بِسْمِ اللهِ الرَّحْمٰنِ الرَّحِيْمِ

فصل ششم – بخش دوم

قانون اول کپلر

۵.۶ قانون اول کپلر: قانون بیضیها

راهکار اثبات

۱ــ تشکیل معادلهٔ دیفرانسیل کلی مدار ذره را در هر میدان نیروی مرکزی و همسانگرد

۲ــ حل این معادلهٔ مداری برای حالت خاص نیروی عکس مجذوری

$$m\ddot{\mathbf{r}} = f(r)\mathbf{e}_r$$

معادلهٔ حرکت در مختصات قطبی

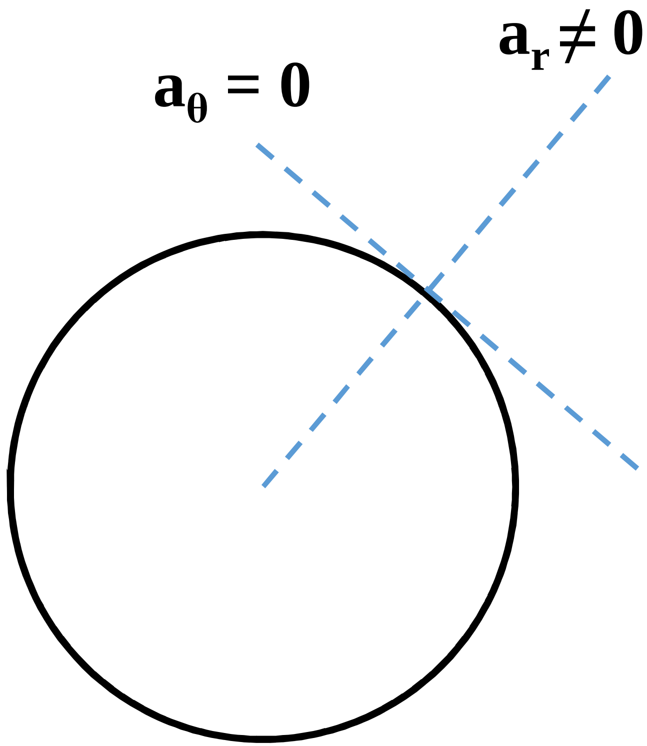
$f(r)$ نیروی مرکزی همسانگرد است که بر ذرهٔ به جرم m وارد می‌آید

نیرو فقط تابعی از فاصلهٔ اسکالر r تا مرکز نیرو

جهت آن در امتداد بردار شعاعی (نیروی مرکزی)

$$m\ddot{\mathbf{r}} = f(r)\mathbf{e}_r$$

$$\mathbf{a} = \underbrace{(\ddot{r} - r\dot{\theta}^{٢})}_{\mathbf{a}_r \text{ مؤلفة شعاعى}}\mathbf{e}_r + \underbrace{(r\ddot{\theta} + ٢\dot{r}\dot{\theta})}_{\mathbf{a}_\theta \text{ مؤلفة عرضى}}\mathbf{e}_\theta$$



$$m(\ddot{r} - r\dot{\theta}^{٢}) = f(r)$$

$$m(٢\dot{r}\dot{\theta} + r\ddot{\theta}) = ٠$$

مؤلفة عرضى $$a_\theta = r\ddot{\theta} + ٢\dot{r}\dot{\theta} = \frac{١}{r}\frac{d}{dt}(r^{٢}\dot{\theta}) = ٠$$

$$\frac{d}{dt}(r^{٢}\dot{\theta}) = ٠ \rightarrow r^{٢}\dot{\theta} = \text{ثابت} = l$$

$$|l| = \frac{L}{m} = |\mathbf{r} \times \mathbf{v}|$$ تکانهٔ زاویه‌ای در واحد جرم

l یک کمیت پایسته است

تکانهٔ زاویه‌ای ذرهٔ متحرک تحت تأثیر نیروی مرکزی ثابت باقی می‌ماند.

با معلوم فرض‌کردن تابع نیروی شعاعی یعنی $f(r)$

می‌توانیم زوج معادلات دیفرانسیل را حل کنیم تا r و θ را به‌صورت تابعی از t به‌دست آوریم.

یافتن معادلهٔ مدار، مسیر در فضای (مدار) مستقل از زمان، t،

تغییر متغیر $r = \frac{1}{u}$

مشتق گیری نسبت به زمان

$$\dot{r} = -\frac{1}{u^2}\dot{u} = -\frac{1}{u^2}\dot{\theta}\frac{du}{d\theta} = -l\frac{du}{d\theta}$$

$$r^2\dot{\theta} = \text{ثابت} = l \quad \xrightarrow{r = \frac{1}{u}} \quad \dot{\theta} = lu^2$$

$$\dot{r} = -\frac{1}{u^2}\dot{u} = -\frac{1}{u^2}\dot{\theta}\frac{du}{d\theta} = -l\frac{du}{d\theta}$$

مشتق گیری
نسبت به زمان

$$\ddot{r} = -l\frac{d}{dt}\frac{du}{d\theta} = -l\frac{d\theta}{dt}\frac{d}{d\theta}\frac{du}{d\theta} = -l\dot{\theta}\frac{d^2u}{d\theta^2} = -l^2u^2\frac{d^2u}{d\theta^2}$$

ثابت

$$\frac{d}{dt} = \frac{d}{d\theta}\frac{d\theta}{dt}$$

$$\dot{\theta} = lu^2$$

$$m(\ddot{r} - r\dot{\theta}^2) = f(r)$$

بازنویسی معادله
برحسب u

$$m\left[-l^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}(l^2u^4)\right] = f(u^{-1})$$

$$m\left[-l^2u^2\frac{d^2u}{d\theta^2}-\frac{1}{u}(l^2u^4)\right]=f(u^{-1})$$

تقسیم دو طرف

بر ml^2u^2

$$\frac{d^2u}{d\theta^2}+u=-\frac{1}{ml^2u^2}f(u^{-1})$$

معادلهٔ دیفرانسیل مدار ذرهٔ متحرک تحت تأثیر نیروی مرکزی

با حل این معادله، u (و بنابراین r) را به‌صورت تابعی از θ به‌دست می‌آوریم

برعکس، اگر معادلهٔ قطبی مدار، یعنی $r=r(\theta)=u^{-1}$ در دست باشد،
با مشتق‌گیری و دستیابی به‌عبارت $d^2u/d\theta^2$ و واردکردن آن در معادلهٔ دیفرانسیل،
تابع نیرو به دست می‌آید.

This equation is a differential equation that may be solved for $u(\theta)$ and hence for $r(\theta)$, which describes the orbit of a particle moving under a central force $F(r)\hat{\mathbf{r}}$. On the other hand, if the orbit of the particle is given in polar coordinates $r(\theta)$, this differential equation may be solved to find the form of the force law $F(r)$. A special case is in order. For $L = 0$, the above equations do not hold. From Eq. (7.40), $mr^2\dot{\theta} = L = 0$ means that since $m \neq 0$, $r \neq 0$, therefore, $\dot{\theta} = 0$, or $\theta =$ constant, which implies the path of the particle is a straight line passing through the origin.

