بسم الله الرحمن الرحيم

عملگر تاو

تاو هر بردار عبارت است از حد نسبت انتگرال سطحی ضرب برداری‌آن بردار در بردار یکهٔ عمود برسطح بسته (درجهت خارج‌آن سطح)، به حجم محصور به‌وسیلهٔ‌آن سطح، وقتی‌که حجم به‌سمت صفر میل‌کند؛ یعنی

$$\operatorname{curl}\mathbf{F} = \lim_{V \to 0} \frac{1}{V} \oint_S \mathbf{n} \times \mathbf{F}\, da$$

در مختصات دکارتی

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

در مختصات استوانه ای

$$\nabla \times \mathbf{A} = \frac{1}{\rho} \begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial \mathbf{z}} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

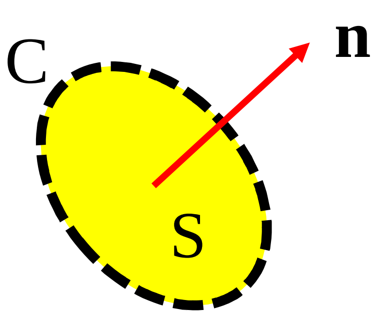
در مختصات کروی

$$\nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin A_\phi \end{vmatrix}$$

قضیه استوکس

Stokes's Theorem

قضیهٔ استوکس . انتگرال خطی یك بردار روی یك منحنی بسته برابر است با انتگرال سطحی مؤلفهٔ قائم تاو آن بردار روی هر سطحی که توسط این منحنی محصور شده باشد؛



$$\oint_C \mathbf{F} \cdot d\mathbf{l} = \int_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} \, da$$

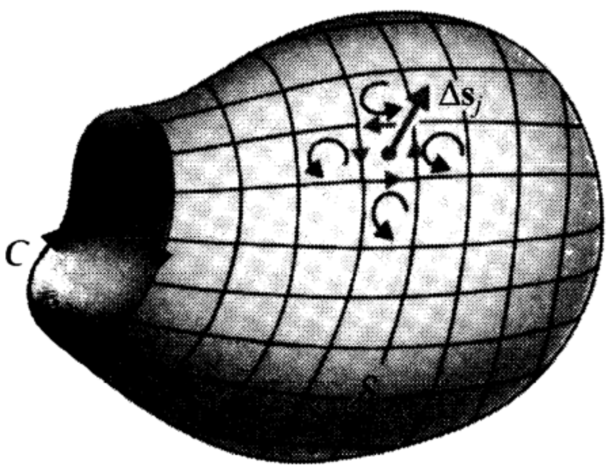
C منحنی بستهٔ پیرامون سطح S

در این قضیه یک انتگرال سطحی از تاو یک بردار به یک انتگرال خطی از بردار تبدیل می شود و بالعکس

کاربرد قضیه استوکس برای یک سطح بسته

If the surface integral of $\nabla \times \mathbf{A}$ is carried over a closed surface, there will be no surface-bounding external contour, and Eq. (2–143) tells us that

$$\oint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = 0 \qquad (2\text{–}144)$$



for any closed surface S. The geometry in Fig. 2–32 is chosen deliberately to emphasize the fact that a nontrivial application of Stokes's theorem always implies *an open surface with a rim.*

In Section 2–7 we stated that a net outward flux of a vector **A** through a surface bounding a volume indicates the presence of a source. This source may be called a ***flow source***, and div **A** is a measure of the strength of the flow source. There is another kind of source, called ***vortex source***, which causes a circulation of a vector field around it. The ***net circulation*** (or simply ***circulation***) of a vector field around a *closed path* is defined as the scalar line integral of the vector over the path. We have

$$\oint_C \mathbf{A} \cdot d\ell .$$

Circulation of **A** around contour

Equation (2–124) is a mathematical definition. The physical meaning of circulation depends on what kind of field the vector **A** represents. If **A** is a force acting on an object, its circulation will be the work done by the force in moving the object once around the contour; if **A** represents an electric field intensity, then the circulation will be an electromotive force around the closed path, as we shall see later in the book. The familiar phenomenon of water whirling down a sink drain is an example of a *vortex sink* causing a circulation of fluid velocity. A circulation of **A** may exist

