بسم الله الرحمن الرحیم

فصل سوم

حل مسائل الکتروستاتیک

(بخش پنجم)

۱۲.۳ دستگاه رساناها.ضرایب پتانسیل

فرض کنید N رسانا باوضع هندسی ثابت وجود دارند. همچنین فرض کنید تمام رساناها بدون بارند، بجز رسانای j ام که دارای بار Q_{j0} است، جواب مناسب معادلهٔ لاپلاس را درفضای خارج از سطوح رساناها با نماد $\varphi^{(j)}(x, y, z)$ و پتانسیل هریك از رساناها را با $\varphi_1^{(j)}, \varphi_2^{(j)} \ldots \varphi_j^{(j)}, \ldots \varphi^{(j)}_N$ نشان می‌دهیم

$$Q_j \to \lambda Q_j$$

تابع پانسیل در همه جا در λ ضرب می شود

معادله لاپلاس برقرار است

پتانسیل هررسانا متناسب است با Q_j، بار رسانای

$$\varphi_i^{(j)} = p_{ij} Q_j, \quad (i = ۱,\ ۲,\ \cdots,\ N)$$

p_{ij} ثابتی است که فقط به‌شکل هندسی بستگی دارد.

$$\varphi_i = p_{ij} Q_j + p_{ik} Q_k, \quad (i = ۱,\ ۲,\ \cdots,\ N)$$

جواب یگانه معادله لاپلاس

پتانسیل در رسانای i را ناشی از بارها روی رساناهای j و k نشان می دهد

ضرایب پتانسیل p_{ij} ارتباط بین هندسه رسانای i و j را بیان می نماید

به طورکلی اگر رساناهای مختلف از ۱ تا N باردار باشند. پتانسیل در محل رسانای i ام ، به صورت رابطه زیر بیان می شود:

$$\varphi_i = \sum_{j=1}^{N} p_{ij} Q_j$$

رابطهٔ خطی میان پتانسیل و بار

ضرایب p_{ij} ضرایب پتانسیل

آرایهٔ (ماتریس) این ضرایب متقارن است، یعنی $p_{ij} = p_{ji}$.

۱۳۰۳ جوابهای معادلهٔ پواسون

اینك به‌مسئله‌ای از الکتروستاتیك می‌پردازیم که درآن قسمتی از بار (بار مشخص شده ازقبل) باتابع معلوم $\rho(x, y, z)$ داده شده‌است وبقیهٔ بار (بار القایی) روی سطوح رساناها قراردارد. چنین مسئله‌ای احتیاج به‌حل معادلهٔ پواسون دارد. جواب عمومی این معادله را می‌توان به‌صورت حاصل جمع انتگرالی ازنوع معادلهٔ (۱۰۳) روی بارمشخص شده‌ازقبل وجواب عمومی معادلهٔ لاپلاس نوشت. اما جواب معادلهٔ لاپلاس را باید طوری انتخاب کرد که کل پتانسیل درتمام شرایط مرزی مسئله صدق کند.

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho\, dv'}{|\mathbf{r}-\mathbf{r}'|} + \text{solution of Laplace's equation}$$

حل معادله پواسون به روش مستقیم:

