بِسْمِ اللَّهِ الرَّحْمَٰنِ الرَّحِيمِ

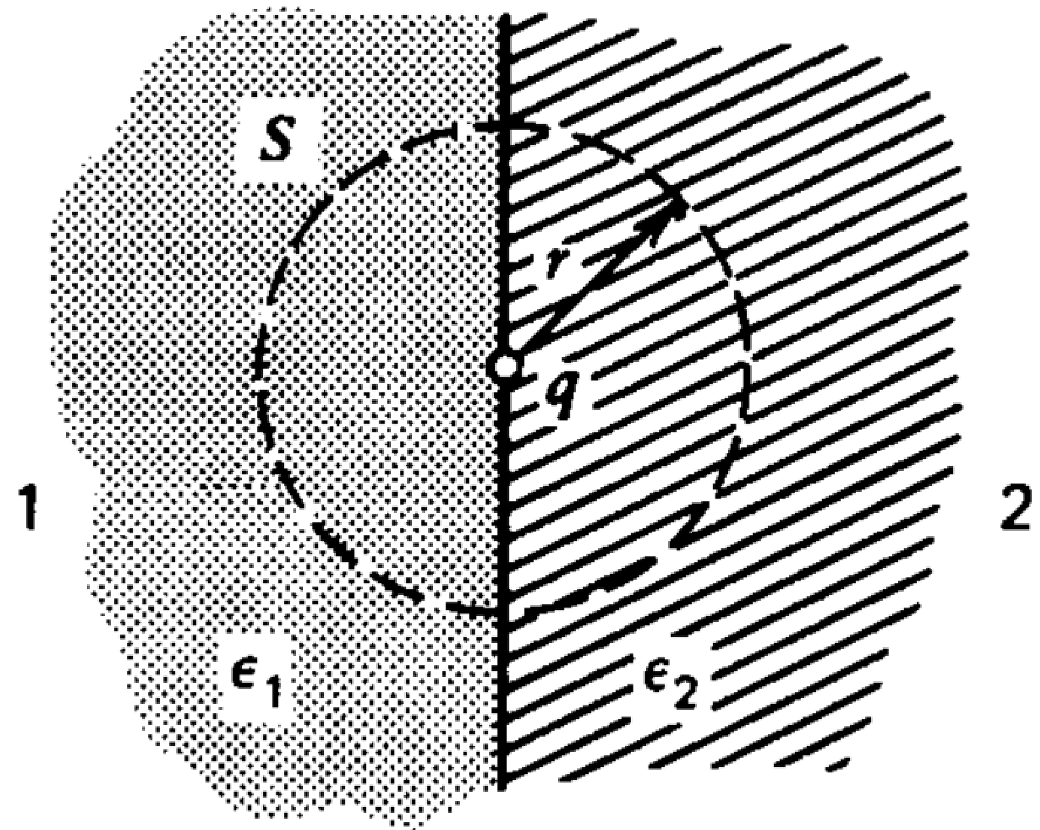
فصل چهارم

# میدان الکتروستاتیک در محیط های دی الکتریک

بخش سوم

## Example 45 Point Charge on a Plane Interfacc - Laplace's Equation In One Dimension

This example deals with a situation where a point charge ***q* is placed on the plane interface of** two homogeneous infinite dielectrics1 and 2 with permittivities $\boldsymbol{\varepsilon_1}$ ***and*** $\boldsymbol{\varepsilon_2}$***, respectively.***



