_3, P_6, C_6\}, \\ n\lfloor \frac{2m+2}{5} \rfloor, & \text{otherwise.} \end{cases}$
- $\beta(K_{n_1,n_2,\ldots,n_t} \circ H) = \begin{cases} n\lfloor \frac{2m+2}{5} \rfloor + t - j - 1, & \text{if } H = P_2 \text{ and } j \neq t, \\ n(m-1) + t - j - 1, & \text{if } H = C_3 \text{ and } j \neq t, \\ n(m-1), & \text{if } H = C_3 \text{ and } j = t, \\ n\lfloor \frac{2m+2}{5} \rfloor + n - j - 1, & \text{if } H \in \{P_3, P_6, C_6\} \text{ and } j \neq t, \\ n\lfloor \frac{2m+2}{5} \rfloor + n - t, & \text{if } H \in \{P_3, P_6, C_6\} \text{ and } j = t, \\ n\lfloor \frac{2m+2}{5} \rfloor + n - t, & \text{if } m \geq 7 \text{ and } r \text{ is odd,} \\ n\lfloor \frac{2m+2}{5} \rfloor, & \text{otherwise,} \end{cases}$

*where* $n_1, n_2, \ldots, n_j$ *are at least* 2, $n_{j+1} = \cdots = n_t = 1$, *and* $\sum_{i=1}^{t} n_i = n$.

Corollary 2.5. *Let* $m = 5k + r$. *If* $H \in \{\overline{P}_m, \overline{C}_m\}$, *then for all* $n \geq 2$,

- $\beta(K_n \circ H) = \begin{cases} n\lfloor \frac{2m+2}{5} \rfloor + n - 1, & \text{if } H \neq \overline{C}_3 \text{ and } r \text{ is odd,} \\ 2n, & \text{if } H = \overline{C}_3, \\ n\lfloor \frac{2m+2}{5} \rfloor & \text{otherwise.} \end{cases}$
- $\beta(K_{n_1,n_2,\ldots,n_t} \circ H) = \begin{cases} n\lfloor \frac{2m+2}{5} \rfloor + n - t, & \text{if } H = \overline{P}_2, \\ n(m-1) + n - t, & \text{if } H = \overline{C}_3, \\ n\lfloor \frac{2m+2}{5} \rfloor + n - j - 1, & \text{if } H \in \{\overline{P}_3, \overline{P}_6, \overline{C}_6\} \text{ and } j \neq t, \\ n\lfloor \frac{2m+2}{5} \rfloor + n - t, & \text{if } H \in \{\overline{P}_3, \overline{P}_6, \overline{C}_6\} \text{ and } j = t, \\ n\lfloor \frac{2m+2}{5} \rfloor + t - j - 1, & \text{if } m \geq 7,\ r \text{ is odd, and } j \neq t, \\ n\lfloor \frac{2m+2}{5} \rfloor, & \text{otherwise,} \end{cases}$

*where* $n_1, n_2, \ldots, n_j$ *are at least* 2, $n_{j+1} = \cdots = n_t = 1$, *and* $\sum_{i=1}^{t} n_i = n$.

Corollary 2.6. *For* $n \geq 2$,

- $\beta(K_n \circ K_m) = nm - 1$
- $\beta(P_n \circ K_m) = \begin{cases} n(m-1), & \text{if } n \geq 3, \\ n(m-1)+1, & \text{if } n = 2. \end{cases}$
- $\beta(C_n \circ K_m) = \begin{cases} n(m-1), & \text{if } n \geq 4, \\ n(m-1)+2, & \text{if } n = 3. \end{cases}$
- $\beta(K_{n_1,n_2,\ldots,n_t} \circ K_m) = \begin{cases} n(m-1) + t - j - 1, & \text{if } j \neq t, \\ n(m-1), & \text{if } j = t, \end{cases}$
  *where* $n_1, n_2, \ldots, n_j$ *are at least* 2, $n_{j+1} = \cdots = n_t = 1$, *and* $\sum_{i=1}^{t} n_i = n$.
- $\beta(K_n \circ \overline{K}_m) = n(m-1)$
- $\beta(P_n \circ \overline{K}_m) = \begin{cases} n(m-1), & \text{if } n \neq 3, \\ n(m-1)+1, & \text{if } n = 3. \end{cases}$
- $\beta(C_n \circ \overline{K}_m) = \begin{cases} n(m-1), & \text{if } n \neq 4, \\ n(m-1)+2, & \text{if } n = 4. \end{cases}$

- $\beta(K_{n_1,n_2,\ldots,n_t} \circ \overline{K}_m) = n(m-1)+n-t$, *where* $n_1, n_2, \ldots, n_j$ *are at least* 2, $n_{j+1} = \cdots = n_t = 1$, *and* $\sum_{i=1}^{t} n_i = n$.

Corollary 2.7. *Let* $m_1, \ldots, m_q \geq 2$, $m_{q+1} = \cdots = m_s$, *and* $m = \sum_{i=1}^{s} m_i$. *Then for* $n \geq 2$,

- $\beta(K_{n_1,n_2,\ldots,n_t} \circ K_{m_1,\ldots,m_s}) = \begin{cases} n(m-q), & \text{if } q = s, \\ n(m-q-1), & \text{if } q \neq s \text{ and } j = t, \\ n(m-q-1)+t-j-1, & \text{otherwise,} \end{cases}$

  *where* $n_1, n_2, \ldots, n_j$ *are at least* 2, $n_{j+1} = \cdots = n_t = 1$, *and* $\sum_{i=1}^{t} n_i = n$.
- $\beta(K_{n_1,n_2,\ldots,n_t} \circ \overline{K}_{m_1,\ldots,m_s}) = \begin{cases} n(m-q), & \text{if } q = s, \\ n(m-q)-t, & \text{otherwise,} \end{cases}$

  *where* $\sum_{i=1}^{t} n_i = n$.

## References


1. P. S. Buczkowski, G. Chartrand, C. Poisson and P. Zhang, *On k-dimensional graphs and their bases*, Period. Math. Hungar. **46** (1) (2003) 9–15.
2. R. C. Brigham, G. Chartrand, R. D. Dutton and P. Zhang, *On the dimension of trees*, Discrete Math. **294** (3) (2005) 279–283.
3. J. Caceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara and D. R. Wood, *On the metric dimension of cartesian products of graphs*, SIAM J. Discrete Math. **21** (2) (2007) 423–441.
4. G. Chartrand and P. Zhang, *The theory and applications of resolvability in graphs.* A survey. In Proc. 34th Southeastern International Conf. on Combinatorics, Graph Theory and Computing, Congr. Numer. **160** (2003) 47–68.
5. M. Garey and D. Johnson, *Computers and Intractability: A Guide to the Theory of NO-Completeness*, Freeman, New York, 1979.
6. F. Harary and R. A. Melter, *On the metric dimension of a graph*, Ars Combin. **2** (1976) 191–195.
7. C. Hernando, M. Mora, I. M. Pelayo, C. Seara and D. R. Wood, *Extremal graph theory for metric dimension and diameter*, Electron. J. Combin. **17** (1) (2010) R30.
8. M. Jannesari and B. Omoomi, *The metric dimension of the lexicographic product of graphs*, Discrete Math. **312** (22) (2012) 3349–3356.
9. S. Khuller, B. Raghavachari and A. Rosenfeld, *Landmarks in graphs*, Discrete Appl. Math. **70** (3) (1996) 203–236.
10. P. J. Slater, *Leaves of trees*, In Proc. 6th Southeastern Conf. on Combinatorics, Graph Theory, and Computing, Congr. Numer. **14** (1975) 549–559.


E-mail: mjannesari@shahreza.ac.ir

# Inequalities on Energy of Graphs and Matrices

Mohammad Reza Oboudi*
Department of Mathematics, Shiraz University, Shiraz, Iran

Abstract. Let $D$ be a symmetric matrix. The energy of $D$ is defined as the sum of the absolute values of its eigenvalues. In addition, the energy of a simple graph $G$ is defined as the energy of the adjacency matrix of $G$. We study the energy of matrices, in particular the energy of graphs, and obtain some inequalities for them.
**Keywords:** Energy of graphs, Energy of matrices.
**AMS Mathematical Subject Classification [2010]:** 05C31, 05C50, 15A18.