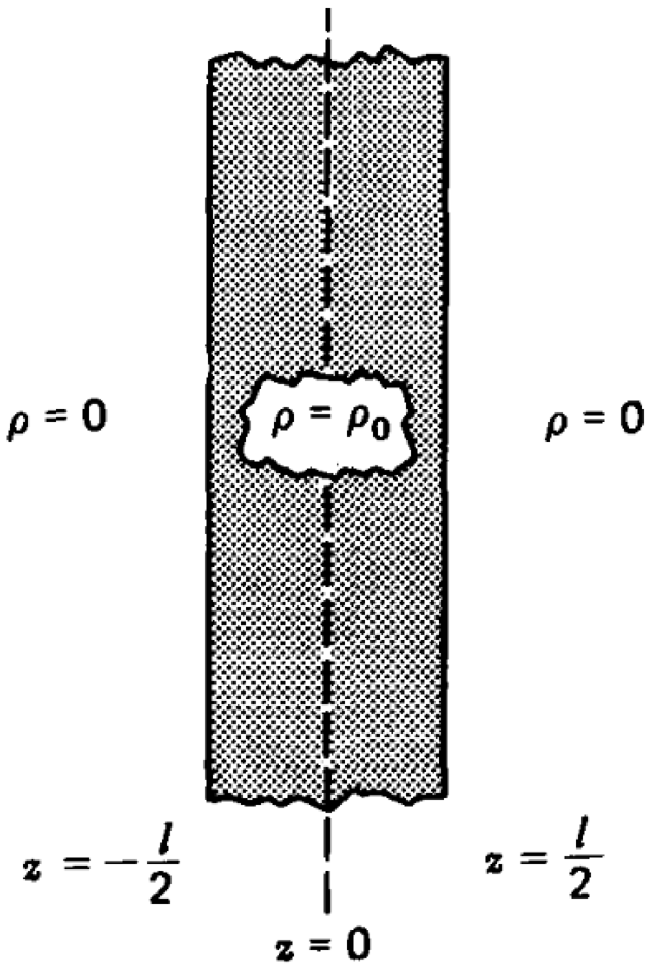
این مورد وقتی پیش می‌آید که هم ρ و هم φ فقط توابعی از یک متغیرمستقل باشند.

به‌عنوان مثالی ازاین مورد، فرض می‌کنیم دردستگاه مختصات کروی ρ فقط تابعی ازمتغیر r است وفرض می‌کنیم تمامی بارباتقارن کروی توزیع شده است

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right)=-\frac{1}{\varepsilon_0}\rho(r)$$

فرض می‌کنیم که بارکل محدود است، یعنی یا بار تا بی‌نهایت ادامه ندارد و یاچگالی بار برای مقادیر بزرگ r سریعاً به‌سمت صفرمیل کند.

مثال) یک تسمه نازک به ضخامت l (در امتداد محور z) و به طول بی نهایت با توزیع بار یکنواخت داریم



$$\begin{cases} \rho = \rho_0 & \text{for } -\frac{l}{2} \le z \le \frac{l}{2} \\ \rho = 0 & \text{for } |z| > \frac{l}{2} \end{cases}$$

۱- $$\Phi = \Phi(z) \quad \rightarrow \quad \nabla^2 \Phi = \frac{d^2\Phi}{dz^2}$$

۲- مرجع پتانسیل در روی محور تسمه با پتانسیل صفر

$$\Phi(z=0) = 0$$

حل معادله پواسون در نواحی مختلف

Region	Equation	Solution
$z \geq \frac{l}{2}$	$\frac{d^2\Phi_2}{dz^2} = 0$	$\Phi_2 = C_1 z + C_2$
$\lvert z\rvert \leq \frac{l}{2}$	$\frac{d^2\Phi_0}{dz^2} = -\frac{\rho_0}{\varepsilon_0}$	$\Phi_0 = -\frac{\rho_0}{\varepsilon_0}\frac{z^2}{2} + K_1 z + K_2$
$z \leq -\frac{l}{2}$	$\frac{d^2\Phi_1}{dz^2} = 0$	$\Phi_1 = C'_1 z + C'_2$

پیدا کردن ضرایب ثابت معادلات پواسون C_1 ، C_1 ، C_1 ، C_1 ، K_1 و K_2

استفاده از شرایط مرزی در $z = \pm\frac{l}{2}$

$K_2 = 0$ since $\Phi(z=0) = 0$ (1)

1. The first boundary condition is the continuity of the potential across the boundary region. At $z = -l/2$, $\Phi_1 = \Phi_0$, and at $z = l/2$, $\Phi_0 = \Phi_2$,

$$\frac{d^2\Phi_0}{dz^2} = -\frac{\rho_0}{\varepsilon_0} \qquad \Phi_0 = -\frac{\rho_0}{\varepsilon_0}\frac{z^2}{2} + K_1 z + K_2$$

$$\frac{d^2\Phi_1}{dz^2} = 0 \qquad \Phi_1 = C_1' z + C_2'$$

$$\frac{d^2\Phi_2}{dz^2} = 0 \qquad \Phi_2 = C_1 z + C_2$$

$$\frac{d^2\Phi_0}{dz^2} = -\frac{\rho_0}{\varepsilon_0} \qquad \Phi_0 = -\frac{\rho_0}{\varepsilon_0}\frac{z^2}{2} + K_1 z + K_2$$

$$\Rightarrow -\frac{l}{2}C_1' + C_2' = -\frac{l^2}{8}\frac{\rho_0}{\varepsilon_0} - \frac{l}{2}K_1 \quad (2)$$

$$\Rightarrow \frac{l}{2}C_1 + C_2 = -\frac{l^2}{8}\frac{\rho_0}{\varepsilon_0} + \frac{l}{2}K_1 \quad (3)$$