***At points away from the point charge, the*** potential satisfies Laplace's equation;

$$\nabla^2 \Phi = 0$$

$$\Phi_1 = \frac{C_1 q}{r} + B_1 \qquad \textit{in regions 1}$$

$$\Phi_2 = \frac{C_2 q}{r} + B_2 \qquad \textit{in regions 2}$$

Where $C_1$, $C_2$, $B_1$, *and* $B_2$ *are constants.*

## Boundary conditions

(1) potential is due to a localized point charge

$$r \to \infty \qquad \Phi_1 = 0 \quad , \quad \Phi_2 = 0$$

$$\boldsymbol{B}_1 = \boldsymbol{B}_2 = 0$$

2 the potential is continuous at the boundary gives

$$\varphi_1(r) = \varphi_2(r)$$

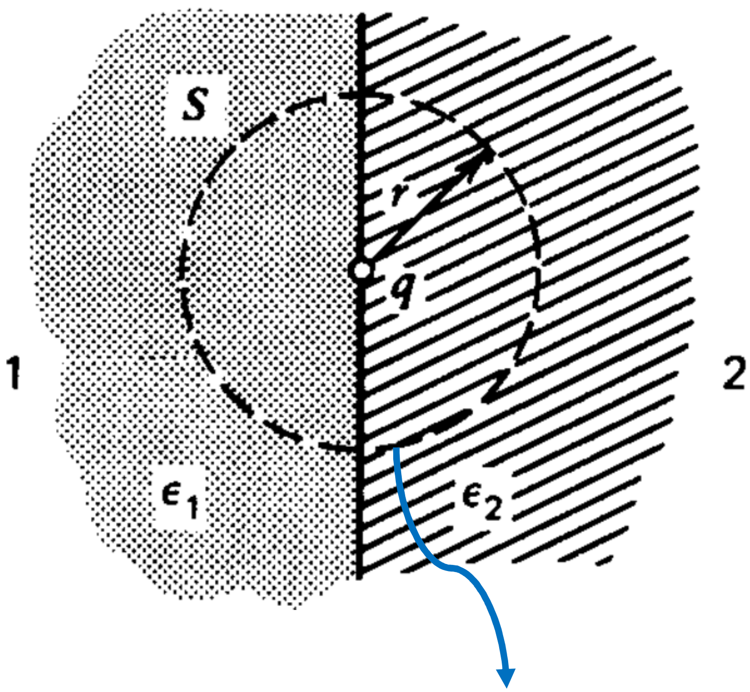
$$\Phi_1 = \frac{C_1 q}{r}$$

$$\Phi_2 = \frac{C_2 q}{r}$$

$$C_1 = C_2 = C$$

?

③ Evaluation of C by using **Gauss' law**




Spherical surface S with its center at the point charge

$$\mathbf{D} = -\varepsilon \nabla \Phi$$

$$\vec{D} = -\varepsilon \vec{\nabla} \Phi$$

$$\vec{D}_1 = \varepsilon_1 \frac{Cq}{r^2} \hat{r} \qquad \vec{D}_2 = \varepsilon_1 \frac{Cq}{r^2} \hat{r}$$

The electric field and displacement & Polarization

$$\oint \vec{D} \,.\, \hat{n} da = q$$

$$\int \vec{D}_1 \,.\, \hat{r} da + \int \vec{D}_2 \,.\, \hat{r} da = q$$

$$\int \varepsilon_1 \frac{Cq}{r^2} \hat{r} \,.\, \hat{r} da + \int \varepsilon_2 \frac{Cq}{r^2} \hat{r} \,.\, \hat{r} da = q$$

$$\varepsilon_1 \frac{Cq}{r^2}(2\pi r^2) + \varepsilon_2 \frac{Cq}{r^2}(2\pi r^2) = q$$

$$2\pi Cq(\varepsilon_1 + \varepsilon_2) = q$$

$$C = \frac{1}{2\pi(\varepsilon_1 + \varepsilon_2)}$$

$$\Phi_1 = \frac{C_1 q}{r} + B_1$$ *in regions 1*

$$\Phi_2 = \frac{C_2 q}{r} + B_2$$ *in regions 2*

$$B_1 = B_2 = 0$$

$$C_1 = C_2 = C = \frac{1}{2\pi(\varepsilon_1 + \varepsilon_2)}$$

$$\Phi(\mathbf{r}) = \frac{1}{2\pi(\varepsilon_1 + \varepsilon_2)} \frac{q}{r}$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{2\pi(\varepsilon_1 + \varepsilon_2)} \frac{q}{r^2}\hat{\mathbf{r}} \qquad \text{for all } r$$

$$\mathbf{D}_i(\mathbf{r}) = \frac{\varepsilon_i}{2\pi(\varepsilon_1 + \varepsilon_2)} \frac{q}{r^2}\hat{\mathbf{r}} \qquad i = 1, 2$$

$$\mathbf{P} = \mathbf{D} - \varepsilon_0 \mathbf{E}$$

$$\mathbf{P}_i(\mathbf{r}) = \frac{\varepsilon_i - \varepsilon_0}{2\pi(\varepsilon_1 + \varepsilon_2)} \frac{q}{r^2}\hat{\mathbf{r}}$$