## 1. Introduction

In this paper the matrices are complex and the graphs are simple. Let $D$ be a square complex matrix. The *trace* and the *determinant* of $B$ are denoted by $tr(B)$ and $det(B)$, respectively. The *energy* of $B$, $\mathcal{E}(B)$, is the sum of the absolute values of its eigenvalues. In other words, if $D$ is an $n \times n$ complex matrix with eigenvalues $\mu_1, \ldots, \mu_n$, then

$$\mathcal{E}(B) = |\mu_1| + \cdots + |\mu_n|.$$

For a graph $G$ with vertices $v_1, \ldots, v_n$, the *adjacency matrix* of $G$, $A(G)$, is the matrix where its $(i, j)$-entry is equal to 1 if $v_i$ and $v_j$ are adjacent, and is 0 otherwise. Since this matrix is symmetric, all of its eigenvalues are real. We call these eigenvalues as the eigenvalues of the graph. There are very kind of associated matrices to graphs and so many kind of energy for graphs, such as Laplacian energy, Seidel energy, distance energy, generalized distance energy, Seidel signless Laplacian energy and minimum covering Seidel energy of graphs. But the most and well known kind of them is the energy of graph as define follows: The *energy* of $G$,$\mathcal{E}(G)$, is defined as the energy of the adjacency matrix of $G$. For more details, see [1]–[12] and the references therein. Here we obtain some bounds for these kind of energy.

For example for two special family of graphs, such as the complete graphs and the complete multipartite graphs it is easy to check the following: The first is that $\mathcal{E}(K_n) = 2n - 2$ ( in fact the eigenvalues of the adjacency matrix of $K_n$ are $n - 1$ and $-1, \ldots, -1$ ($n - 1$ times) and the second is that $\mathcal{E}(K_{m,n}) = 2\sqrt{mn}$, because the eigenvalues of the adjacency matrix of the complete bipartite graph $K_{m,n}$ are $\sqrt{mn}$, $0, \ldots, 0$ ($m + n - 2$ times) and $-\sqrt{mn}$. For more details on energy and spectra of graphs we refer to [1]–[12] and the references therein. In this paper we obtain some bounds for energy of complex matrices and energy of graphs.

*Presenter

## 2. Main Results

Here we find some results related to the energy of matrices and the energy of graphs. We recall some famous inequalities. One of them is related to the a relation between arithmetic mean value and geometric mean value.

THEOREM 2.1. *Let $t \geq 2$ and $y_1 \geq 0, y_2 \geq 0, \ldots, y_t \geq 0$ be some real numbers. suppose that*

$$X = \frac{y_1 + y_2 + \cdots + y_t}{t} \text{ and } Y = \sqrt[t]{y_1 y_2 \ldots y_t}.$$

*In other words, $X$ is the arithmetic average and $Y$ is the geometric average of $y_1, y_2, \ldots, y_t$, respectively. Then $X \geq Y$ and the equality holds if and only if $y_1 = y_2 = \cdots = y_t$.*

The first result is about a relation between energy of a matrix and its determinant.

THEOREM 2.2. *Let $n \geq 3$ and $A$ be a complex matrix with eigenvalues $\mu_1, \mu_2, \ldots,$ $\mu_n$. Then*

$$\mathcal{E}(A) \geq \frac{2}{n-2} \sum_{1 \leq i < j \leq n} \sqrt{|\lambda_i \lambda_j|} - \frac{n}{n-2} \sqrt[n]{|det(A)|}.$$

THEOREM 2.3. *Let $A$ be a square real symmetric matrix such that $tr(A) = 0$. Assume that $A$ has at least two positive eigenvalues and $\lambda_1, \lambda_2, \ldots, \lambda_p$ are all positive eigenvalues of $A$. Then*

$$\mathcal{E}(A) \geq \frac{4}{p-1} \sum_{1 \leq i < j \leq p} \sqrt{\lambda_i \lambda_j},$$

*and the equality holds if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_p$.*

In the next result we find some bounds for the square root of the energy of matrices with trace zero in terms of the square root of their positive eigenvalues.

THEOREM 2.4. *Let $A \neq 0$ be a square real symmetric matrix such that $tr(A) = 0$. Assume that $\lambda_1, \ldots, \lambda_p$ are all positive eigenvalues of $A$. Then*

$$\sqrt{2}\Big(\sqrt{\lambda_1} + \cdots + \sqrt{\lambda_p}\Big) \geq \sqrt{\mathcal{E}(A)} \geq \sqrt{\frac{2}{p}}\left(\sqrt{\lambda_1} + \cdots + \sqrt{\lambda_p}\right).$$

*Moreover in the left hand side the equality holds if and only if $p = 1$ and in the right hand side the equality holds if and only if $p = 1$ or $p \geq 2$ and $\lambda_1 = \cdots = \lambda_p$.*

As some applications of above theorems we find that:

THEOREM 2.5. *Let $G$ be a graph of order $n \geq 3$. Assume that $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of $G$. Then*

$$\mathcal{E}(G) \geq \frac{2}{n-2} \sum_{1 \leq i < j \leq n} \sqrt{|\mu_i \mu_j|} - \frac{n}{n-2} \sqrt[n]{|det(G)|}.$$

THEOREM 2.6. *Let $G$ be a connected graph of order $n$. Assume that $G$ has at least two positive eigenvalues and $\mu_1, \mu_2, \ldots, \mu_p$ are all positive eigenvalues of $G$. Then*

$$\mathcal{E}(G) > \frac{4}{p-1} \sum_{1 \le i < j \le p} \sqrt{\mu_i \mu_j}.$$

**Acknowledgement**

This research was in part supported by Iran National Science Foundation (INSF) under the contract No. 98001945.

## References

1. A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*, Springer, New York, 2012.
2. D. M. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs Theory and Application*, Academic Press, New York, 1979.
3. D. M. Cvetković, P. Rowlinson and S. Simić, *An Introduction to the Theory of Graph Spectra*, London Mathematical Society Student Texts, Vol. 75, Cambridge University Press, Cambridge, 2010.
4. I. Gutman, *Bounds for total $\pi$-electron energy,* Chem. Phys. Lett. **24** (1974) 283–285.
5. I. Gutman, *Oboudi-type bounds for graph energy,* Math. Interdisc. Res. **4** (2019) 151–155.
6. I. Gutman, *The energy of a graph,* Ber. Math. Statist. Sekt. Forsch. Graz. **103** (1978) 1–22.
7. V. Nikiforov, *The energy of graphs and matrices,* J. Math. Anal. Appl. **326** (2007) 1472–1475.
8. M. R. Oboudi, *A new lower bound for the energy of graphs*, Linear Algebra Appl. **580** (2019) 384–395.
9. M. R. Oboudi, *Characterization of graphs with exactly two non-negative eigenvalues*, Ars Math. Contemp. **12** (2017) 271–286.
10. M. R. Oboudi, *Energy and Seidel energy of graphs*, MATCH Commun. Math. Comput. Chem. **75** (2016) 291–303.
11. M. R. Oboudi, *On the difference between the spectral radius and maximum degree of graphs*, Algebra Discrete Math. **24** (2017) 302–307.
12. J. H. Smith, *Symmetry and multiple eigenvalues of graphs,* Glas. Mat. Jugosl. **12** (32) (1977) 3–8.

E-mail: mr_oboudi@yahoo.com; mr_oboudi@shirazu.ac.ir

# The Forgotten Coindex of Several Random Models

Ramin Kazemi*
Faculty of Sciences, Imam Khomeini International University, Qazvin, Iran

Abstract. The forgotten coindex of a graph $G$ is defined as

$$\overline{F}(G) = \sum_{uv \notin E(G)} [deg(u)^2 + deg(v)^2],$$

where $deg(u)$ is the degree of the vertex $u$ of $G$. In this article, we investigate the forgotten coindex of several random models, including random recursive trees, random heap-ordered trees, and random $d$-ary increasing trees.
**Keywords:** Forgotten coindex, Random trees, Mean.
**AMS Mathematical Subject Classification [2010]:** 05C05, 60F05.

## 1. Introduction

Let $G$ be a simple connected graph. The vertex set and the edge set of graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. Two vertices in $G$ which are connected by an edge are called adjacent vertices. The number of vertices adjacent to a given vertex $v$ is the degree of $v$ and is denoted by $deg(v)$.