2. Remembering that there is no surface charge density (in this problem), the normal components of **E**, $-\partial\Phi/\partial z$, are everywhere continuous; hence, from the boundary at $z = -l/2$ and at $z = l/2$, we find that

$$at\ z = -\frac{l}{2}: \quad \frac{\partial \Phi_0}{\partial z} = \frac{\partial \Phi_1}{\partial z}$$

$$C_1' = \frac{l}{2}\frac{\rho_0}{\varepsilon_0} + K_1 \qquad (4)$$

$$at\ z = +\frac{l}{2}: \quad \frac{\partial \Phi_2}{\partial z} = \frac{\partial \Phi_0}{\partial z}$$

$$C_1 = -\frac{l}{2}\frac{\rho_0}{\varepsilon_0} + K_1 \qquad (5)$$

3. The symmetry of the configuration requires that the electric field be zero at $z = 0$ [that is, $\mathbf{E}(z) = -\mathbf{E}(-z)$]; therefore $K_1 = 0$. Solving the above equations simultaneously results in the following expressions for the potentials:

$$\Phi_0 = -\frac{\rho_0}{\varepsilon_0}\frac{z^2}{2} + K_1 z + K_2$$

$$E(z) = -\frac{\partial \Phi_0}{\partial z} = \frac{\rho_0}{\varepsilon_0} z + K_1$$

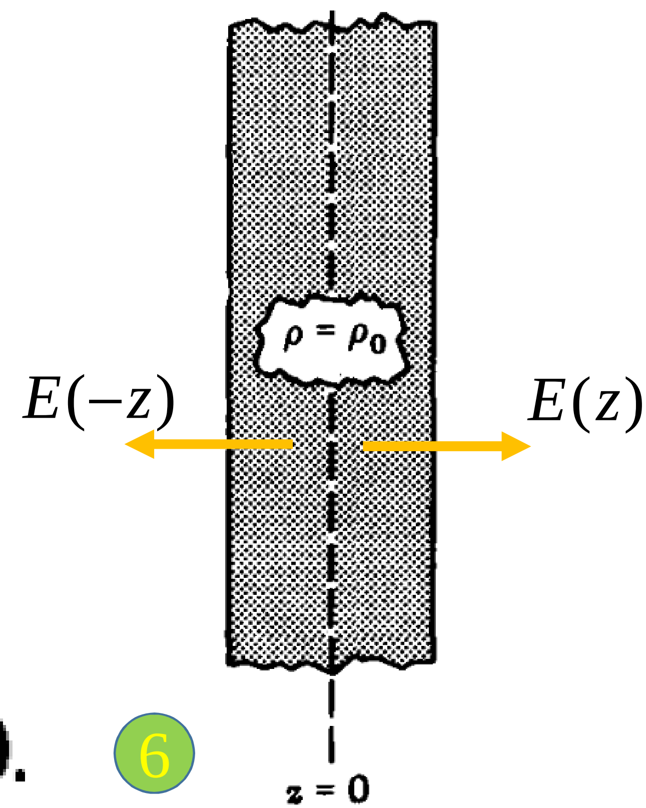
$$E(z) = -E(-z)$$

$$\frac{\rho_0}{\varepsilon_0} z + K_1 = -[\frac{\rho_0}{\varepsilon_0}(-z) + K_1]$$