مثال ۱.۵.۶

ذره‌ای در میدانی مرکزی در مدار مارپیچی حرکت می‌کند: تابع این نیرو را تعیین کنید.

$$r = c\theta^2$$

$$r = \frac{1}{u} \longrightarrow u = \frac{1}{c\theta^2}$$

مشتق گیری نسبت به زمان

$$\frac{du}{d\theta} = \frac{-2}{c}\theta^{-3} \qquad \frac{d^2u}{d\theta^2} = \frac{6}{c}\theta^{-4} = 6cu^2$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{ml^2u^2}f(u^{-1}) \longrightarrow 6cu^2 + u = -\frac{1}{ml^2u^2}f(u^{-1})$$

$$f(u^{-1}) = -ml^2(6cu^4 + u^3)$$

$$f(r) = -ml^2\left(\frac{6c}{r^4} + \frac{1}{r^3}\right)$$

نیرو ترکیبی از قانون عکس مکعب و عکس توان چهارم است.

مثال ۲.۵.۶

در مثال ۱.۵.۶، تعیین کنید زاویهٔ θ چگونه با زمان تغییر می‌کند.

$l = r^{2}\dot{\theta}$ کمیتی ثابت

$$\dot{\theta} = lu^{2} = l\frac{1}{c^{2}\theta^{4}}$$

$$\dot{\theta} = \frac{d\theta}{dt}$$

$$\theta^{4} d\theta = \frac{l}{c^{2}}dt$$

انتگرال گیری

$$\frac{\theta^{5}}{5} = lc^{-2}t$$

ثابت انتگرال‌گیری صفر

در $t = 0$ داریم: $\theta = 0$.

$$\theta = \alpha t^{1/5}$$

$\alpha = \text{ثابت} = (5lc^{-2})^{1/5}$.

مدارهای نیروی مرکزی و پتانسیل موثر

حرکت یک ذره تحت تاثیر یک نیروی مرکزی، دارای یک حرکت دوبعدی هست که می توان آن را به دو حرکت یک بعدی تبدیل کرد

$$E = \frac{1}{2} m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r) = K_{\text{rad}} + V_{\text{cent}}(r) + V(r)$$

انرژی جنبشی حرکت شعاعی	انرژی جنبشی حرکت زاویه ای

$$E = \tfrac{1}{2} m\dot{r}^2 + V_{\text{eff}}(r)$$

$$V_{\text{eff}}(r) = V_{\text{cent}}(r) + V(r) = \frac{L^2}{2mr^2} + V(r)$$

The dependence of total energy E on the variables r and $\dot{r}$ is similar to the motion of a particle in one dimension (as discussed in Section 2.5) if we replace x by r, $\dot{x}$ by $\dot{r}$, and $V(x)$ by $V_{\text{eff}}(r)$. Hence the energy diagram method discussed in Chapter 2 can be applied here.

مثالی از نیروی مرکزی

We shall now apply the energy diagram method, making plots of V_{eff} versus r, to the two commonly encountered force laws: (1) the isotropic harmonic force law, and (2) the inverse-square force law. The salient features of central force motion can be well understood by first considering the case of a harmonic oscillator, as shown in Fig. 7.9(a). For this case,

مثالی از یک نیروی مرکزی

$$F(r) = -kr \quad \text{or} \quad V(r) = \tfrac{1}{2}kr^2$$

$$V_{\mathrm{eff}}(r) = V_{\mathrm{cent}}(r) + V(r) = \frac{L^2}{2mr^2} + \frac{1}{2}kr^2$$

$$V_{\text{eff}}(r) = V_{\text{cent}}(r) + V(r) = \frac{L^2}{2mr^2} + \frac{1}{2}kr^2$$

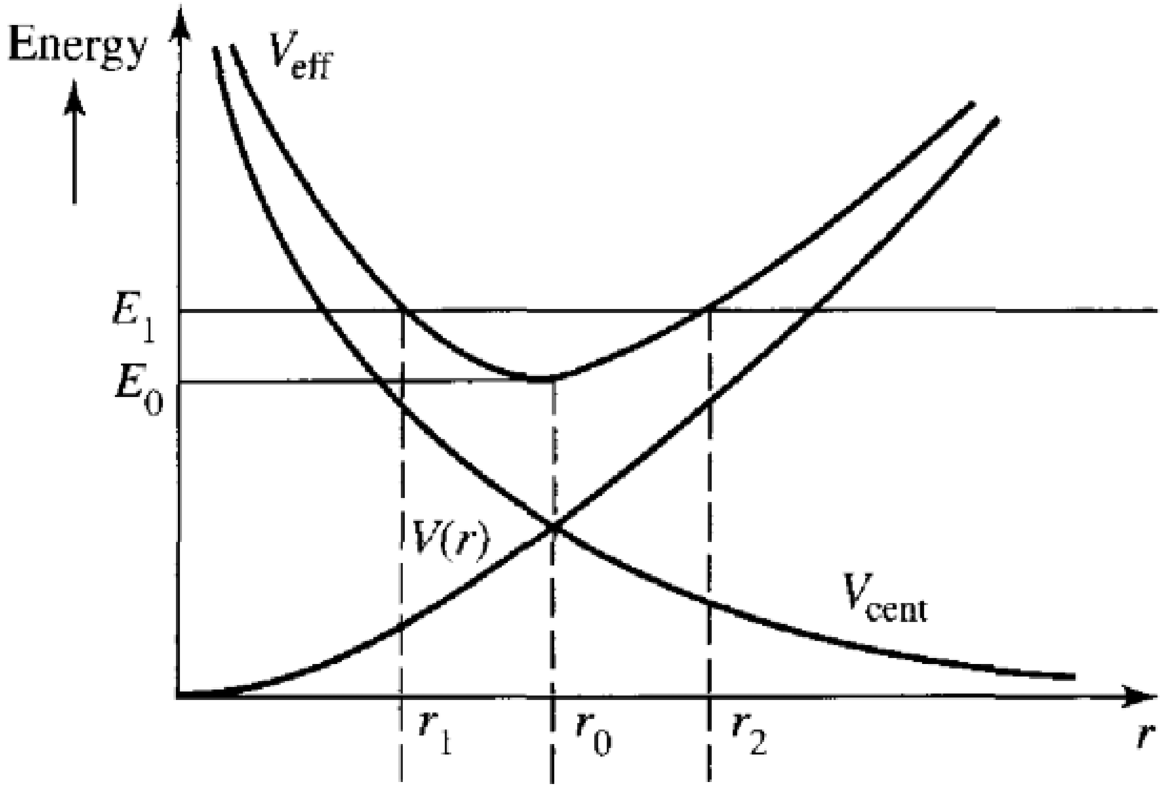


Figure 7.9(a) Graphs of $V(r)$, V_{cent}, and V_{eff} for an isotropic harmonic force law.