A topological index for a (chemical) graph $G$ is a numerical quantity invariant under automorphisms of $G$. Topological indices and graph invariants based on the vertex degrees are widely used for characterizing molecular graphs, establishing relationships between structure and properties of molecules, predicting biological activity of chemical compounds, and making their chemical applications. The first Zagreb index of $G$ is defined as

$$M_1(G) = \sum_{v \in V(G)} deg(v)^2.$$

Followed by $M_1$, Furtula and Gutman [2] introduced forgotten topological index (also called F-index) which was defined as

$$F(G) = \sum_{v \in V(G)} deg(v)^3.$$

They [2] raised that the predictive ability of this index is almost similar to that of first Zagreb index and for the acentric factor and entropy, and both of them obtain correlation coefficients larger than 0.95. De et al. [1] and Khaksari et al. [6], separately introduced the forgotten coindex of a graph $G$ as

$$\overline{F}(G) = \sum_{uv \notin E(G)} [deg(u)^2 + deg(v)^2].$$

*Presenter

Equivalently,

$$\overline{F}(G) = \sum_{u \in V(G)} \overline{deg}(u) deg(u)^2,$$

where $\overline{deg}(u)$ denotes the degree of the vertex $u$ in the complement of $G$.

Here, we give the mean of the forgotten coindex of several random tree models, including random recursive trees, random heap-ordered trees, and random $d$-ary increasing trees.

## 2. Random Models

There are several tree models, namely so called recursive trees (RT), $d$-ary increasing trees (DIT) and heap-ordered trees (HOT), which turned out to be appropriate in order to describe the behavior of a lot of quantities in various applications. All above families can be considered as increasing trees (labelled trees labelled by distinct integers of the set $\{1, 2, ..., n\}$ such that each sequence of labels along any path starting at the root is increasing) [7].

**2.1. Random Recursive Trees.** Every order-$n$ recursive tree can be obtained uniquely by attaching $n$th vertex to one of the $n-1$ vertices in a tree of order $n-1$. We assume that all trees of order $n$ are considered to appear equally likely (random tree). At step 1 the process starts with the root. At step $i$ the $i$th vertex is attached to any previous vertex $v$ of the already grown tree $T$ of order $i-1$ with probability $p_i(v) = \frac{1}{i-1}$.

**2.2. Random $d$-Ary Increasing Trees.** For any fixed integer $d \geq 2$, each vertex has no more than $d$ children. In a $d$-ary increasing tree, the number of nodes can be attached to node $v$ of out-degree $odeg(v)$ is $d - odeg(v)$ (For a vertex, the number of tail ends adjacent to a vertex is its outdegree and is denoted by $odeg(v)$). At step 1 the process starts with the root. At step $i$ the $i$th vertex is attached to a previous node $v$ of the already grown $d$-ary increasing tree $T$ of order $i-1$ with probability $p_i(v) = \frac{d-odeg(v)}{(d-1)(i-1)+1}$.

**2.3. Random Heap-Ordered Trees.** The process starts with the root labelled by 1. At step $i+1$ the node with label $i+1$ is attached to any previous node $v$ (with degree $deg(v)$) of the already grown heap-ordered tree of order $i$ with probability $p(v) = \frac{deg(v)}{2i-1}$. Let $M^H_{1,n}$ and $F^H_n$ be the first Zagreb index and forgotten topological index of a random heap-ordered tree, respectively. Also, let $\overline{F}^H_n$ be its forgotten coindex.

**2.4. Equalities.** Note that $\overline{deg}(u) = n - 1 - deg(u)$ which implies that

$$\overline{F}(G) = (n-1)M_1(G) - F(G). \tag{1}$$

Let $M_{1,n}$ and $F_n$ be the first Zagreb index and forgotten topological index of a random tree, respectively. Also, let $\overline{F}_n$ be its forgotten coindex. Let $\mathcal{F}_n$ be the sigma-field generated by the first $n$ stages of these trees. Let $U_n$ be a randomly chosen vertex belonging to a random tree of order $n$. Then,

$$M_{1,n}|\mathcal{F}_{n-1} = M_{1,n-1} + 2d_{U_{n-1}} + 2|\mathcal{F}_{n-1}, \tag{2}$$

$$F_n|\mathcal{F}_{n-1} = F_{n-1} + 3d^2_{U_{n-1}} + 3d_{U_{n-1}} + 2|\mathcal{F}_{n-1}, \tag{3}$$

$$\overline{F}_n|\mathcal{F}_{n-1} = (n-1)M_{1,n}|\mathcal{F}_{n-1} - F_n|\mathcal{F}_{n-1}. \tag{4}$$

## 3. Main Results

Let $M^R_{1,n}$ and $F^R_n$ be the first Zagreb index and forgotten topological index of a random recursive tree, respectively. Also, let $\overline{F}^R_n$ be its forgotten coindex.

THEOREM 3.1. *For a random recursive tree of order $n$,*

$$\mathbb{E}(\overline{F}^R_n) = (n-1)(6n-32) - 4H_{n-1}(n-7) + 6H^2_{n-1} - 6H^{(2)}_{n-1},$$

*where $H_n$ and $H^{(2)}_n$ are the $n$-th harmonic number of order 1 and 2, respectively.*

PROOF. From (2), $\mathbb{E}(M^R_{1,n}) = (n-1)6 - 4H_{n-1}$ since $p_i(v) = \frac{1}{i-1}$. Also, from (3), $\mathbb{E}(F^R_n) = 26(n-1) - 24H_{n-1} - 6H^2_{n-1} + 6H^{(2)}_{n-1}$. Proof is completed by relation (1) and (4) [4]. □

Let $M^D_{1,n}$ and $F^D_n$ be the first Zagreb index and forgotten topological index of a random $d$-ary increasing tree, respectively. Also, let $\overline{F}^D_n$ be its forgotten coindex. For each $n, d \geq 2$, we assume that $q_n = n(d-1)+1$ and define

$$\beta_{n,i} = \frac{\Gamma\left(\frac{nd-n+1}{d-1}\right)}{\Gamma\left(\frac{nd-n+1-i}{d-1}\right)}, \ i \geq 1,$$

where $\Gamma(\cdot)$ is the gamma function. For each $n, d \geq 2$ define

$$\alpha_{n,d} = \frac{2d(n-1)}{q_n} + 1,$$

$$\sigma_{n,d} = \frac{3(d-1)}{q_n}\left(\frac{1}{\beta_{n,2}}\sum_{i=1}^{n-1}\beta_{i+1,2}\alpha_{i,d} - \frac{2}{\beta_{n,1}}\sum_{i=1}^{n-1}\beta_{i+1,1}\frac{d}{q_i} + 3(n-1)\right) + \frac{3}{2}\alpha_{n,d} + 1.$$

THEOREM 3.2. *For each $d$-ary increasing tree of order $n$, we have*

$$\mathbb{E}(\overline{F}^D_n) = \frac{n-1}{\beta_{n,2}}\sum_{i=1}^{n-1}\beta_{i+1,2}\alpha_{i,d} - \frac{2(n-1)}{\beta_{n,1}}\sum_{i=1}^{n-1}\beta_{i+1,1}\frac{d}{q_i} + 3(n-1)^2 - 4(n-1)$$
$$- \sum_{i=1}^{n-1}\left(\frac{\beta_{i+1,3}\sigma_{i,d}}{\beta_{n,3}} + \frac{3\beta_{i+1,2}\left(\alpha_{i,d} - \frac{2d-1}{q_i}\frac{1}{\beta_{i,1}}\sum_{j=1}^{i-1}\beta_{j+1,1}\frac{d}{q_j} + \frac{d}{q_i}\right)}{\beta_{n,2}} - \frac{3\beta_{i+1,1}\frac{d}{q_i}}{\beta_{n,1}}\right).$$

PROOF. From (2), $\mathbb{E}(M^D_{1,n}) = \frac{1}{\beta_{n,2}}\sum_{i=1}^{n-1}\beta_{i+1,2}\alpha_{i,d} - \frac{2}{\beta_{n,1}}\sum_{i=1}^{n-1}\beta_{i+1,1}\frac{d}{q_i} + 3(n-1)$, since $p_i(v) = \frac{d-odeg(v)}{(d-1)(i-1)+1}$. Also, from (3),

$$\mathbb{E}(F^D_n) = \sum_{i=1}^{n-1}\left(\frac{\beta_{i+1,3}\sigma_{i,d}}{\beta_{n,3}} + \frac{3\beta_{i+1,2}(\alpha_{i,d} - \eta_{i,d})}{\beta_{n,2}} - \frac{3\beta_{i+1,1}\frac{d}{q_i}}{\beta_{n,1}}\right) + 4(n-1).$$