$$K_1 = -K_1$$

$$K_1 = 0.$$

6

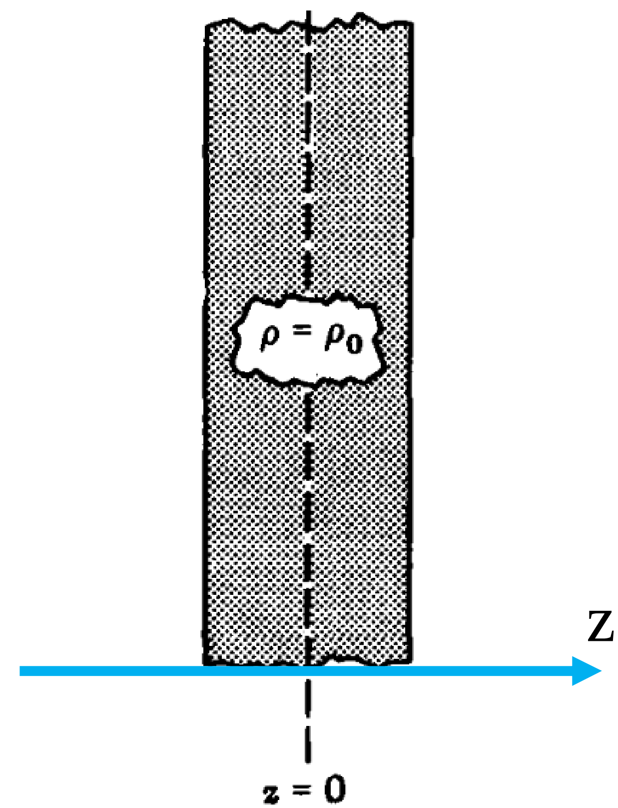


با کمک ٦ رابطه بدست آمده می توان بدست آورد:

$$\Phi_0 = -\frac{\rho_0 z^2}{2\varepsilon_0}$$

$$\Phi_1 = \frac{\rho_0 l}{2\varepsilon_0} z + \frac{\rho_0 l^2}{8\varepsilon_0}$$

$$\Phi_2 = -\frac{\rho_0 l}{2\varepsilon_0} z + \frac{\rho_0 l^2}{8\varepsilon_0}$$



Example 3.17 Uniformly Charged Sphere

We now analyze a case where the charge is distributed in a spherically symmetric way. Let a charge q be distributed over a sphere of radius R with a constant volume charge density ρ, and thus for $r > R$ the charge density is zero.

Poisson's equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = -\frac{\rho}{\varepsilon_0} \qquad r \leq R$$

$$\Phi(r) = \frac{-\rho r^2}{6\varepsilon_0} + \frac{A_1}{r} + B_1$$

Laplace's equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 0 \qquad r > R,$$

$$\Phi(r) = \frac{A_2}{r} + B_2$$

boundary conditions:

(1) $\Phi\ (r \to \infty) = 0$;

(2) $\Phi\ (r \to 0)$ is finite since there are no point charges at the center of the sphere;

(3) the two potentials should match at $r = R$;

(4) the total charge of this distributi is $(4\pi/3)R^3\rho$.

(1) $\Phi \quad (r \to \infty) = 0;$

$$\Phi(r) = \frac{A_2}{r} + B_2 \qquad r > R,$$

$\Rightarrow$ $B_2 = 0.$

(2) $\Phi \quad (r \to 0)$ is finite since there are no point charges at the center of the sphere;

$$\Phi(r) = \frac{-\rho r^2}{6\varepsilon_0} + \frac{A_1}{r} + B_1 \qquad r \leq R$$

$\Rightarrow$ $A_1 = 0.$

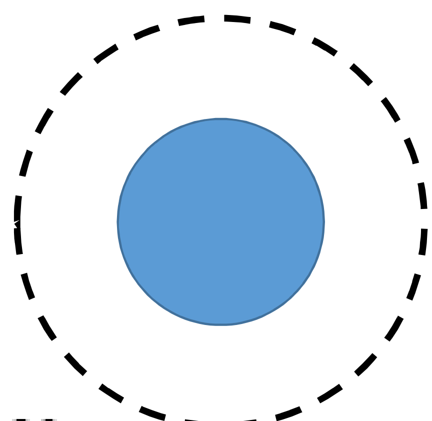
(3) the two potentials should match at $r = R$:

$$\Phi_{\text{int}}(r=R) = \Phi_{ext}(r=R)$$

$$-\frac{\rho R^2}{6\varepsilon_0} + B_1 = \frac{A_2}{R}$$

(4) the total charge of this distributi is $(4\pi/3)R^3\rho$.