**induced charge**

$\rho_p = -\nabla \cdot \mathbf{P}$ is zero for $r \neq 0$

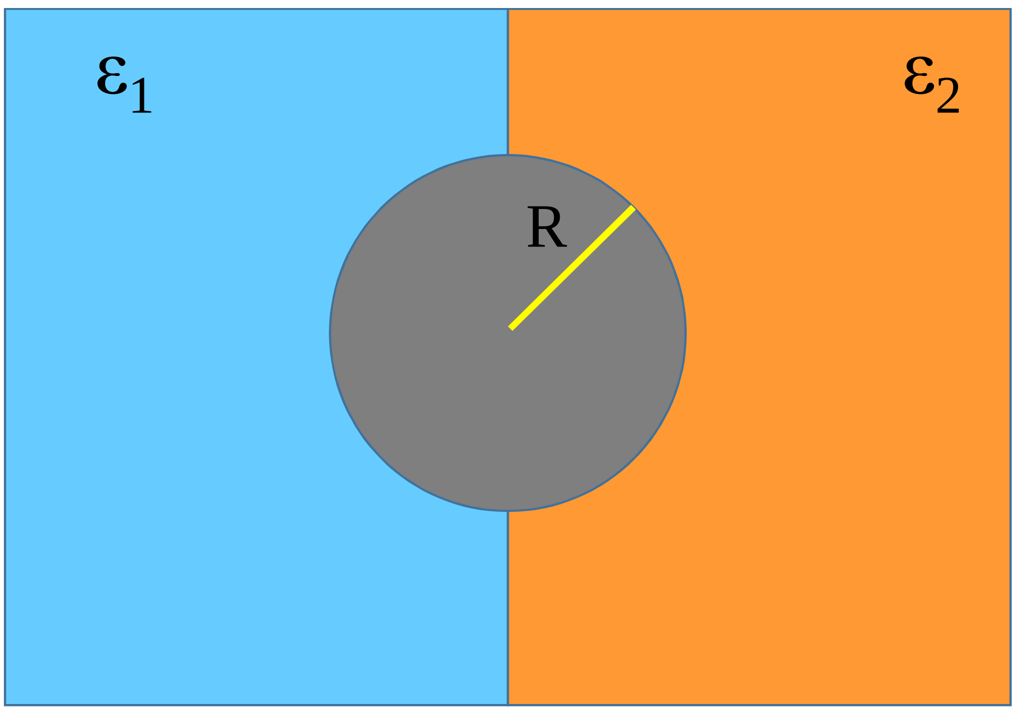
$\sigma_p = \mathbf{P} \cdot \mathbf{n}$ is not zero

$$q_p = -\lim_{a \to 0} \left[ \frac{\varepsilon_1 - \varepsilon_0}{2\pi(\varepsilon_1 + \varepsilon_2)} \frac{q}{a^2} + \frac{\varepsilon_2 - \varepsilon_0}{2\pi(\varepsilon_1 + \varepsilon_2)} \frac{q}{a^2} \right] 2\pi a^2 = -q \left[ 1 - \frac{2\varepsilon_0}{\varepsilon_1 + \varepsilon_2} \right]$$

Where a is taken to be the radius of *q*. *The assignment of a radius to the point charge is just* an intermediate step to facilitate evaluating the induced charge since the final result was derived in the limit of a becoming very small.

**Example: A Conducting Charged Sphere Between Two Dielectrics-One-Dimensional Problem**

sphere of radius ***and center at the plane interface***




the electric field in a conductor is zero

$$\sigma_{1f} = D_{1n} \text{ and } \sigma_{2f} = D_{2n}$$

Charge densities on the two halves of the sphere

## charge density on the surface of the sphere

$\Phi(r) = Cq/r$, where $C = 1/2\pi(\varepsilon_1 + \varepsilon_2)$.

$$\vec{D} = -\varepsilon\vec{\nabla}\Phi$$

$$D_{in} = -\varepsilon_i \frac{\partial\Phi}{\partial r} \quad \text{evaluated at } r = R$$

$$\sigma_{1f} = \frac{q\varepsilon_1}{2\pi R^2(\varepsilon_1 + \varepsilon_2)} \quad \text{and} \quad \sigma_{2f} = \frac{q\varepsilon_2}{2\pi R^2(\varepsilon_1 + \varepsilon_2)}$$

$$Q_f = 2\pi R^2(\sigma_{1f} + \sigma_{2f}) = q$$

polarization charge densities on the surface of the sphere

$$\mathbf{P}_i = \mathbf{D}_i - \varepsilon_0 \mathbf{E},$$

$$\mathbf{P}_1 = \frac{\varepsilon_1 - \varepsilon_0}{2\pi(\varepsilon_1 + \varepsilon_2)} \frac{q}{r^2} \hat{\mathbf{r}}$$

$$\mathbf{P}_2 = \frac{\varepsilon_2 - \varepsilon_0}{2\pi(\varepsilon_1 + \varepsilon_2)} \frac{q}{r^2} \hat{\mathbf{r}}$$