Proof is completed by relation (1) and (4) [4]. □

Suppose that

$$c(n,j,i) := \frac{\Gamma\left(\frac{2n+3+i}{2}\right)}{\Gamma\left(\frac{2n+3-j}{2}\right)}, \ n \geq 3, \ i,j \geq 1.$$

THEOREM 3.3. *For a random heap-ordered tree of order n,*

$$\mathbb{E}(\overline{F}_n^H) = 2(n-1)c(n-1,2,0)\sum_{t=1}^{n-1}\frac{1}{c(t,2,0)} - c(n-1,2,1)\sum_{t=1}^{n-1}\frac{\frac{3}{2t-1}2c(t-1,2,0)\sum_{j=1}^{t-1}\frac{1}{c(j,2,0)}+2}{c(t,2,1)}.$$

PROOF. From (2),

$$\mathbb{E}(M_{1,n}^H) = 2c(n-1,2,0)\sum_{t=1}^{n-1}\frac{1}{c(t,2,0)},$$

since $p_i(v) = \frac{deg(v)}{2i-1}$. Also, from (3),

$$\mathbb{E}(F_n^H) = c(n-1,2,1)\sum_{t=1}^{n-1}\frac{\frac{3}{2t-1}2c(t-1,2,0)\sum_{j=1}^{t-1}\frac{1}{c(j,2,0)}+2}{c(t,2,1)}.$$

Proof is completed by relation (1) and (4) [5]. □

## References

1. N. De, S. M. A. Nayeem and A. Pal, *The F-coindex of some graph operations*, SpringerPlus **5** (2016) 221.
2. B. Furtula and I. Gutman, *A forgotten topological index*, J. Math. Chem. **53** (4) (2015) 1184–1190.
3. I. Gutman and N. Trinajstić, *Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons*, Chem. Phys. Lett. **17** (1972) 535–538.
4. R. Kazemi and A. Behtoei, *The first Zagreb and forgotten topological indices of d-ary trees*, Hacet. J. Math. Stat. **46**(4) (2017) 603–611.
5. R. Kazemi, *Gordon-Scantlebury and Platt indices of random plane-oriented recursive trees*, Math. Interdisc. Res. (2020) inpress.
6. A. Khaksari and M. Ghorbani, *On the forgotten topological index*, Iranian J. Math. Chem. **8** (2017) 327–338.
7. M. Kuba and A. Panholzer, *On the distribution of distances between specified nodes in increasing trees*, Discrete Appl. Math. **158** (5) (2010) 489–506.

E-mail: r.kazemi@sci.ikiu.ac.ir

# Binary Words and Majorization

Farzad Shaveisi*
Department of Mathematics, Faculty of Science, Razi University, Kermanshah, Iran

Abstract. The majorization graph of binary words, denoted by $\mathcal{MG}_n$, is a graph whose vertex set is the set of all non-trivial bianary words with length $n$ and two distinct vertices are adjacent if one of them majorizes the other one. Here, the connectivity and weakly perfecness of $\mathcal{MG}_n$ are studied and graph parameters such as girth, clique and chromatic numbers are determined.
**Keywords:** Majorization graph, Binary word, Weight.
**AMS Mathematical Subject Classification [2010]:** 68Q87, 05C30, 05C15.

## 1. Introduction

We begin with recalling some definitions and notations on graphs. Throughout this paper, a graph $G$ is an undirected simple graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. A graph is said to be *connected* if there exists a path between any two distinct vertices. The *diameter* of a connected graph $G$, denoted by $diam(G)$, is the maximum distance between any pair of vertices of $G$. For disconnected graphs, the diameter is defined to be $\infty$. The girth of $G$ which is the length of a shortest cycle is denoted by $girth(G)$. In directed graphs, we distinguish the *out-degree* $d^+(v)$, the number of edges leaving the vertex $v$, and the *in-degree* $d^-(v)$, the number of edges entering the vertex $v$. The degree of any vertex $v$ equals $d^+(v) + d^-(v)$ is denoted by $d(v)$; maximum and minimum degrees are denoted by $\Delta$ and $\delta$, respectively. Moreover, in this paper, we use the notations $\omega(G), \chi(G)$ and $\chi'(G)$ for the *clique number*, *vertex chromatic number* and *edge chromatic number*, respectively. A cycle with length $n$ is denoted by $C_n$. Every graph with no cycle is called a *forest*. Moreovere, we denote the *induced subgraph* on $X \subset V(G)$, by $G[X]$. A *bipartite graph* is a graph whose vertex set can be divided into two disjoint parts $X$ and $Y$ such that both of the induced subgraphs $G[X]$ and $G[Y]$ have no edges. Moreover, a *complete bipartite graph* is a bipartite graph in which every vertex of one part is joined to every vertex of the other part. If the size of one of the parts is 1, then it is said to be a *star graph*. For undefined terminologies the reader is referred to [1] and [2].

Let $u = u_1u_2 \dots u_n$ and $v = v_1v_2 \dots v_n$ be two distinct binary words. We say $v$ *majorizes* $u$ and write $u \preceq v$ if $u_i \leq v_i$, for every $i$, $1 \leq i \leq n$. The *majorization graph* of binary words, denoted by $\mathcal{MG}_n$, is a simple graph whose vertex set is the set of all non-trivial bianary words with length $n$ except $\mathbf{1}$ and $\mathbf{0}$ and to distinct vertices $u, v$ are adjacent if either $u \preceq v$ or $v \preceq u$. Here, by $\mathbf{1}$ ($\mathbf{0}$), we mean the all ones (zeros) binary word.

*Presenter

In Section 2, it is shown that the majorization graph $\mathcal{MG}_n$ is connected for every $n \geq 3$ with diameter at most 5; its girth is at most 6. Also, we show that paths, stars and cycles don't happen as majorization graphs, except $C_6$.

In Section 3, degrees of vertices are computed for finding the size (number of edges) of the graph.

Finally, in Section 4, it is shown that the clique and (vertex) chromatic number of $\mathcal{MG}_n$, both are $n-1$ which implies the weakly perfectness of graph; moreover, we prove that the majorization graph of binary words is of Class one.

## 2. The Majorization Graph and its Connectivity

We start by the following main definition.

Definition 2.1. Let $n$ be a positive integer. The majorization digraph of binary words, denoted by $\overrightarrow{\mathcal{MG}_n}$ is a directed graph whose vertex set is the set of all binary words (sequences) of length $n$ except the words $\mathbf{0}$, $\mathbf{1}$ and for any two distinct vertices $v, w \in \{0,1\}^n$, there is an arc from $v$ to $w$ if $v$ majorizes $w$. Also, the underlying graph is called the majorization graph and it is denoted by $\mathcal{MG}_n$

Lemma 2.2. *The graph $\overrightarrow{\mathcal{MG}_n}$ contains at least one arc if and only if $n \geq 3$.*

Theorem 2.3. *For any positive integer $n \geq 3$, $\mathcal{MG}_n$ is a connected graph whose diameter and girth are at most 5 and 6, respectively.*

The following proposition shows that paths and stars are not majorization graphs of binary words.

Proposition 2.4. *The majorization graph $\mathcal{MG}_n$ is neither path nor star graph.*

Now, from the previous results, we can deduce the following immediate corollary.

Proposition 2.5. *The only cycle which can be a majorization graph of binary words is $\mathcal{MG}_3 \cong C_6$.*

## 3. Degrees of the Vertices and Counting the Edges

In this section, the degree of any vertex of the majorization graph of binary words of length $n$ is determined; moreovere, a formula for the number of edges of this graph is presented. Recall that the *weight* of a binary word $b$, $wt(b)$, is the number of bits of b equal to 1.

Proposition 3.1. *Let $n \geq 3$ and $b$ be a vertex of $\mathcal{MG}_n$. Then about the degree of vertices in graph $\mathcal{MG}_n$, we have*

i) $d^+(b) = 2^{wt(b)} - 2$; $d^-(b) = 2^{n-wt(b)} - 2$.
ii) $d(b) = 2^{wt(b)} + 2^{n-wt(b)} - 4$.
iii) $\Delta = 2^{n-1} - 2$; $\delta = 2^{\lceil \frac{n}{2} \rceil} + 2^{\lfloor \frac{n}{2} \rfloor} - 4$.