The plots of $V(r)$, V_{cent}, and V_{eff} are shown in Fig. 7.9(a). V_{eff} has a minimum at r_0. For a given total energy E (greater than the minimum energy $E_0 = [V_{eff}(r)]_{min}$), the particle oscillates between two extreme values of r, $r_1 = r_{min}$ and $r_2 = r_{max}$; that is, $r_{min} < r < r_{max}$. The two points are the turning points in motion. At these points the radial velocity is zero; that is, $\dot{r} = 0$. These turning points are the roots of the equation [energy conservation, Eq. (7.68)], with $\dot{r} = 0$; that is,

$$E - V(r) - \frac{L^2}{2mr^2} = 0 \tag{7.73}$$

These two radial distances, r_1 and r_2, define two circles of radii r_1 and r_2 about the force center in the plane of the orbit. And it is the angular motion that restricts the motion of the particle within these two circles (discussed later).

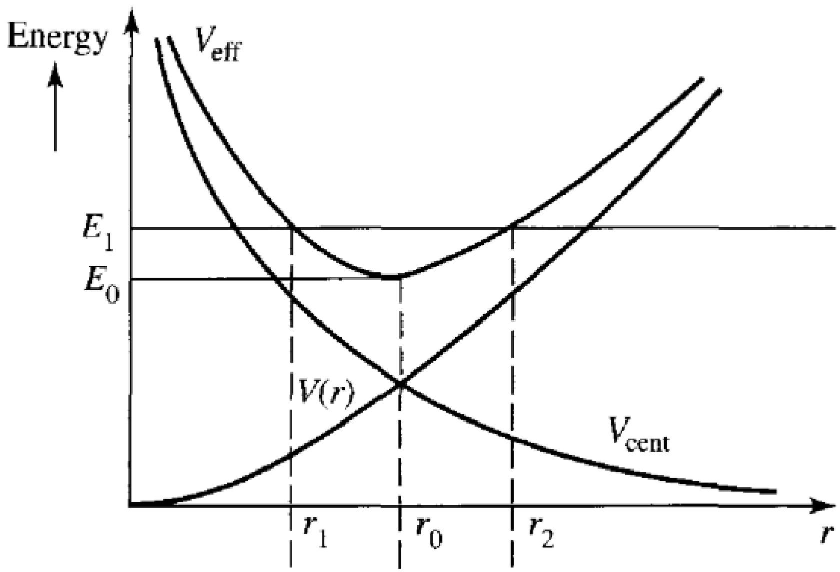


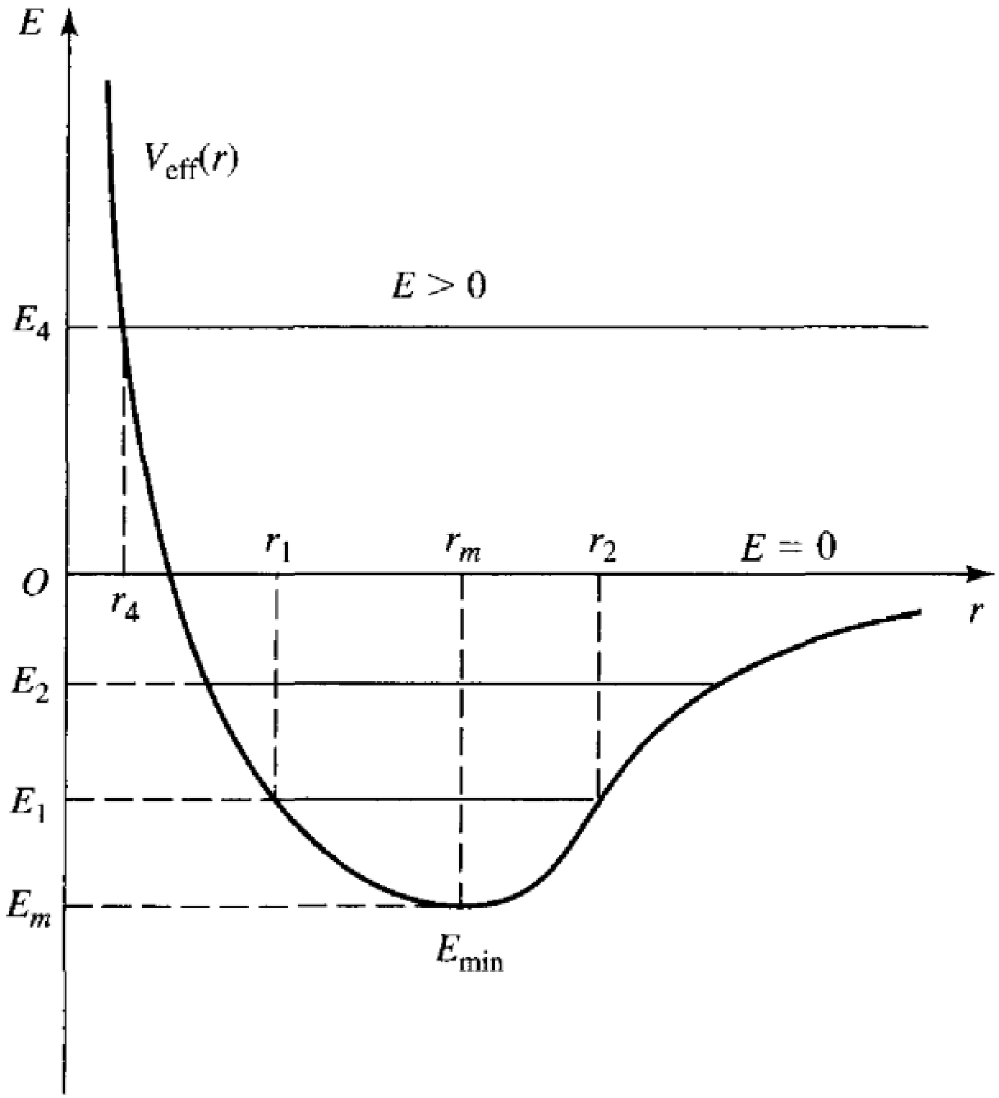
Figure 7.9(a) Graphs of $V(r)$, V_{cent}, and V_{eff} for an isotropic harmonic force law.

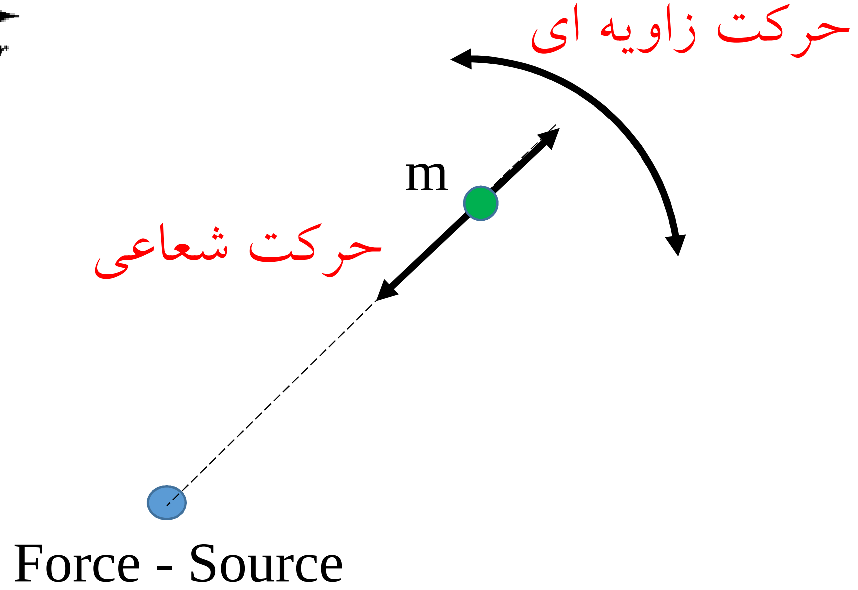
مثالی از پتانسیل موثر یک نیروی مرکزی گرانشی