Surface densities

$$\sigma_{1p} = \mathbf{P}_1 \cdot \hat{\mathbf{n}}\big|_{r=R} = -\frac{q(\varepsilon_1 - \varepsilon_0)}{2\pi R^2(\varepsilon_1 + \varepsilon_2)}$$

$$\sigma_{2p} = \mathbf{P}_2 \cdot \hat{\mathbf{n}}\big|_{r=R} = -\frac{q(\varepsilon_2 - \varepsilon_0)}{2\pi R^2(\varepsilon_1 + \varepsilon_2)}$$

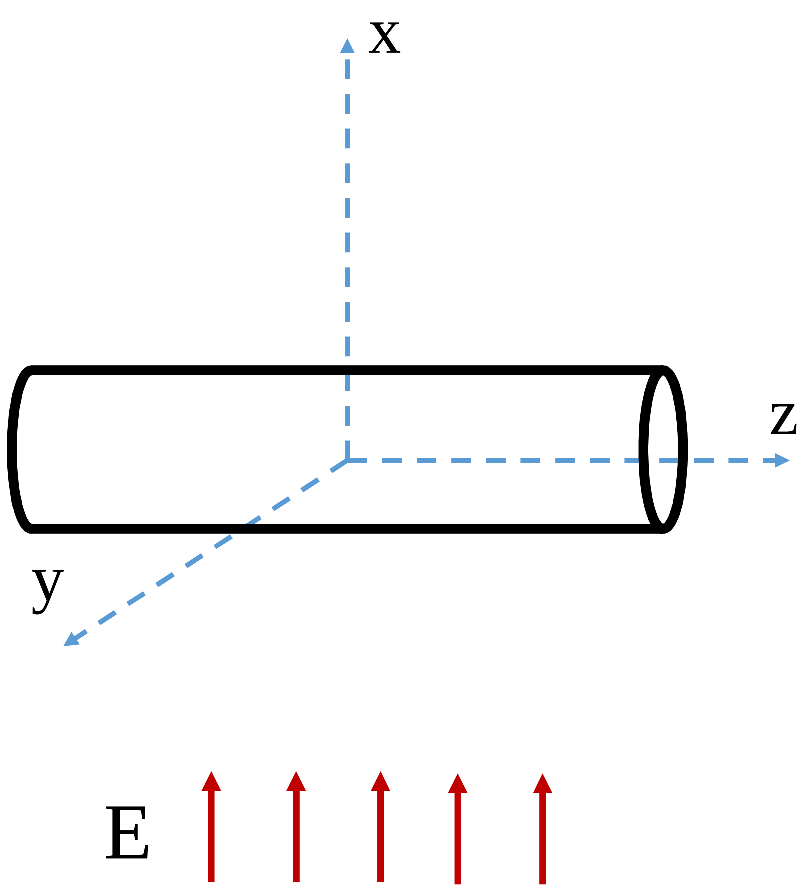
total polarization charge on the sphere

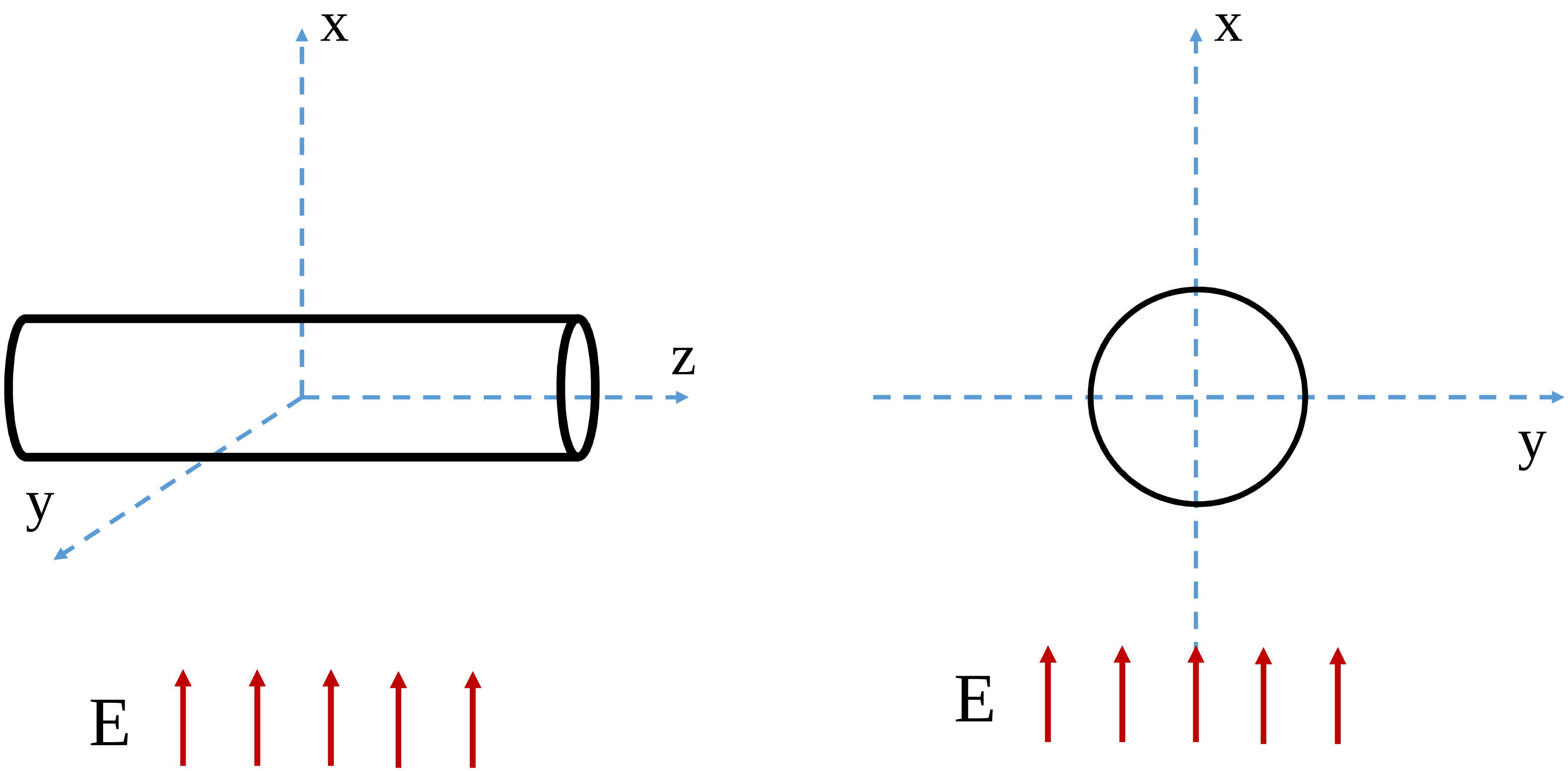
$$q_p = 2\pi R^2(\sigma_{1p} + \sigma_{2p}).$$

$$q_p = -q\left(1 - \frac{2\varepsilon_0}{\varepsilon_1 + \varepsilon_2}\right)$$

Example: A Long, Dielectric Cylinder in an Electric Field- Two-Dimensional Problem

Consider a long, dielectric cylinder of permittivity ε ***placed in a uniform electric field*** that is normal to its axis. We choose a cylindrical coordinate system with the origin taken at the axis of the cylinder and the **x axis along the electric field.**




x
z
y
E
x
y
E

Since there is no free charge on the cylinder, the potential in the ***x-y plane satisfies*** Laplace's equation.

The potentials $\mathbf{\Phi_1}$***, and*** $\mathbf{\Phi_2}$ ***inside and outside the cylinder, respectively,*** depend on ρ and φ**.**