نیروی پایستار

An immediate utilization of Stokes' theorem is the derivation of the criterion for determining whether a field is *conservative* or not. If $\oint \mathbf{A} \cdot d\mathbf{r} = 0$, for all possible closed paths, in a region of space, then it follows that $\nabla \times \mathbf{A} = 0$ everywhere in this region. The converse is also true. Such a vector is called a conservative vector. We may therefore summarize the criteria that determine whether or not a vector field **A** is conservative in a region of space. If in some simply connected region one of the following relations holds,

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = 0 \qquad \text{for arbitrary } C$$

$$\nabla \times \mathbf{A} = 0 \tag{1.52}$$

$$\mathbf{A} = \nabla f \qquad \text{for some scalar function } f \text{ (see Example 1.2)}$$

then **A** is a conservative field. One of these criteria being true throughout a simply connected region of space implies that the other two also are true.

EXAMPLE

Show that $\nabla \times \mathbf{A} = 0$ if

a) $\mathbf{A} = \mathbf{a}_\phi(k/r)$ in cylindrical coordinates, where k is a constant, *or*

b) $\mathbf{A} = \mathbf{a}_R f(R)$ in spherical coordinates, where $f(R)$ is any function of the radial distance R.

a) In cylindrical coordinates the following apply:

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \mathbf{a}_r & \mathbf{a}_\phi r & \mathbf{a}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \frac{1}{r}\begin{vmatrix} \mathbf{a}_r & \mathbf{a}_\phi r & \mathbf{a}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ 0 & k & 0 \end{vmatrix} = 0$$

b) In spherical coordinates the following apply:

$$\nabla \times \mathbf{A} = \frac{1}{R^2 \sin\theta} \begin{vmatrix} \mathbf{a}_R & \mathbf{a}_\theta R & \mathbf{a}_\phi R \sin\theta \\ \dfrac{\partial}{\partial R} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_R & RA_\theta & R \sin\theta A_\phi \end{vmatrix}$$

$$= \frac{1}{R^2 \sin\theta} \begin{vmatrix} \mathbf{a}_R & \mathbf{a}_\theta R & \mathbf{a}_\phi R \sin\theta \\ \dfrac{\partial}{\partial R} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ f(R) & 0 & 0 \end{vmatrix} = 0.$$

A curl-free vector field is called an ***irrotational*** or a ***conservative field***. We will see in the next chapter that an electrostatic field is irrotational (or conservative). The

۷.۱ عملگر دیفرانسیلی برداری ∇

$$\text{grad} = \nabla$$

$$\nabla\varphi = \mathbf{i}\frac{\partial\varphi}{\partial x} + \mathbf{j}\frac{\partial\varphi}{\partial y} + \mathbf{k}\frac{\partial\varphi}{\partial z}$$

$$\text{div} = \nabla\cdot$$

$$\nabla\cdot\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\text{curl} = \nabla\times$$

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

قضیه‌های انتگرالی مهم

$$\int_{a\,C}^{b} \nabla\varphi \cdot d\mathbf{l} = \int_{a}^{b} d\varphi = \varphi\Big|_{a}^{b} = \varphi_b - \varphi_a$$

$$\int_{S} \nabla\times\mathbf{F}\cdot\mathbf{n}\,da = \oint_{C} \mathbf{F}\cdot d\mathbf{l}$$

$$\int_{V} \nabla\cdot\mathbf{F}\,dv = \oint_{S} \mathbf{F}\cdot\mathbf{n}\,da$$

∇ یك عملگر خطی است و اگر a و b کمیتهای نرده‌ای ثابت

$$\nabla(a\varphi + b\psi) = a\nabla\varphi + b\nabla\psi$$

$$\nabla \cdot (a\mathbf{F} + b\mathbf{G}) = a\nabla \cdot \mathbf{F} + b\nabla \cdot \mathbf{G}$$

$$\nabla \times (a\mathbf{F} + b\mathbf{G}) = a\nabla \times \mathbf{F} + b\nabla \times \mathbf{G}$$