حرکت شامل

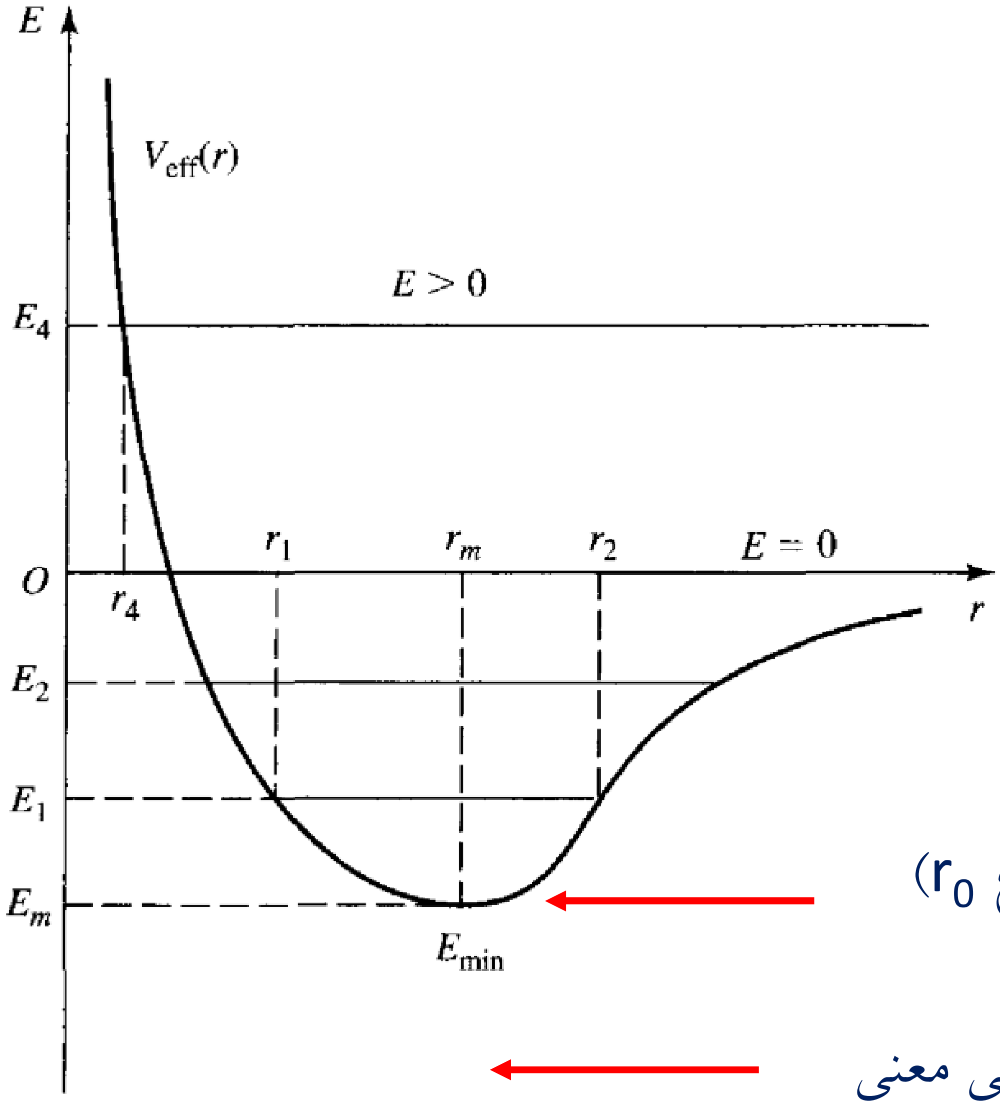
حرکت شعاعی وابسته به انرژی متفاوت

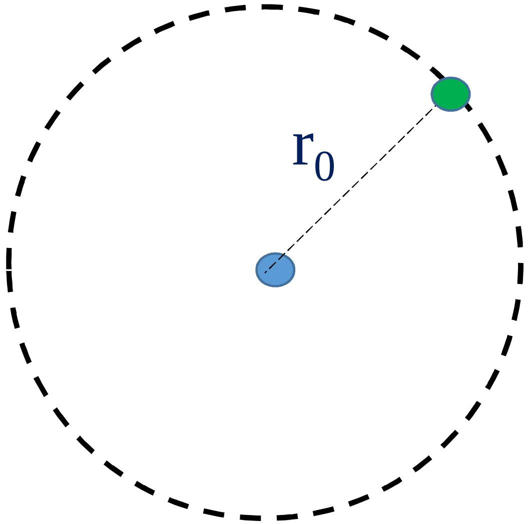
و حرکت زاویه ای با تکانه زاویه ای ثابت





$E = E_m$





$E = E_m$

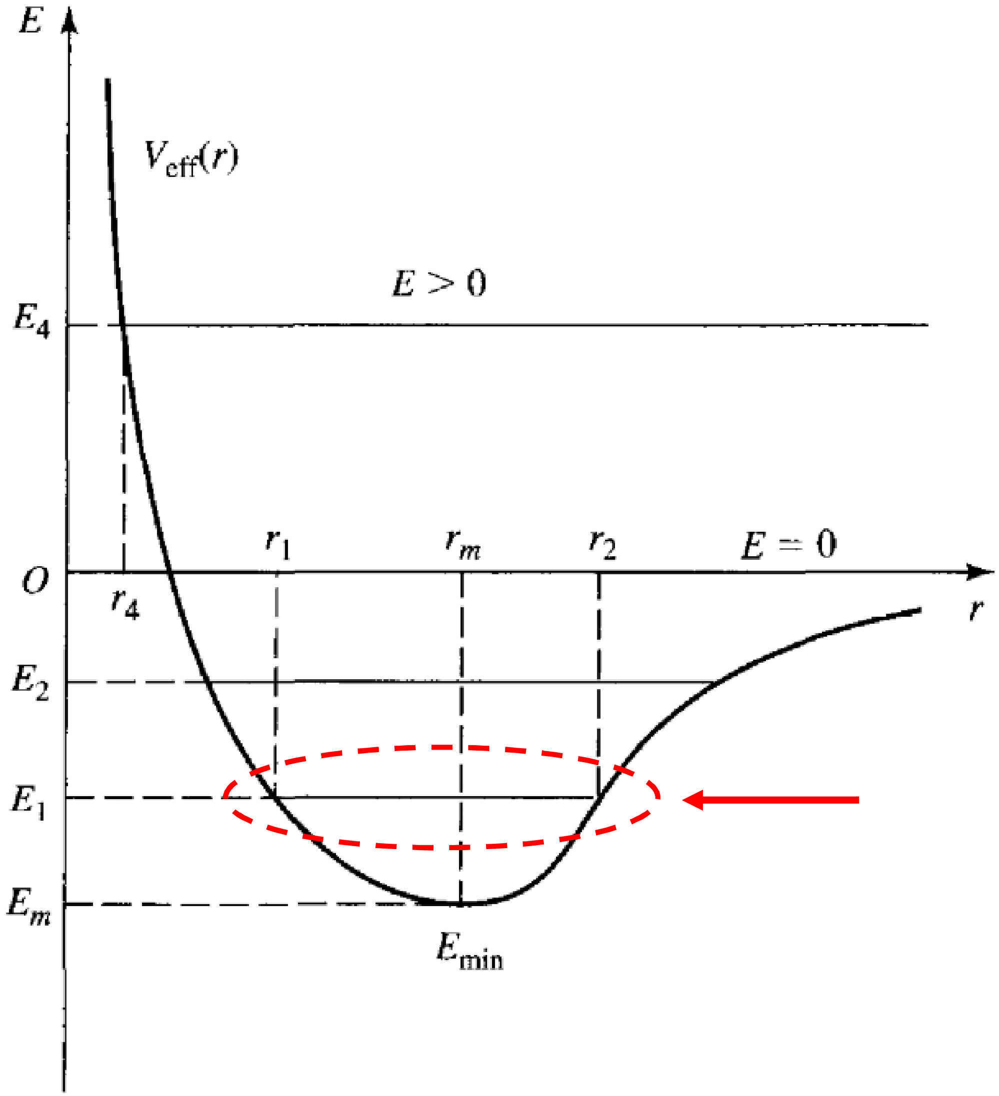
حرکت در شعاع ثابت (مسیر دایره ای با شعاع r_0)
و با سرعت زاویه ای ثابت

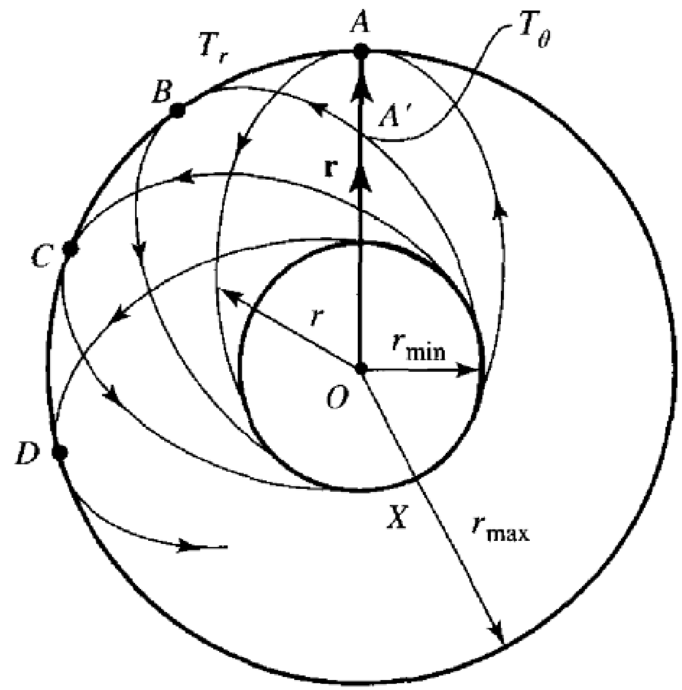
حرکت با این انرژی بی معنی

$E_m < E < 0$

Now suppose a particle has energy between $E = 0$ and $E = E_m$, say E_1, as shown in Fig. 7.11. The radial motion of the particle will be confined to the values of $r = r_1 = r_{\min}$ and $r = r_2 = r_{\max}$. The points r_1 and r_2 are the turning points. Actually, the motion is confined between the areas of two circles of radii r_1 and r_2, as shown in Fig. 7.12. The motion is periodic with the radial time period T_r, which is the time the particle takes to go from $r_{\min}$ to $r_{\max}$ and then back to $r_{\min}$. In Fig. 7.12, T_r is equal to the time in going from A to B. Furthermore, the orbit must be such that it is tangent to both the circles at the