$$\oint \mathbf{E}\cdot\hat{\mathbf{n}}\, da = \frac{4\pi R^3}{3\varepsilon_0}\rho$$



gaussian surface

$$r > R, \qquad \Phi(r) = \frac{A_2}{r} + B_2$$

$$E = -\frac{d\Phi}{dr} \quad \Rightarrow \quad \mathbf{E} = A_2 \hat{\mathbf{r}}/r^2$$

$$\oint \vec{E} \,.\, \hat{n} da = \frac{4\pi R^3}{3\varepsilon_0} \rho$$

$$\oint \frac{A_2}{r^2} \hat{r} \,.\, \hat{r} da = \frac{4\pi R^3}{3\varepsilon_0} \rho \quad \rightarrow \quad \frac{A_2}{R^2} \oint da = \frac{4\pi R^3}{3\varepsilon_0} \rho \quad \rightarrow \quad \frac{A_2}{R^2} (4\pi R^2) = \frac{4\pi R^3}{3\varepsilon_0} \rho$$

$$A_2 = \frac{R^3}{3\varepsilon_0} \rho$$

$$\Phi(r) = \frac{\rho R^2}{2\varepsilon_0}\left(1 - \frac{r^2}{3R^2}\right) \qquad r < R$$

$$\Phi(r) = \frac{\rho R^3}{3\varepsilon_0}\frac{1}{r} \qquad r > R$$

Equation (3.141) indicates that the potential inside the sphere is a quadratic function of r with the potential at the origin larger than that at the edge of the sphere. It is to be noted that the electric field is continuous at $r = R$. For $r \leq R$, $\mathbf{E} = (\rho r/3\varepsilon_0)\hat{\mathbf{r}}$ and for $r \geq R$, $\mathbf{E} = \rho R^3/(3\varepsilon_0 r^2)\hat{\mathbf{r}}$, giving $\mathbf{E} = \rho R/(3\varepsilon_0)\hat{\mathbf{r}}$ at $r = R$. This continuity is a direct result of the absence of surface charge at $r = R$.

Exponential Charge Distribution

Consider a spherically symmetric charge distribution (of total charge q), which has the radial dependence $\rho(r) = \rho_0 e^{-\alpha r}$. We mention here that this density describes the electronic charge distribution in the ground state of hydrogen. The potential at an arbitrary r satisfies Poisson's equation:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = \frac{-\rho(r)}{\varepsilon_0}$$

$$\frac{d\Phi}{dr} = -\frac{1}{r^2}\int \frac{\rho(r)}{\varepsilon_0} r^2\, dr$$

$$\frac{d\Phi}{dr} = \frac{\rho_0}{\varepsilon_0 \alpha^3} e^{-\alpha r}\left(\frac{2}{r^2} + \frac{2\alpha}{r} + \alpha^2\right) + \frac{C}{r^2}$$

$$\Phi(r) = \frac{q}{4\pi\varepsilon_0 r} e^{-\alpha r} - \frac{\alpha q}{8\pi\varepsilon_0} e^{-\alpha r} - \frac{C}{r} + D$$

boundary conditions:

charge density goes to zero as $r \to \infty$ ⇒ $D = 0$

Gauss' law on a closed shell of radius r and center at the origin.

$$\begin{cases} E = -\dfrac{d\Phi}{dr} \\ \dfrac{d\Phi}{dr} = \dfrac{\rho_0}{\varepsilon_0 \alpha^3} e^{-\alpha r}\left(\dfrac{2}{r^2} + \dfrac{2\alpha}{r} + \alpha^2\right) + \dfrac{C}{r^2} \end{cases}$$

$$\oint \mathbf{E} \cdot \hat{\mathbf{n}}\, da = \int \rho\, dv$$

$$\oint \mathbf{E} \cdot \hat{\mathbf{n}}\, da = 4\pi\rho_0 \int_0^r e^{-\alpha r} r^2\, dr$$