In the classical graph theory, the handshaking lemma states that the number of edges in a simple graph equals the sum of its degrees. By using the previous proposition, handshaking lemma and a simple computation, one can prove the following result.

Theorem 3.2. *The majorization graph $\mathcal{MG}_n$ has $3(3^{n-1} - 2^n + 1)$ edges.*

## 4. The Coloring of Majorization Graph

In this section, the clique number, the vertex chromatic number and the edge chromatic number of the majorization graph of binary words are determined and it is shown that these parameters are depend only on the length of binary words, considered.

Recall that in any graph $G$, the clique number of $G$ does not exceed its vertex chromatic number. A graph $G$ is said to be *weakly perfect* if $\omega(G) = \chi(G)$.

THEOREM 4.1. *For any positive integer* $n \geq 3$, $\mathcal{MG}_n$ *is a weakly perfect graph whose clique number is* $n-1$.

Vizing's Theorem (See [3, p. 16]) states that if $G$ is a simple graph, then either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$. In the first case it is said that $G$ is of Class 1 and in the second case, the graph is called Class 2. Here, it is shown that all majorization graphs of binary words are Class 1 graphs. First of all, we recall the following lemma.

LEMMA 4.2. [1, Corollary 5.4] *Let $G$ be a simple graph. Suppose that for every vertex $u$ of maximum degree, there exists an edge $\{u, v\}$ such that $\Delta(G) - d(v) + 2$ is more than the number of vertices with maximum degree in $G$. Then $\chi'(G) = \Delta(G)$.*

THEOREM 4.3. *For any positive integer* $n \geq 3$*, the graph* $\mathcal{MG}_n$ *is Class* 1.

Finally, from the previous theorem and Proposition 3.1, we obtain the following immediate corollary.

COROLLARY 4.4. *For any positive integer* $n \geq 3$, $\chi'(\mathcal{MG}_n) = 2^{n-1} - 2$.

## References

1. L. W. Beineke and B. J. Wilson, *Selected Topics in Graph Theory*, Academic Press Inc., London, 1978.
2. J. A. Bondy and U. S. R. Murty, *Graph Theory*, Graduate Texts in Mathematics, Vol. 244, Springer, New York, 2008.
3. H. P. Yap, *Some Topics in Graph Theory*, London Mathematical Society Lecture Note Series, Vol. 108, Cambridge University Press, Cambridge, 1986.

E-mail: f.shaveisi@razi.ac.ir

# A Novel Method for Finding PI Index of Polyomino Chains and its Extremals

Zahra Yarahmadi*
Department of Mathematics, Faculty of Science, Khorramabad Branch, Islamic Azad University, Khorramabad, I. R. Iran

Abstract. The PI index of a graph $G$ is the sum of the number of edges which are not equidistant to $u$ and $v$. In this paper the PI index of polyomino chains by different method is computed. Then first, second extremal of polyomino chains with respect to the PI index are also determined.
**Keywords:** PI index, Polyomino chain.
**AMS Mathematical Subject Classification [2010]:** 92E10, 05C35.

## 1. Introduction and Preliminaries

A graph $G$ consists of a set of vertices $V(G)$ and a set of edges $E(G)$. If the vertices $u, v \in V(G)$ are connected by an edge $e$ then we write $e = uv$. We will write $|G|$ and $\|G\|$ for the number of vertices and edges of $G$, respectively. A topological index is a numerical quantity related to a graph which is invariant under graph automorphisms. Let $Top(G)$ be a topological index of a graph $G$, for every graph $H$ isomorphic to $G$, we have $Top(G) = Top(H)$. The Wiener index is one of the oldest and most studied topological indices, see [5]. Another topological index was introduced in [1, 2] and named it Padmakar-Ivan index. They abbreviated this new topological index as PI. Let $G$ be a simple connected graph. The PI index of graph G is defined as follows:

$$PI(G) = \sum_{e=uv\in E(G)} [m_u(e|G) + m_v(e|G)],$$

where for edge $e = uv$, $m_u(e|G)$ is the number of edges of $G$ lying closer to $u$ than $v$, $m_v(e|G)$ is the number of edges of $G$ lying closer to $u$ than $v$ and summation goes over all edges of $G$. The edges equidistant from $u$ and $v$ are not consider for the calculation of PI index. In [6], authors obtained PI index of this class of graphs. In this paper, we recalculate the PI index of polyomino chains of by different method. In addition, we determine upper and lower bounds for PI index, this method is able to obtain second extremal polyomino chains with respect to PI index. Let $G$ be a graph and $X \subseteq V(G)$. The subgraph of $G$ induced by $X$ will be denoted by $\langle X\rangle$.

For an edge $e = uv$ of a graph $G$ set,

$$G_u(e) = \{x \in V(G)|\ d_G(x, u) < d_G(x, v)\},$$
$$G_v(e) = \{x \in V(G)|\ d_G(x, v) < d_G(x, u)\}.$$

*Presenter

It is easy to see, $G_u(e)$ is the set of vertices closer to $u$ than to $v$ while $G_v(e)$ consists of those vertices that are closer to $v$. Note that the roles of $G_u(e)$ and $G_v(e)$ would be interchanged if the edge $e$ would be considered as $e = vu$. Since these two sets will always be considered in pairs, this imprecision in the definition will cause no problem. Observe that if $G$ is bipartite then for any edge $e$ of $G$, $G_u(e)$ and $G_v(e)$ form a partition of $V(G)$. If $G$ is bipartite graph, then the number of edges in the subgraph of $G$ induced by $G_u(e)$ ($G_v(e)$) is equal to $m_u(e|G)$ ($m_v(e|G)$). Now, the $PI$ index of $G$ is defined as:

$$PI(G) = \sum_{e=uv\in E(G)} \big[\|\langle G_u(e)\rangle\| + \|\langle G_v(e)\rangle\|\big].$$

Let $G$ be a graph, then we say that a partition $E_1, \ldots, E_t$ of $E(G)$ is a $PI-$partition of $G$ if for any $i$, $1 \leqslant i \leqslant t$ , and for any $e, f \in E_i$, we have $G_u(e) = G_u(f)$ and $G_v(e) = G_v(f)$.

Let $E_1, \ldots, E_t$ be a PI-partition of a bipartite graph. It is called an ordered PI-partition, when for each $1 \leqslant i, j \leqslant t$, such that $i \leq j$, then $|E_i| \leq |E_j|$. Now suppose that $E_1, \ldots, E_t$ be an ordered PI-partition of a bipartite graph, we introduce the PI-partition sequence as $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t$, such that $\varepsilon_i = |E_i|$ for all $1 \leqslant i \leqslant t$.

In what follows, we use the method which Klavžar mentioned in [4]. By this method, it is possible to obtain first, second of polyomino chains with respect to the PI index.

Lemma 1.1. [4] *Let $E_1, \ldots, E_t$ be a $PI-$partition of a bipartite graph $G$. Then*

$$PI(G) = |E(G)|^2 - \sum_{i=1}^{t} |E_i|^2.$$

Now we recall some concept that will be used in this paper. A $k-$polyomino system is a finite $2-$connected plane graph such that each interior face (also called cells is surrounded by a regular $4k-$cycle of length one. In other words, it is an edge-connected union of cells. A $k-$polyomino system with $n$ cells is denoted by $B_{n,k}$ For the origin of polyominoes see [3].

Lemma 1.2. *For any $k-$polyomino $B_{n,k}$, the number of vertices and edges are computed as follows:*

$$|V(B_{n,k})| = (4k-2)n + 2,$$
$$|E(B_{n,k})| = (4k-1)n + 1.$$

Let $E_1, \ldots, E_t$ be a PI-partition of a $k-$polyomino chain $B_{n,k}$ with $n$ cells, it is easy to see that $t = (2k-1)n + 1$ and $\sum_{i=1}^{t} |E_i| = |E(B_{n,k})|$. Also if $E_1, \ldots, E_t$ is a PI-partition of a $k-$polyomino chain $B_{n,k}$. Then one can see that for each $E_i$, $1 \leq i \leq t$ there is $e_i \in E(B_{n,k})$ such that $E_i = \{e \in E(B) | \, e \| e_i \}$. In Figure 1, PI-partition of a $k-$polyomino system is marked by dashed lines.