turning points, such as A on the outer circle and X on the inner circle, respectively. The reason is that at the turning points the radial velocity $\dot{r}$ must be zero, while the tangential velocity cannot be zero because of angular momentum. As shown in Fig. 7.12, the orbit at A is tangent to the outer circle, and when it reaches X, it is tangent to the inner circle. The orbit continues and once again becomes tangent to the outer circle at B. The time it takes for the particle motion to go from A to X and then to B is equal to the *radial time period* T_r. Note that vector **r** is continuously changing direction. The time it takes to turn through 2π angle is called the *angular time period* T_θ, also called the *characteristic time period* or the *revolution time period.* In Fig. 7.12, to start with, vector **r** was equal to OA, and after turning through 2π angle, it is at OA'. The time it takes for vector **r** to reach OA' is equal to the period T_θ.

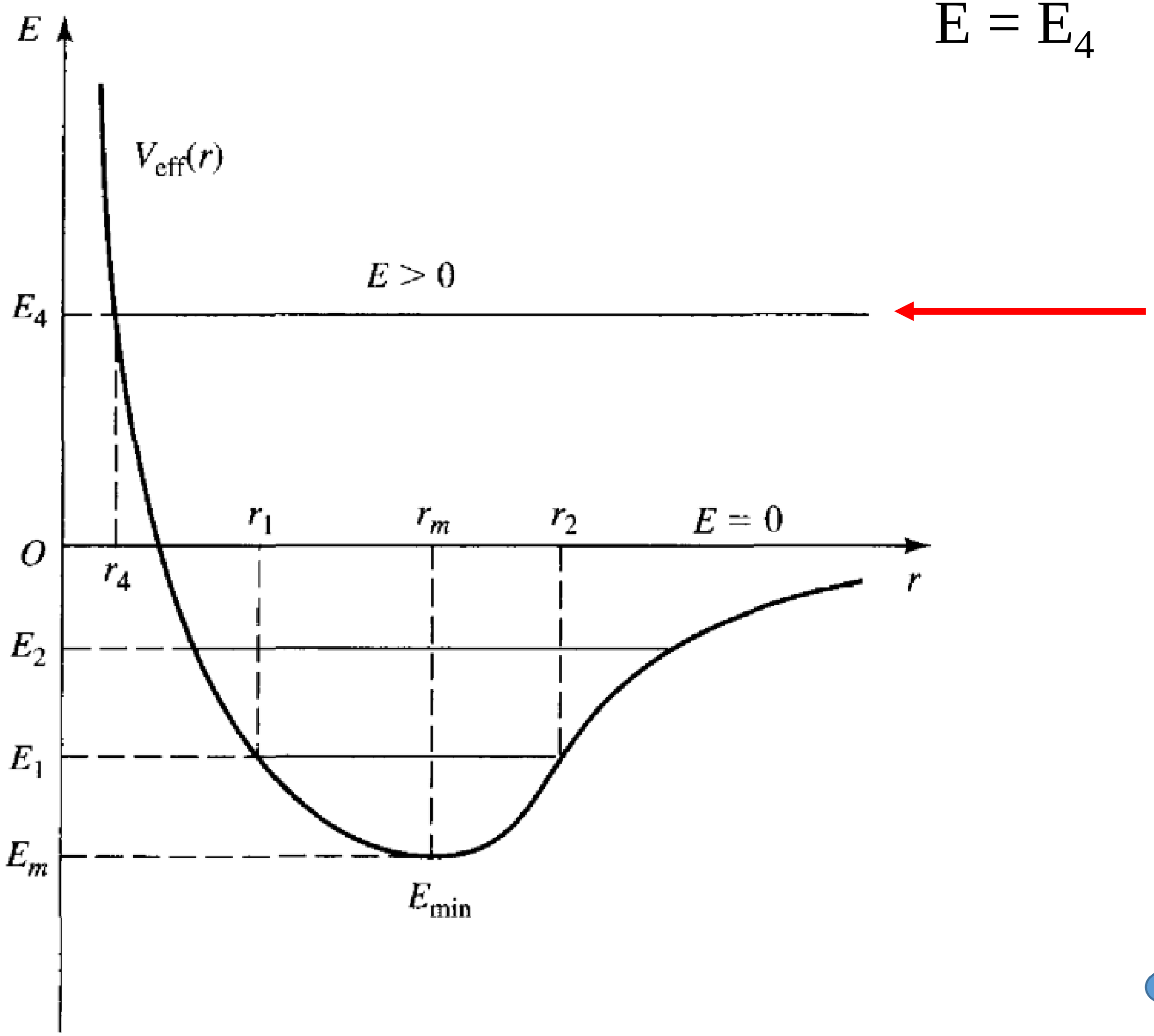
حرکت شعاعی ذره مقید به شعاع های r_1 و r_2 هست