$$C = q/4\pi\varepsilon_0\, .$$

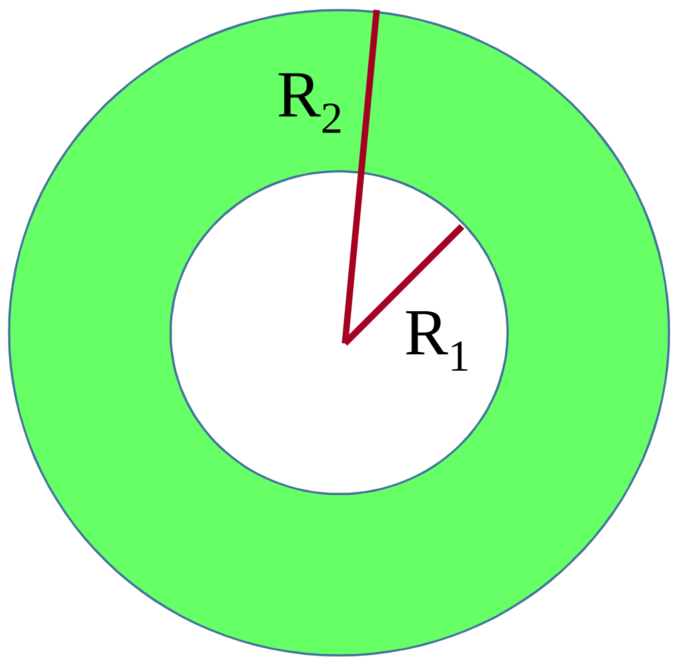
با جایگذاری مقادیر ثابت C و D

$$\Phi(r) = \frac{q}{4\pi\varepsilon_0 r}(1 - e^{-\alpha r}) - \frac{q\alpha}{8\pi\varepsilon_0} e^{-\alpha r}$$

$$\mathbf{E} = \frac{q\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2}[1 - (\alpha r + 1)e^{-\alpha r}] - \frac{q\alpha^2}{8\pi\varepsilon_0} e^{-\alpha r}\hat{\mathbf{r}}$$

Localized, Nonuniform, Spherically Symmetric Distribution

Let us now consider a localized, nonuniform, spherically symmetric charge distribution. Consider a concentric shell of charge of radii R_1 and R_2 ($R_2 > R_1$). The charge density is given by $\rho = \beta/r^2$ where β is a constant and r is the distance from the center of the shell.



This example involves three regions of space

Laplace's equation $\nabla^2\Phi_1 = 0$ for $R < R_1$

Poisson's equation $\nabla^2\Phi_2 = -\rho/\varepsilon_0$ for $R_1 < r < R_2$

Laplace's equation $\nabla^2\Phi_3 = 0$ for $r > R_2$

Laplace's equation $\nabla^2\Phi_1 = 0$

$$\Phi_1(r) = \frac{A_1}{r} + B_1 \qquad r < R_1$$

Laplace's equation $\nabla^2\Phi_3 = 0$

$$\Phi_3(r) = \frac{A_3}{r} + B_3 \qquad r > R_2$$

The potential inside the shell satisfies the following radial equation.

Poisson's equation $\nabla^2\Phi_2 = -\rho/\varepsilon_0$ for $R_1 < r < R_2$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi_2}{dr}\right) = -\frac{\beta}{\varepsilon_0 r^2} \qquad R_1 < r < R_2$$

integrated twice

$$\Phi_2(r) = \frac{-\beta}{\varepsilon_0}\ln r - \frac{A_2}{r} + B_2$$

six constants A_i and B_i

boundary conditions:

1 charge distribution is bounded, then the potential vanishes as $r \to \infty$,

$$\Phi_3(r) = \frac{A_3}{r} + B_3 \qquad r > R_2$$

$$\Rightarrow \quad B_3 = 0.$$

2 the potential be finite as $r \to 0$,

$$\Phi_1(r) = \frac{A_1}{r} + B_1 \qquad r < R_1$$

$$\Rightarrow \quad A_1 = 0.$$

③ The continuity of the potential at $r = R_1$ and $r = R_2$

$$\Phi_1(r = R_1) = \Phi_2(r = R_1)$$

$$\Phi_1(r) = \frac{A_1}{r} + B_1$$

$$\Phi_2(r) = \frac{-\beta}{\varepsilon_0} \ln r - \frac{A_2}{r} + B_2$$

$$\Rightarrow \quad B_1 = -\frac{\beta}{\varepsilon_0} \ln R_1 - \frac{A_2}{R_1} + B_2$$