عملگرهای مرکب

۱– عملگر لاپلاسی (واگرایی میدان نرده ای)

۲– عملگر تاو شیب میدان نرده ای

۳– عملگر واگرایی تاو میدان برداری

۴– عملگر تاو تاو میدان برداری

۱– عملگر لاپلاسی (واگرایی میدان نرده ای)

واگرایی شیب یك میدان نرده‌ای. $\nabla \cdot \nabla = \nabla^{۲}$

$$\nabla^2 f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f$$

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

۲– عملگر تاو شیب میدان نرده ای

تاو شیب هرمیدان نرده‌ای صفر است.

$$\nabla\times\nabla=0$$

اثبات ۱:

$$\nabla\times(\nabla\varphi)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z}\end{vmatrix}=\mathbf{i}\left(\frac{\partial^2\varphi}{\partial y\,\partial z}-\frac{\partial^2\varphi}{\partial z\,\partial y}\right)+\cdots=0$$

اثبات ۲:

by Stokes's theorem:

$$\int_S[\nabla\times(\nabla V)]\cdot d\mathbf{s}=\oint_C(\nabla V)\cdot d\boldsymbol{\ell}=\oint_C dV=0.$$

A converse statement of Identity I can be made as follows: ***If a vector field is curl-free, then it can be expressed as the gradient of a scalar field.*** Let a vector field be $\mathbf{E}$. Then, if $\nabla \times \mathbf{E} = 0$, we can define a scalar field V such that

$$\mathbf{E} = -\nabla V.$$

irrotational (a conservative) vector field can always be expressed as the gradient of a scalar field.

۳– عملگر واگرایی تاو میدان برداری

واگرایی هرتاو نیز برابر صفر است.

$$\nabla\cdot\nabla\times\mathbf{F}=\nabla\times\nabla\cdot\mathbf{F}=0$$

$$\nabla\cdot(\nabla\times\mathbf{F})=\frac{\partial}{\partial x}\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)+\cdots=0$$

Equation (1.66) is important in magnetostatics since the divergence of the magnetic field $\mathbf{B}$ is known to be zero ($\nabla\cdot\mathbf{B}=0$), then it allows casting of the magnetic field $\mathbf{B}$ in terms of a vector potential $\mathbf{A}$:

$$\mathbf{B}=\nabla\times\mathbf{A}.$$

۴– عملگر تاو تاو میدان برداری

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^{۲}\mathbf{F}$$

$$\nabla \times (\nabla \times \mathbf{f}) = \nabla(\nabla \cdot \mathbf{f}) - \nabla \cdot (\nabla \mathbf{f})$$

where $\nabla \mathbf{f}$ is a second-rank tensor or dyadic (see Example 1.3). In cartesian coordinates we have $\nabla \cdot (\nabla \mathbf{f}) = (\nabla \cdot \nabla)\mathbf{f} = \nabla^2 \mathbf{f}$ where ∇^2 is the laplacian operator.

$$\nabla \cdot \nabla\varphi = \nabla^2\varphi$$ (۱.۱.۱)

$$\nabla \cdot \nabla \times \mathbf{F} = 0$$ (۲.۱.۱)

$$\nabla \times \nabla\varphi = 0$$ (۳.۱.۱)

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$$ (۴.۱.۱)

$$\nabla(\varphi\psi) = (\nabla\varphi)\psi + \varphi\nabla\psi$$ (۵.۱.۱)

$$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{F} \times (\nabla \times \mathbf{G}) + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{G} \times (\nabla \times \mathbf{F})$$ (۶.۱.۱)

$$\nabla \cdot (\varphi\mathbf{F}) = (\nabla\varphi) \cdot \mathbf{F} + \varphi\nabla \cdot \mathbf{F}$$ (۷.۱.۱)