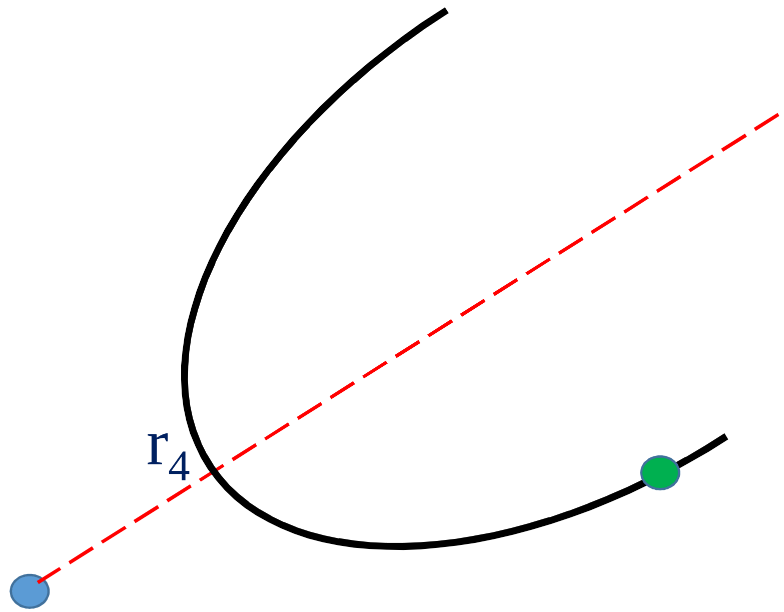




حرکت شعاعی با یک نقطه بازگشت

$E = E_4$





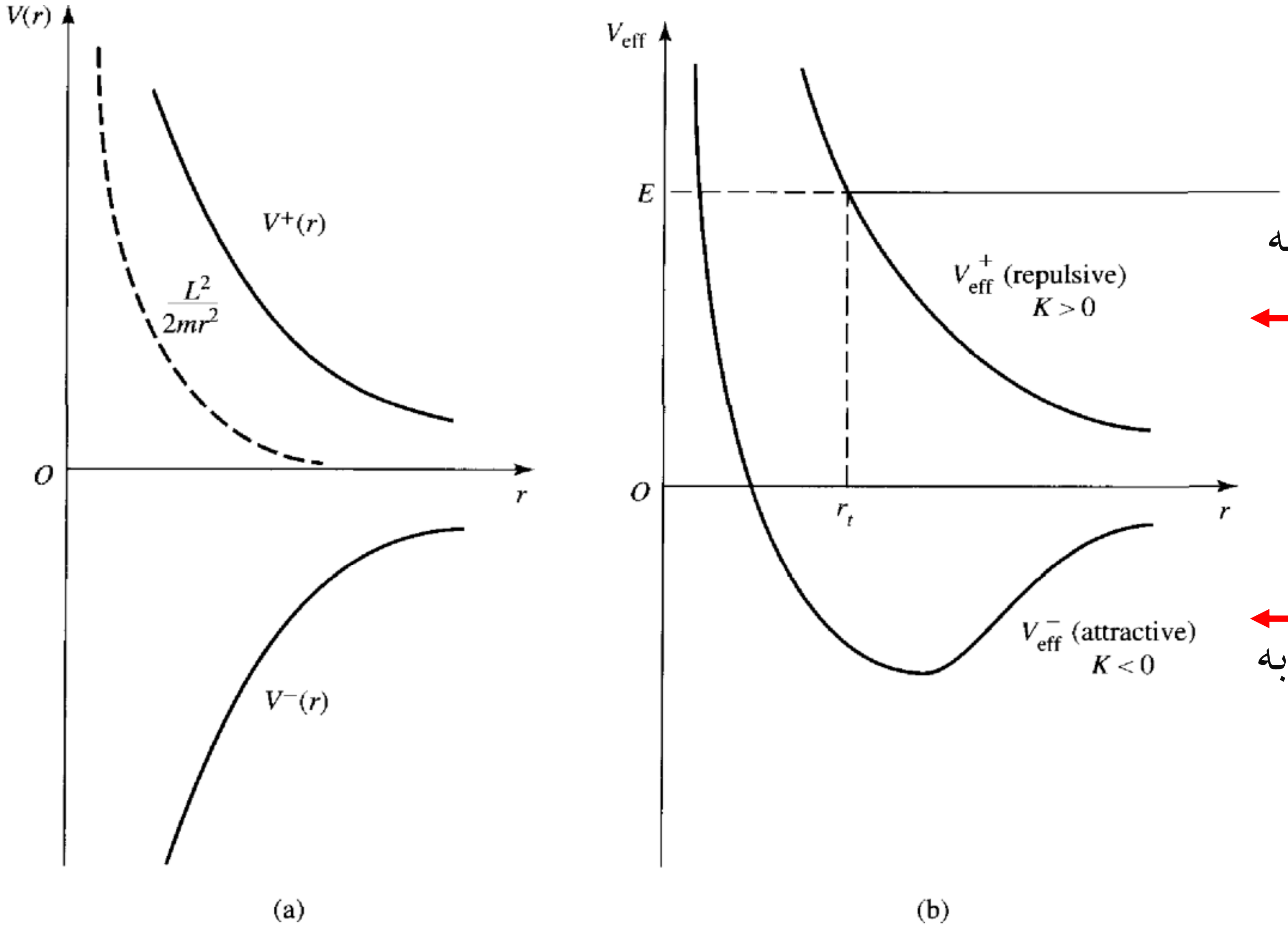
مثالی از پتانسیل موثر یک نیروی مرکزی دفعی