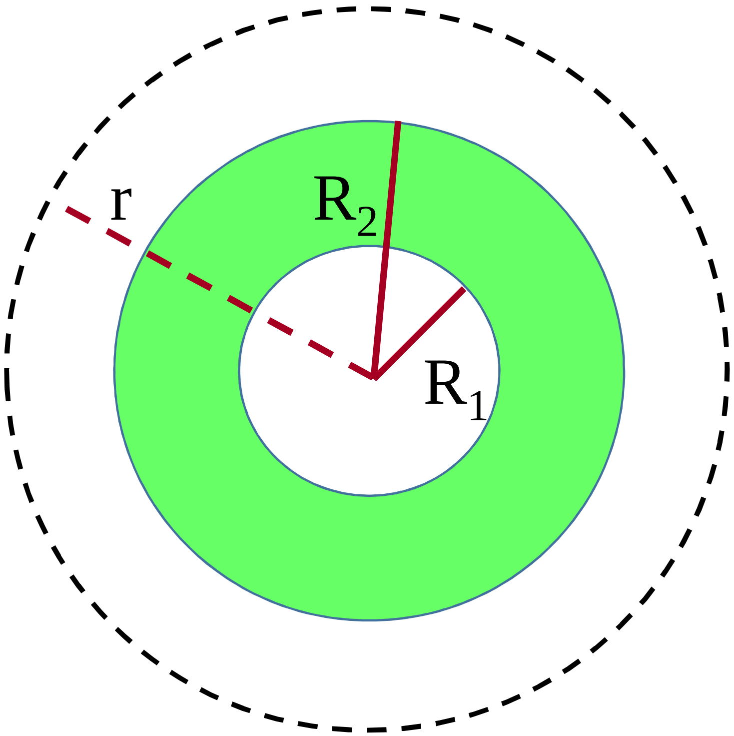
$$\Phi_2(r = R_2) = \Phi_3(r = R_2)$$

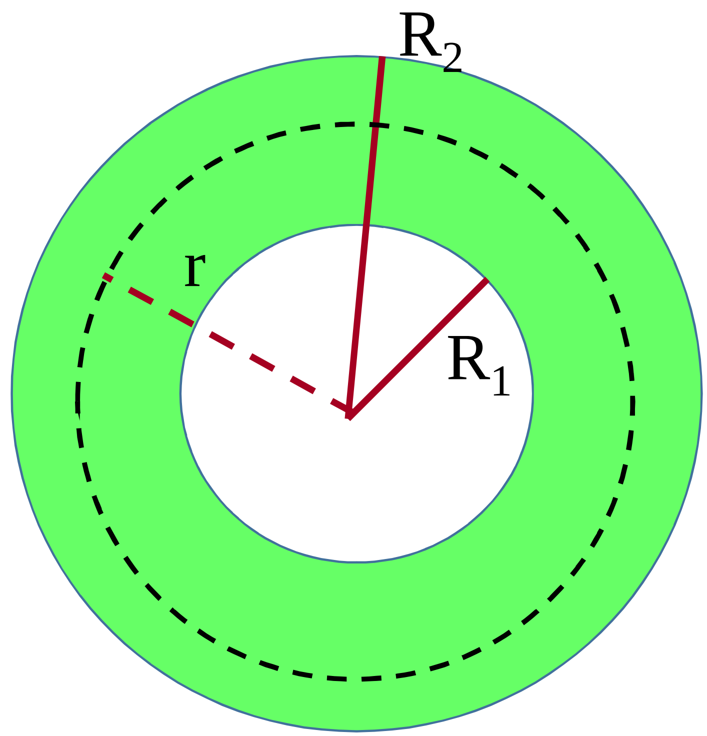
$$\Phi_2(r) = \frac{-\beta}{\varepsilon_0} \ln r - \frac{A_2}{r} + B_2$$

$$\Phi_3(r) = \frac{A_3}{r} + B_3$$

$$\Rightarrow \quad \frac{A_3}{R_2} = -\frac{\beta}{\varepsilon_0} \ln R_2 - \frac{A_2}{R_2} + B_2$$