For calculating the PI index of a k-polyomino chain, we introduce some concepts. The linear chain $L_{n,k}$ of $k-$polyomino with $n$ cells is the $k-$polyomino chain with the PI-partition sequence as $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t$, such that $\varepsilon_i = 2$ for $1 \leq i \leq t-1$ and $\varepsilon_t = n + 1$, In Figure 2, the linear chain $L_{n,2}$ and $L_{n,1}$ are shown.

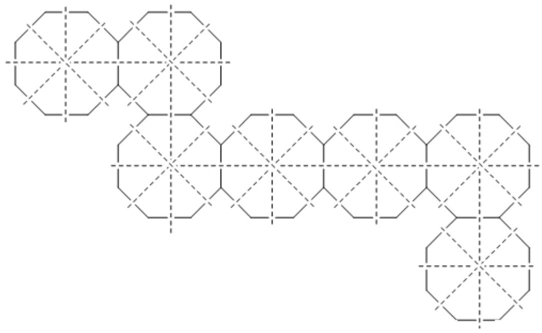


FIGURE 1. PI-partition of a $k-$polyomino.

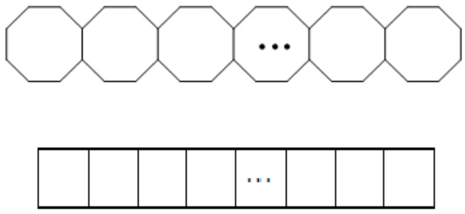


FIGURE 2. The linear chain $L_{n,2}$ and $L_{n,1}$.

A zigzag chain $Z_{n,k}$ of $k-$polyomino with $n$ cells is the $k-$polyomino chain with the PI-partition sequence as $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t$, such that $\varepsilon_i = 2$ for $1 \leq i \leq t-n+1$ and $\varepsilon_i = 3$, for $t+n-1 \leq i \leq t$, see Figure 3, for $Z_{6,2}$, $Z_{7,2}, Z_{6,1}$, $Z_{7,1}$.

A segment of a $k-$polyomino chain is a maximal linear chain in the polyomino chain, including the kinks and/or terminal $4k-$cycles at its end. The number of $4k-$cycles in a segment $S$ is called its length and is denoted by $l(S)$. For any segment $S$ of a polyomino chain with $n$, $4k-$cycles one has $1 \leq l(S) \leq n$.
In this paper, we study on $k-$polyomino chain when, $k = 1$ and call them polyomino chain. In polyomino chain, each interior face (or say a cell) is surrounded by a regular square. We denote $B_{n,1}$, $L_{n,1}$ and $Z_{n,1}$ by $B_n$, $L_n$ and $Z_n$ respectively. Moreover, $V(B_n)| = 2n + 2$, and $|E(B_n)| = 3n + 1$.

## 2. Main Results

The result of the following theorem was obtained by Xu and Chen in [6], but we have proved it by interesting method in this article. Moreover, we have been able to obtain the first and second extremals with this new method.

THEOREM 2.1. *Let $B_n$ be a polyomino chain with $n$ squares and consisting of $r$ segments $S_1, S_2, \ldots, S_r$, $(r \geq 1)$ with lengths $l_1, l_2, \ldots, l_r$. Then*

$$PI(B_n) = 9n^2 + r - 1 - \sum_{i=1}^{r} l_i^2.$$

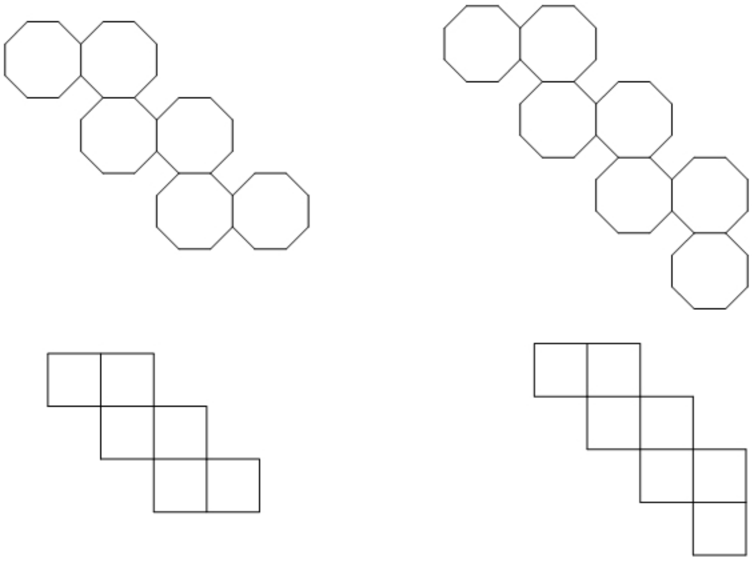


FIGURE 3. The zigzag chain $Z_{6,2}$ and $Z_{7,2}$, $Z_{6,1}$, $Z_{7,1}$.

Particularly, for a linear chain $L_n$ with $n$ squares, we have $r = 1$ and $l_1 = n$. For a zigzag chain $Z_n$ with $n$ squares, $n$ is even and we have $r = \frac{n}{2}$ and $l_i = 2$ for $i = 1, \ldots, \frac{n}{2}$. Let $\mathbf{B_n}$ be the set of all polyomino chains with $n$ squares. For odd number $n$, denote $\widehat{\mathbf{Z}}_\mathbf{n}$, be the subset of $\mathbf{B_n}$ contains all polyomino chains with $[\frac{n-1}{2}]$ segments such that one of the segments has the length 3 and another segments are the length 2, obviously $|\widehat{\mathbf{Z}}_\mathbf{n}| = [\frac{n-1}{2}]$. We call the elements of $\widehat{\mathbf{Z}}_\mathbf{n}$, semi zigzag chain.

COROLLARY 2.2. *The PI index of linear chain, zigzag chain and semi zigzag chain are computed as follows:*

i) $PI(L_n) = 8n^2$,
ii) $PI(Z_n) = 9n^2 - 3n + 2$,
iii) $PI(\widehat{Z}_n) = 9n^2 - 3n$, *for all* $\widehat{Z}_n \in \widehat{\mathbf{Z}}_\mathbf{n}$.

In the following theorem upper and lower bound are obtained and first extremal polyomino chains are determined.

THEOREM 2.3. *For any polyomino chain $B_n$ with $n$ squares,*

i) $PI(L_n) \leq PI(B_n)$.
ii) *If $n$ is even, then $PI(B_n) \leq PI(Z_n)$, and the equality holds if and only if $B_n = Z_n$.*
iii) *If $n$ is odd, then $PI(B_n) \leq PI(\widehat{Z}_n)$, for all $\widehat{Z}_n \in \widehat{\mathbf{Z}}_\mathbf{n}$. This bounds can be achieved if and only if there exists $\widehat{Z}_n \in \widehat{\mathbf{Z}}_\mathbf{n}$ such that $B_n = \widehat{Z}_n$.*

Now set denote $\mathbf{L'_n}$ be the subset of $\mathbf{B_n}$ contains all polyomino chains with two segments such that the length of one of them is 2 and the length of another is $n-2$, obviously $|\mathbf{L'_n}| = 2$ for $n \geq 4$. Now define $\mathbf{Z'_n}$ as a subset of $\mathbf{B_n}$ contains all polyomino chain with $\frac{n}{2} - 1$ segments such that there are $i, j$ such that $|l_i| = |l_j| = 3$

and the length of another stairs is 2 and $|\mathbf{Z'_n}| = \binom{\frac{n}{2}-1}{2}$. Also $\widehat{\mathbf{Z}'}_{\mathbf{n}}$ is be the subset of $\mathbf{B_n}$ contains all polyomino chains with $\frac{n-3}{2}$ stairs such that there are $i, j, k$ such that $|l_i| = |l_j| = |l_k| = 3$ and the length of another stairs is 2 and $|\widehat{\mathbf{Z}'}_{\mathbf{n}}| = \binom{\frac{n-3}{2}}{3}$.