$$V(r) = \frac{k}{r} > 0 \quad , \quad k > 0$$

$$r \uparrow \quad \rightarrow \quad V \downarrow$$

Let us consider the case of a repulsive force $F(r) = K/r^2$ and $V(r) = K/r$, where K is positive for repulsive potential. Hence $V(r)$ is positive and decreases monotonically with increasing r. The resultant effective potential will always be positive; hence there will be no bounded motion. These points are illustrated in Fig. 7.13. For a given value of L, Fig. 7.13(a) shows the plots of $V_{cent}(r)$ ($= L^2/2mr^2$), $V^+(r)$, which is a repulsive potential, and $V^-(r)$, which is of the same magnitude as $V^+(r)$ except that it is an attractive potential. Figure 7.13(b) shows the plots of

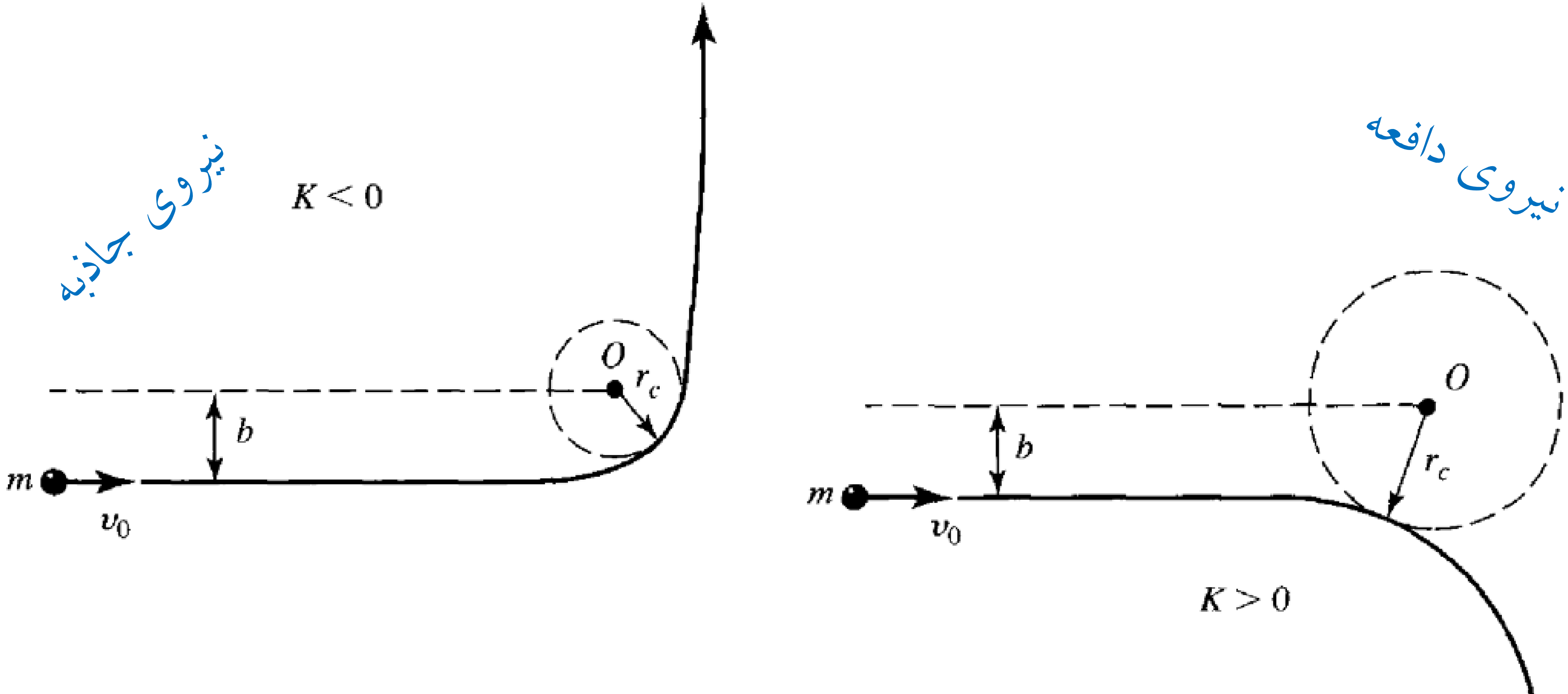
$$V^+_{eff} = V^+(r) + \frac{L^2}{2mr^2}, \qquad \text{repulsive force}$$



پتانسیل موثر نیروی دافعه

پتانسیل موثر نیروی جاذبه

Figure 7.13 (a) Plots of attractive potential $V^{-}(r)$, repulsive potential $V^{+}(r)$, and V_{cent} ($= L^2/2mr^2$) versus r. (b) Plots of attractive effective potential V_{eff}^{-} and repulsive effective V_{eff}^{+} versus r.



Suppose a particle with energy E is at infinity and is traveling toward the center of force O. At such large distances, both $V(r)$ and $L^2/2mr^2$ are zero; hence the particle travels in a straight line with a speed $v_0 = (2E/m)^{1/2}$. The particle misses the center of force by a distance b, called the *impact parameter*, as shown in Fig. 7.14. Thus the angular momentum L of the particle is

$$L = mv_0 b \tag{7.78}$$

نیروی تابع قانون عکسی مجذوری (بررسی کیفی)

$$\mathbf{F}(r) = \frac{K}{r^2}\,\hat{\mathbf{r}} \quad \text{or} \quad F(r) = \frac{K}{r^2}$$

while its potential energy is given by

$$V(r) = -\int_{r_s}^{r} F(r)\,dr = -\int_{r_s}^{r} \frac{K}{r^2}\,dr$$

Assuming $r_s = \infty$ and $V(\infty) = 0$,

$$V(r) = \frac{K}{r}$$

$$V(r) = \frac{K}{r} \tag{7.80}$$

where $K < 0$ for an attractive force and $K > 0$ for a repulsive force. Two important cases of an inverse-square force are: (1) gravitational force, which is always attractive, and the quantity K and the constant G, which are

$$K = -Gm_1m_2 \quad \text{and} \quad G = 6.67 \times 10^{-11}\ \text{N-m}^2/\text{kg}^2 \tag{7.81}$$

and (2) the coulomb force, for which the quantity K and the constant ϵ_0 are

$$K = \frac{1}{4\pi\epsilon_0} q_1 q_2 \tag{7.82}$$

where $$\epsilon_0 = 8.85 \times 10^{-12}\ \text{C}^2/\text{N-m}^2$$

ϵ_0 is the permittivity of free space. If q_1 and q_2 have the same signs, the force is repulsive and $K > 0$ (positive), while if q_1 and q_2 have opposite signs, the force is attractive and $K < 0$ (negative).