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - (\nabla \times \mathbf{G}) \cdot \mathbf{F}$$ (۸.۱.۱)

$$\nabla \times (\varphi\mathbf{F}) = (\nabla\varphi) \times \mathbf{F} + \varphi\nabla \times \mathbf{F}$$ (۹.۱.۱)

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = (\nabla \cdot G)\mathbf{F} - (\nabla \cdot \mathbf{F})\mathbf{G} + (G \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$$ (۱۰.۱.۱)

اتحادهای دیفرانسیلی برداری

اگر تابع $\mathbf{F} = \mathbf{r}$ باشد

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$$

$$\nabla \cdot \mathbf{r} = ۳$$

$$\nabla \times \mathbf{r} = ۰$$

$$\mathbf{G} \cdot \nabla \mathbf{r} = \mathbf{G}$$

$$\nabla^{۲} \mathbf{r} = ۰$$

قضیه گرین

$$\int_{v} (\psi\nabla^{2}\varphi - \varphi\nabla^{2}\psi)\, dv = \oint_{s} (\psi\nabla\varphi - \varphi\nabla\psi) \cdot \mathbf{n}\, da$$

اثبات: در قضیه واگرایی به جای F به صور زیر می گذاریم

$$\int_{v} \nabla \cdot \mathbf{F}\, dv = \oint_{s} \mathbf{F} \cdot \mathbf{n}\, da$$

$$\mathbf{F} = \psi\nabla\varphi - \varphi\nabla\psi$$

قضیه گرین

قضیه‌های انتگرالی برداری

$$\int_s \mathbf{n} \times \nabla\varphi \, da = \oint_c \varphi \, d\mathbf{l}$$

$$\int_v \nabla\varphi \, dv = \oint_s \varphi\mathbf{n} \, da$$

$$\int_v \nabla \times \mathbf{F} \, dv = \oint_s \mathbf{n} \times \mathbf{F} \, da$$

$$\int_v (\nabla \cdot \mathbf{G} + \mathbf{G} \cdot \nabla)\mathbf{F} \, dv = \oint_s \mathbf{F}(\mathbf{G} \cdot \mathbf{n}) \, da$$

Conservative Nature of Radial Vectors—Potential Functions

Consider a radial vector field given in spherical coordinates by $\mathbf{A} = f(r)\hat{\mathbf{r}}$ where $f(r)$ is a scalar function that depends on r only. We will show below that this vector is conservative. The criteria that determine whether or not a vector is conservative are given in Eq. (1.52). Substituting $A_r = f(r)$, and $A_\theta = A_\phi = 0$ in $\nabla \times \mathbf{A}$ in spherical coordinates (Eq. 1.47) immediately gives $\nabla \times \mathbf{A} = 0$; thus indicating that $\mathbf{A}$ is conservative.

Radial vector fields are of importance to electrostatics since the electric field produced by a point charge is radial, and hence it is conservative. Because of the importance of this property, we will examine the conservative nature of these vectors from the point of view of the last criterion of Eq. (1.52). If $\mathbf{A}$ is conservative, then it must be written as the gradient of a scalar function Φ; that is, $\mathbf{A} = \nabla\Phi$.