Theorem 2.4. *Let $B_n \in \boldsymbol{B_n}$ The following statements are hold:*

i) *If $B_n \neq L_n$, then $PI(L'_n) \leq PI(B_n)$ and equality holds if and only if $B_n \in \boldsymbol{L'_n}$.*

ii) *If $B_n \neq Z_n$, then $PI(B_n) \leq PI(Z'_n)$ and equality holds if and only if $B_n \in \boldsymbol{Z'_n}$.*

iii) *$B_n \in \widehat{\boldsymbol{Z}'}_{\boldsymbol{n}}$, then $PI(B_n) \leq PI(\widehat{Z}'_n)$ and equality holds if and only if $B_n \in \widehat{\boldsymbol{Z}'}_{\boldsymbol{n}}$.*

## References

1. P. V. Khadikar, *On a novel structure descriptor PI*, Nat. Acad. Sci. Lett. **23** (2000) 113–118.
2. P. V. Khadikar, S. Karmarkar and V. K. Agrawal, *PI index of polyacenes and its use in developing QSPR*, Nat. Acad. Sci. Lett. **23** (2000) 124–128.
3. D. A. Klarner, *Polyominoes,* In: J. E. Goodman and J. O'Rourke (Eds.), Handbook of Discrete and Computational Geometry, CRC Press LLC, Boca Raton, Florida, 1997.
4. S. Klavžar, *On the PI index: PI-partitions and Cartesian product graphs*, MATCH Commun. Math. Comput. Chem. **57** (2007) 573–586.
5. H. Wiener, *Structural determination of the paraffin boiling points*, J. Am. Chem. Soc. **69** (1) (1947) 17–20.
6. L. Xu and S. Chen, *The PI index of polyomino chains*, Appl. Math. Lett. **21** (2008) 1101–1104.

E-mail: z.yarahmadi@gmail.com

# Contributed Poster

Logic

# A Note on Finite Version of the Thin Set Theorem

Amir Khamseh*
Department of Mathematics, Kharazmi University, Tehran, Iran

ABSTRACT. The Thin Set Theorem states that for all $f : \mathbb{N}^k \to \mathbb{N}$, there exists an infinite set $A \subseteq \mathbb{N}$ such that $f[A^k] \neq \mathbb{N}$. In this talk, we study the finite version of the Thin Set Theorem in Peano Arithmetic, PA. Moreover, we present some problems concerning this theorem.
**Keywords:** Ramsey theory, Peano arithmetic, Thin set theorem, Free set theorem.
**AMS Mathematical Subject Classification [2010]:** 03B30, 03F30, 03C62.

## 1. Introduction

We begin by recalling some notions and definitions. If $X$ is a set and $n$ is a natural number, then $[X]^n$ denotes the collection of subsets of $X$ of cardinality $n$. We will identify a natural number $n$ with the set $\{0, \ldots, n-1\}$. Also we shall use $\mathbb{N}$ to denote the set of natural numbers as well as its cardinality, in the arrow notation below. If $n$, $k$, and $c$ are either $\mathbb{N}$ or elements of $\mathbb{N}$, $X \to (k)^n_c$ means that whenever $f : [X]^n \to c$ there is $H \subseteq X$ with $|X| \geq k$ such that $f$ is constant on $[H]^n$. In this case we say that $H$ is *homogeneous* for $f$. Using these definitions we can state the Infinite Ramsey Theorem and its finite version as follows.

THEOREM 1.1. *For any* $n, c \in \mathbb{N}$, $\mathbb{N} \to (\mathbb{N})^n_c$.

THEOREM 1.2. *For any* $n, c, k \in \mathbb{N}$, *there is an* $m \in \mathbb{N}$ *such that* $m \to (k)^n_c$.

Note that Infinite Ramsey Theorem is a statement in the second-order language of arithmetic while Finite Ramsey Theorem can be formulated in the first-order language of arithmetic and is provable in Peano Arithmetic. For more information see [7]. Let us recall a variant of Theorem 1.2, the Paris-Harrington principle. A set $H \subseteq \mathbb{N}$ is called *relatively large* if $|H| \geq \min(H)$. The notation $X \to_* (k)^n_c$ means that in addition the homogeneous set is relatively large. Paris-Harrington Principle (denoted PH) is the following statement:

For any $n, c, k \in \mathbb{N}$, there is an $m \in \mathbb{N}$ such that $m \to_* (k)^n_c$.

Paris and Harrington showed [8] that PH is not provable in PA.

Another important PA-unprovable statement was introduced by Kanamori and McAloon [6]. Let $X \to (k)^n_{reg}$ mean that whenever $f : [X]^n \to \mathbb{N}$ is regressive, that is $f(x_1, \ldots, x_n) \leq x_1$ for all $x_1 < \cdots < x_n$ from $X$, then there is $H \subseteq X$ with cardinality $k$ such that for all $x_1 < \cdots < x_n$ from $H$, $f(x_1, \ldots, x_n)$ only depends on $x_1$. Such $H$ is called min-homogeneous for $f$. Kanamori-McAloon Principle (denoted KM) is the following statement:

*Presenter

For any $n, k \in \mathbb{N}$, there is an $m \in \mathbb{N}$ such that $m \to (k)^n_{reg}$.

Kanamori and McAloon showed [6] that KM is not provable in PA.

For more information we refer the reader to [1] and [2]. Let us now consider a weak form of Ramsey Theorem called Thin Set Theroem. It was introduced in the literature on the partition calculus in combinatorial set theory, in [3]. The Thin Set Theorem for unorderd tuples states that for any $n, c$, $\mathbb{N} \xrightarrow{thin} (\mathbb{N})^n_c$. Here $\xrightarrow{thin}$ means a weak form of homogeneity that at least one color is omitted. A variant of this statement was introduced in [5], where the authors consider the ordered tuples instead of unordered tuples. Then the Thin set theorem (for ordered tuples) reads for all $f : \mathbb{N}^k \to \mathbb{N}$, there exists an infinite set $A \subseteq \mathbb{N}$ such that $f[A^k] \neq \mathbb{N}$. A study of the Thin Set Theorem in mathematical logic and reverse mathematics appeared for the first time in [5]. Note that it is obtained from a well known theorem for $\mathbb{N}$ called Free Set Theorem.

Theorem 1.3. (Free Set Theorem for $\mathbb{N}$) *Let $k \geq 1$ and $f : \mathbb{N}^k \to \mathbb{N}$. There exists infinite $A \subseteq \mathbb{N}$ such that for all $x \in A^k$, if $f(x_1, \ldots, x_k) \in A$ then $f(x_1, \ldots, x_k) \in \{x_1, \ldots, x_k\}$.*

Observe that if $A$ is the infinite set obtained from the Free Set Theorem, then $A \setminus \{\min(A)\}$ satisfies the Thin Set Theorem. It is well-known that the Free Set Theorem is a consequence of the infinite Ramsey Theorem and that the Thin Set Theorem is not provable in $\mathsf{ACA}_0$. However the Free Set Theorem, and hence the Thin Set Theorem, for arity 1, is provable in $\mathsf{RCA}_0$; for more details see [4]. For an analysis of mathematical statements in subsystems of second order arithmetic such as $\mathsf{RCA}_0$ and $\mathsf{ACA}_0$, see [9]. Here we state some open questions concerning the Thin Set Theorem and the Free Set Theorem.

Question 1.4. For fixed exponents, is $\mathsf{ACA}_0$ obtained from the Thin Set Theorem or Free Set Theorem over $\mathsf{RCA}_0$? More generally, are they equivalent to $\mathsf{ACA}_0$ over $\mathsf{RCA}_0$?

Question 1.5. For arbitrary exponents, is Ramsey's Theorem obtained from the Thin Set Theorem or Free Set Theorem?

Question 1.6. Does the Thin Set Theorem imply the Free Set Theorem over $\mathsf{RCA}_0$?

## 2. The Finite Version

In this section we prove that the finite version of the Thin Set Theorem with a largeness condition implies the KM and hence is not provable in PA. We use a generalization of some graphs called *shift graphs*. The shift graph $G_n$ is the graph whose vertex set is the set of 2-subsets of $\{1, 2, \ldots, n\}$, there being an edge joining two pairs $\{i, j\}$ and $\{k, l\}$, where $i < j$ and $k < l$, if and only if $j = k$. It is easy to see that the chromatic number of this graph is equal to $\chi(G_n) = \lceil \log_2 n \rceil$. There is a natural generalization of a shift graph as follows. Fix integers $n$ and $k$ with $1 \leq k < n$. An ordered pair $(A, B)$ of $k$-element sets is called a $(k, n)$-*shift pair* if there exists a $(k+1)$-element subset $C = \{i_1 < i_2 < \cdots < i_{k+1}\} \subseteq \{1, 2, \ldots, n\}$ so that $A = \{i_1, i_2, \ldots, i_k\}$ and $B = \{i_2, i_3, \ldots, i_{k+1}\}$. Then the $(k, n)$-*shift graph* $S(k, n)$ is the graph whose vertex set consists of all $k$-element subsets of $\{1, 2, \ldots, n\}$ with a $k$-element set $A$ adjacent to a $k$-element set $B$ exactly when $(A, B)$ is a $(k, n)$-shift pair. Clearly a $(1, n)$-shift graph is just a shift graph. Similarly, a $(2, n)$-shift graph is called a *double shift* graph. It is well-known [10] that for each $k \geq 3$, there exists positive constant $c_k$ such that

$$\chi(S(k, n)) \geq c_k \log^{(k)} n, \tag{1}$$

for all $n \geq k+1$. For $h, n, e$, let $*(n) = \max\{h^{e-1}(n-e)+2, R_2^e(n-1)+1\}$. Remember that $X \overset{thin}{\longrightarrow}_* (k)_c^n$ means that whenever $f : [X]^n \longrightarrow c$ there is a subset $H \subseteq X$ such that $f([H]^n) \neq c$ and $|H| \geq \max\{k, *(\min(H))\}$.