As before, without actually solving the equations of motion, we can determine the nature of the orbits by discussing the effective potential, which for the inverse-square force is [Eqs. (7.70) and (7.80)]

$$V_{\text{eff}}(r) = \frac{K}{r} + \frac{L^2}{2mr^2} \tag{7.83}$$

The plots of $V_{\text{eff}}(r)$ versus r for four situations ($K < 0, L = 0$; $K < 0, L \neq 0$; $K = 0, L \neq 0$; $K > 0$, $L \neq 0$) are shown in Fig. 7.16 and Fig. 7.17. As discussed earlier for $K > 0$, E is always positive and there is no periodic motion in r. The same is the case for $K = 0$, except that the turning point for a given value of E and L occurs at a smaller value of r than for $K > 0$. The motion in both cases is unbound. For an attractive force ($K < 0$), both bound ($L \neq 0$) and unbound ($L = 0$) motions are possible. The latter corresponds to the one-dimensional motion of a falling body.

For $K < 0$ and $L \neq 0$, then motion of the particle is unbound if $E > 0$, and the turning poin. occurs at a distance smaller than for the case $K = 0$. The motion is bound and periodic if $E < 0$. The minimum in the effective potential energy curve is given by the condition that at the equilibrium point $dV_{\text{eff}}/dr = 0$:

$$\frac{d}{dr} V_{\text{eff}}(r) = \frac{d}{dt}\left(\frac{K}{r} + \frac{L^2}{2mr^2}\right)\bigg|_{r=r_0} = -\frac{K}{r_0^2} - \frac{L^2}{mr_0^3} = 0$$

That is,

$$r_0 = -\frac{L^2}{mK} \tag{7.84}$$

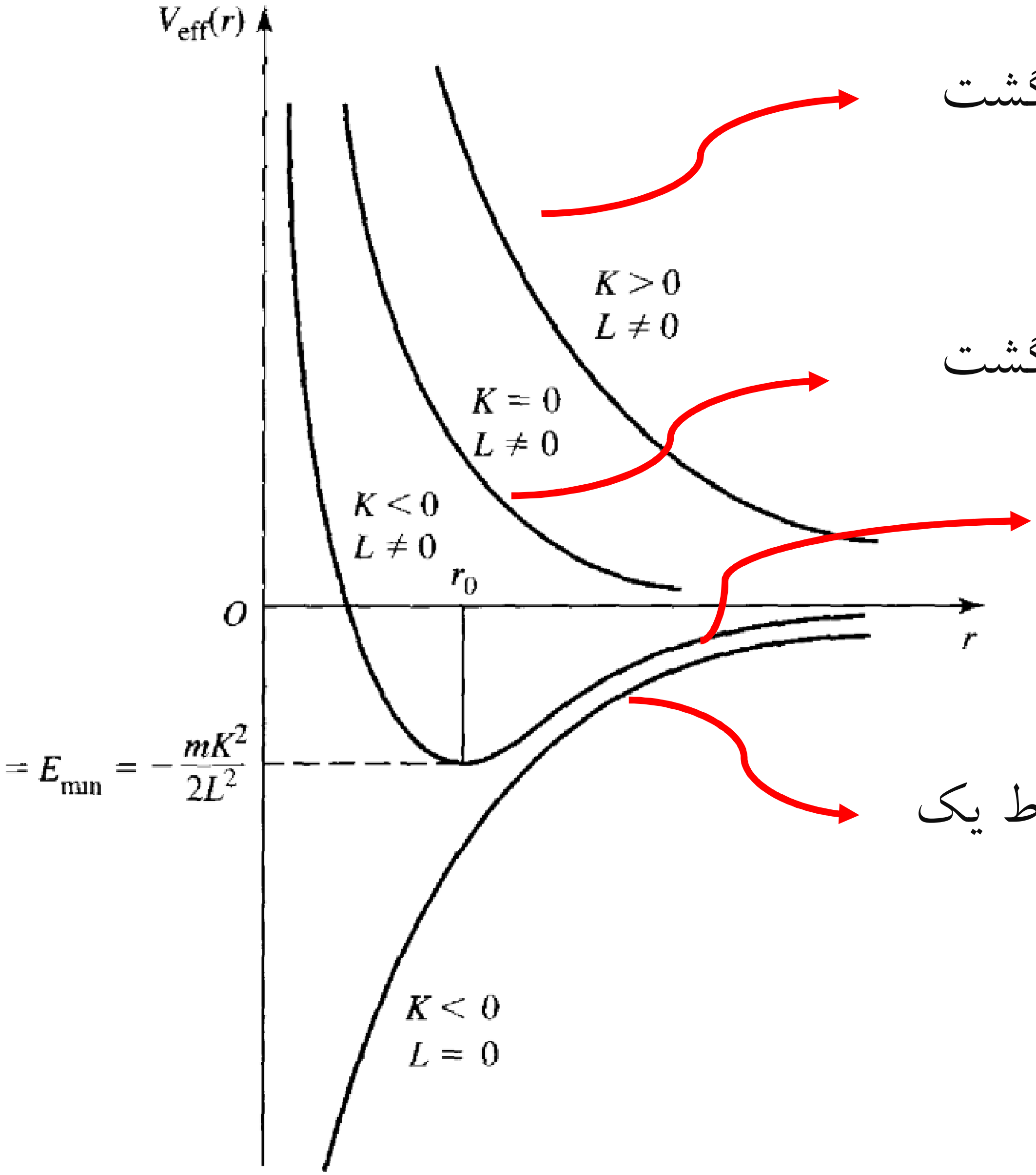
while the value of V_{eff} at $r = r_0$ is obtained by substituting for r_0 from Eq. (7.84) into Eq. (7.83); that is,

$$V_{\text{eff}}(r_0) = \frac{K}{r_0} + \frac{L^2}{2mr_0^2} = \frac{K}{-L^2/mK} + \frac{L^2}{2m(-L^2/mK)^2}$$

or

$$V_{\text{eff}}(r_0) = -\frac{1}{2}\frac{mK^2}{L^2} \tag{7.85}$$

as shown in Figs. 7.16 and 7.17. Thus for $E = E_0 = E_{\text{min}} = V_{\text{eff}}(r_0)$, as given by Eq. (7.85) a particle moves in a circle of radius r_0 $(= -L^2/mK)$ given by Eq. (7.84). But if the energy is less than 0 but greater than $-mK^2/2L^2$, that is, $-mK^2/2L^2 < E < 0$, the coordinate r oscillates between two turning points, as shown in Fig. 7.18. For all negative values of E and $L \neq 0$, the orbit of the particle is an ellipse. If the value of E is very close to the minimum value of E_{min}, the period



حرکت غیرمقید و شامل فقط یک نقطه بازگشت

حرکت غیرمقید و شامل فقط یک نقطه بازگشت

تنها حالت مقید و ذره در حرکت نوسانی
بین دو شعاع

حرکت غیرمقید و بدون چرخش مانند سقوط یک
جسم به سمت مبدا

Figure 7.16 Plots of $V_{\text{eff}}(r)$ versus r for different values of K and L. For $E = E_{\min} = -mK^2/2L^2$, the particle moves in a circle of radius r_0.