4 **charge distribution and the total charge be given.**





$r > R_2$ $\oint \vec{E} \cdot \hat{n} da = \frac{q}{\varepsilon_0}$

$R_1 < r < R_2$ $\oint \vec{E} \cdot \hat{n} da = \frac{q}{\varepsilon_0}$

$$r > R_2 \qquad \oint \vec{E} \,.\, \hat{n} da = \frac{q}{\varepsilon_0}$$

$$\frac{q}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int \rho dv = \frac{1}{\varepsilon_0}\int_{r=R_1}^{r=R_2} \frac{\beta}{r^2} 4\pi r^2 dr = \frac{4\pi\beta}{\varepsilon_0}\int_{r=R_1}^{r=R_2} dr = \frac{4\pi\beta}{\varepsilon_0}(R_2 - R_1)$$

$$\vec{E} = -\vec{\nabla}\Phi_3 \quad \rightarrow \quad \vec{E} = \frac{A_3}{r^2}\hat{r}$$

$$\oint \vec{E} \,.\, \hat{n} da = \oint \frac{A_3}{r^2}\hat{r} \,.\, \hat{r} da = \frac{A_2}{r^2}\oint da = \frac{A_3}{r^2}(4\pi r^2) = 4\pi A_3$$

$$\boldsymbol{A_3 = \beta(R_2 - R_1)\varepsilon_0 .}$$

$$R_1 < r < R_2 \qquad \oint \vec{E} \cdot \hat{n} da = \frac{q}{\varepsilon_0}$$

$$\frac{q}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int \rho dv = \frac{1}{\varepsilon_0}\int_{r=R_1}^{r=r} \frac{\beta}{r^2} 4\pi r^2 dr = \frac{4\pi\beta}{\varepsilon_0}\int_{r=R_1}^{r=r} dr = \frac{4\pi\beta}{\varepsilon_0}(r - R_1)$$

$$\mathbf{E} = -\nabla\Phi_2 = \left(\frac{\beta}{\varepsilon_0 r} + \frac{A_2}{r^2}\right)\hat{\mathbf{r}}$$

$$\oint \vec{E} \cdot \hat{n} da = \oint \left(\frac{\beta}{\varepsilon_0 r} + \frac{A_2}{r^2}\right)\hat{r} \cdot \hat{r} da = \left(\frac{\beta}{\varepsilon_0 r} + \frac{A_2}{r^2}\right)(4\pi r^2) = \frac{4\pi\beta}{\varepsilon_0} r + 4\pi A_2$$

⇨ $$A_2 = \beta R_1/\varepsilon_0 .$$

$A_1 = 0.$ $A_2 = \beta R_1/\varepsilon_0.$ $A_3 = \beta(R_2 - R_1)\varepsilon_0.$

B$_1$= ? **B$_2$= ?** $B_3 = 0.$

$$\frac{A_3}{R_2} = -\frac{\beta}{\varepsilon_0}\ln R_2 - \frac{A_2}{R_2} + B_2$$

$$B_1 = -\frac{\beta}{\varepsilon_0}\ln R_1 - \frac{A_2}{R_1} + B_2$$

$$B_2 = \beta/\varepsilon_0(1 + \ln R_2)$$

$$B_1 = \beta \ln(R_2/R_1)/\varepsilon_0.$$

$$\Phi_1(r) = \frac{\beta}{\varepsilon_0} \ln \frac{R_2}{R_1}$$

$$\Phi_2(r) = \frac{\beta}{\varepsilon_0} \left(1 - \ln \frac{r}{R_2} - \frac{R_1}{r} \right)$$

$$\Phi_3(r) = \frac{\beta}{\varepsilon_0} \frac{R_2 - R_1}{r}$$

Charge Distribution Due to a Given Potential

Since Poisson equation relates the potential to the charge density, then it can be used to determine the charge density of a given potential. Consider the potential $\Phi(r) = (q/4\pi\varepsilon_0 r)e^{-\alpha r}$.

$$\rho = -\varepsilon_0 \nabla^2 \Phi(r)$$

$$\rho = -\frac{1}{4\pi}\nabla^2\left[\frac{q}{r} + \frac{q(e^{-\alpha r} - 1)}{r}\right]$$

$$\rho = q\delta(\mathbf{r}) - \frac{q}{4\pi}\alpha^2 \frac{e^{-\alpha r}}{r}$$