THEOREM 2.1. *For $a$, $k$, $e$, and $b$ satisfying*

$$[a+e, b] \overset{thin}{\longrightarrow}_* (R_2^e(e+k-1)+1)_3^{e+1},$$

*we have* $[a, b] \longrightarrow (k)_{reg}^e$.

PROOF. Let $f : [[a, b]]^e \longrightarrow b$ be regressive. Define $g' : [[a+e, b]]^{e+1} \longrightarrow 3$ by

$$g'(x_0, \ldots, x_e) = \begin{cases} 0, & \text{if } f(x_0-e, x_1-e, \ldots, x_{e-1}-e) \\ & \quad = f(x_0-e, x_2-e, \ldots, x_e-e), \\ 1, & \text{if } f(x_0-e, x_1-e, \ldots, x_{e-1}-e) \\ & \quad < f(x_0-e, x_2-e, \ldots, x_e-e), \\ 2, & \text{if } f(x_0-e, x_1-e, \ldots, x_{e-1}-e) \\ & \quad > f(x_0-e, x_2-e, \ldots, x_e-e). \end{cases}$$

Then by $[a+e, b] \overset{thin}{\longrightarrow}_* (R_2^e(e+k-1)+1)_3^{e+1}$, there exists $H' \subseteq [a+e, b]$ so that $|H'| \geq R_2^e(e+k-1)+1$, $|H'| \geq *(\min(H'))$, and $g'([H']^{e+1}) \neq \{1, 2, 3\}$. Let $H'' = H' - e = \{h' - e : h' \in H'\}$ and $x_0 = \min(H'')$. Then $\min(H') = x_0 + e$ and so $|H''| = |H'| \geq R_2^e(e+k-1)+1$, $|H''| = |H'| \geq R_2^e(x_0+e-1)+1$, and $|H''| = |H'| \geq h^{e-1}(x_0)+2$. Moreover $H''$ is a thin set for the function

$$g''(x_0, \ldots, x_e) = \begin{cases} 0, & \text{if } f(x_0, x_1, \ldots, x_{e-1}) \\ & \quad = f(x_0, x_2, \ldots, x_e), \\ 1, & \text{if } f(x_0, x_1, \ldots, x_{e-1}) \\ & \quad < f(x_0, x_2, \ldots, x_e), \\ 2, & \text{if } f(x_0, x_1, \ldots, x_{e-1}) \\ & \quad > f(x_0, x_2, \ldots, x_e), \end{cases}$$

i.e. $g''([H'']^{e+1}) \neq \{1, 2, 3\}$. Let us first prove that $g''([H'']^{e+1}) \neq \{1, 2\}$. Suppose, contrary to our claim, that $g''([H'']^{e+1}) = \{1, 2\}$. Let $n = h^{e-1}(x_0)+2$ and write $H''$ as $\{x_0, x_1, \ldots, x_n, \ldots\}$. Color the vertices of the $(e-2, n)$-shift graph $S(e-2, n)$ by

$$c(k_1, \ldots, k_{e-1}) = f(x_0, x_{k_1}, \ldots, x_{k_{e-1}}) < x_0.$$

Two adjacent vertices $(k_1, k_2, \ldots, k_{e-1})$ and $(k_2, \ldots, k_{e-2}, k_e)$ have different colors if and only if $f(x_0, x_{k_1}, \ldots, x_{k_{e-1}}) \neq f(x_0, x_{k_2}, \ldots, x_{k_e})$ if and only if $g''(x_0, x_{k_1}, \ldots, x_{k_{e-1}}, x_{k_e}) \neq 0$, which holds by the assertion. Hence $\chi(S(e-2, n)) < x_0$, contrary to (1). Therefore, either $g''([H'']^{e+1}) = \{0, 1\}$, or $g''([H'']^{e+1}) = \{0, 2\}$. We assume that $g''([H'']^{e+1}) = \{0, 1\}$ and leave the other case to the reader. Now let $g : [H'' - \{x_0\}]^e \longrightarrow \{0, 1\}$ be defined by

$$g(x_1, \ldots, x_e) = g''(x_0, x_1, \ldots, x_e).$$

By $|H'' - \{x_0\}| \geq \max\{R_2^e(e+k-1), R_2^e(x_0+e-1)\}$, there is $H \subseteq H'' - \{x_0\}$ such that $g$ is constant on $[H]^e$ and $|H| \geq \max\{k+e-1, x_0+e-1\}$. Let the value of $g$ on $[H]^e$ be 1. Then writing $H$ as $\{t_1, t_2, \ldots, t_{x_0+e-1}, \ldots\}$, we have

$$f(x_0, t_1, \ldots, t_{e-1}) < f(x_0, t_2, \ldots, t_e) < \cdots < f(x_0, t_{x_0+e-1}),$$

which implies that $f(x_0, t_i, \ldots, t_{i+e-1}) \geq x_0$ for some $i$, contrary to $f$ being regressive. Hence $g$ has constant value 0 on $[H]^e$. Let $H_0$ be the first $k-1$ elements of $H$ plus $x_0$. Then $H_0$ is min-homogeneous for $f$ since $g$ has value 0 on $[H]^e$. □

Corollary 2.2. *The finite version of Thin Set Theorem with a largeness condition is not provable in* PA*.*

Question 2.3. Discuss the provability of finite version of the Thin Set Theorem in fragments of PA.

## References

1. A. Bovykin, *Brief introduction to unprovability*, Logic Colloquium 2006, 38–64, Lect. Notes Log., 32, Assoc. Symbol. Logic, Chicago, IL, 2009.
2. A. Bovykin, *Several Proofs of PA-Unprovability*, Logic and its Applications, 29–43, Contemp. Math., 380, Amer. Math. Soc., Providence, RI, 2005.
3. P. Erdös, A. Hajnal and R. Rado, *Partition relations for cardinal numbers*, Acta Math. Acad. Sci. Hungar. **16** (1965) 93–196.
4. H. Friedman, *Boolean relation theory and incompleteness*, Lecture Notes in Logic, 2015.
5. H. Friedman and S. G. Simpson, *Issues and problems in reverse mathematics*, Computability theory and its applications (Boulder, CO, 1999), 127–144, Contemp. Math., 257, Amer. Math. Soc., Providence, RI, 2000.
6. A. Kanamori and K. McAloon, *On Gödel incompleteness and finite combinatorics*, Ann. Pure Appl. Logic **33** (1) (1987) 23–41.
7. R. Kaye, *Models of Peano Arithmetic*, Oxford Logic Guides, Vol 15. Clarendon Press, Oxford University Press, Oxford and New York, 1991.
8. J. Paris and L. Harrington, *A mathematical incompleteness in Peano arithmetic*, Handbook of mathematical logic, 1133–1142, Stud. Logic Found. Math., 90, North-Holland, Amsterdam, 1977.
9. S. G. Simpson, *Subsystems of second order arithmetic*, 2nd ed., Perspectives in Logic. Cambridge University Press, Cambridge; Association for Symbolic Logic, Poughkeepsie, NY, 2009.
10. W. T. Trotter, *Graphs and partially ordered sets: Recent results and new directions*, Surveys in graph theory (San Francisco, CA, 1995). Congr. Numer. **116** (1996) 253–278.

E-mail: khamseh@khu.ac.ir

# Author Index

# Author Index

www.aimc51.kashanu.ac.ir