Gradient of a Vector—Dyadics

This example deals with the gradient of a vector, which will be useful when we deal with forces on electric dipoles placed in external electric fields. Consider a vector $\mathbf{E} = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}$. Formally, we can define $\nabla\mathbf{E}$ as follows:

$$\nabla\mathbf{E} = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)(E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}})$$

Expanding, we get

$$\begin{aligned}\nabla\mathbf{E} = &\left(\frac{\partial E_x}{\partial x}\hat{\mathbf{x}}\hat{\mathbf{x}} + \frac{\partial E_y}{\partial x}\hat{\mathbf{x}}\hat{\mathbf{y}} + \frac{\partial E_z}{\partial x}\hat{\mathbf{x}}\hat{\mathbf{z}}\right) + \left(\frac{\partial E_x}{\partial y}\hat{\mathbf{y}}\hat{\mathbf{x}} + \frac{\partial E_y}{\partial y}\hat{\mathbf{y}}\hat{\mathbf{y}} + \frac{\partial E_z}{\partial y}\hat{\mathbf{y}}\hat{\mathbf{z}}\right)\\ &+ \left(\frac{\partial E_x}{\partial z}\hat{\mathbf{z}}\hat{\mathbf{x}} + \frac{\partial E_y}{\partial z}\hat{\mathbf{z}}\hat{\mathbf{y}} + \frac{\partial E_z}{\partial z}\hat{\mathbf{z}}\hat{\mathbf{z}}\right)\end{aligned} \tag{1.76}$$

The quantities $\hat{\mathbf{x}}\hat{\mathbf{x}}$, $\hat{\mathbf{x}}\hat{\mathbf{y}}$,… are called *unit dyads*. Note that $\hat{\mathbf{x}}\hat{\mathbf{y}}$, for example, is not the same as $\hat{\mathbf{y}}\hat{\mathbf{x}}$; thus we have nine different unit dyads in the gradient. A quantity that can be expanded in the form

$$\mathbf{\Phi} = a_{11}\hat{\mathbf{x}}\hat{\mathbf{x}} + a_{12}\hat{\mathbf{x}}\hat{\mathbf{y}} + a_{13}\hat{\mathbf{x}}\hat{\mathbf{z}} + a_{21}\hat{\mathbf{y}}\hat{\mathbf{x}} + a_{22}\hat{\mathbf{y}}\hat{\mathbf{y}} + a_{23}\hat{\mathbf{y}}\hat{\mathbf{z}} + a_{31}\hat{\mathbf{z}}\hat{\mathbf{x}} + a_{32}\hat{\mathbf{z}}\hat{\mathbf{y}} + a_{33}\hat{\mathbf{z}}\hat{\mathbf{z}} \tag{1.77}$$

is called a dyadic, and the nine coefficients a_{ij} are its components.

In previous sections we mentioned that *a divergenceless field is solenoidal* and *a curl-free field is irrotational.* We may classify vector fields in accordance with their being solenoidal and/or irrotational. A vector field $\mathbf{F}$ is

1. Solenoidal and irrotational if

$$\nabla \cdot \mathbf{F} = 0 \qquad \text{and} \qquad \nabla \times \mathbf{F} = 0.$$

EXAMPLE: A static electric field in a charge-free region.

2. Solenoidal but not irrotational if

$$\nabla \cdot \mathbf{F} = 0 \qquad \text{and} \qquad \nabla \times \mathbf{F} \neq 0.$$

EXAMPLE: A steady magnetic field in a current-carrying conductor.

3. Irrotational but not solenoidal if

$$\nabla \times \mathbf{F} = 0 \qquad \text{and} \qquad \nabla \cdot \mathbf{F} \neq 0.$$

EXAMPLE: A static electric field in a charged region.

4. Neither solenoidal nor irrotational if

$$\nabla \cdot \mathbf{F} \neq 0 \qquad \text{and} \qquad \nabla \times \mathbf{F} \neq 0.$$

EXAMPLE: An electric field in a charged medium with a time-varying magnetic field.

Helmholtz's Theorem

The most general vector field then has both a nonzero divergence and a nonzero curl, and can be considered as the sum of a solenoidal field and an irrotational field.

Helmholtz's theorem states that a general vector function **F** can be written as the sum of the gradient of a scalar function and the curl of a vector function. Thus

$$\mathbf{F} = -\nabla V + \nabla \times \mathbf{A}. \tag{2-160}$$