potentials not to depend on ***z because the cylinder is*** long.

Therefore, the potentials are given by the cylindrical harmonics

$$\Phi = \sum_{n=1}^{\infty} (A_n \cos(n\phi) + B_n \sin(n\phi))\rho^n$$

$$+ \sum_{n=1}^{\infty} (A'_n \cos(n\phi) + B'_n \sin(n\phi))\rho^{-n} + A_0 + A'_0 \ln \rho$$

## Boundary conditions

1 Inside dielectric cylinder:

The potential $\Phi_1(\rho,\varphi)$ should not blow up as $\rho \to 0$ . This implies that it should not have terms of radial dependence $\frac{1}{\rho^n}$ and In $\rho$.

$$\Phi = \sum_{n=1}^{\infty} (A_n \cos(n\phi) + B_n \sin(n\phi))\rho^n$$

$$+ \underbrace{\sum_{n=1}^{\infty} (A'_n \cos(n\phi) + B'_n \sin(n\phi))\rho^{-n}}_{0} + A_0 + \underbrace{A'_0 \ln \rho}_{0}$$

2 Outside dielectric cylinder :

Far away from the cylinder, the potential should reduce to a uniform electric field in the $x$ ***direction.***

$$\Phi_2(\rho, \phi) = -E_0 \rho \cos\phi + V_0, \text{ where } V_0 \text{ is a constant.}$$

$\Phi_2(\rho,\varphi)$ should not have terms of radial dependence $\rho^n$ *where* $n > 1$

$$\Phi = \sum_{n=1}^{\infty} \underbrace{(A_n \cos(n\phi) + B_n \sin(n\phi))}_{0 \text{ for } n > 1} \rho^n + \sum_{n=1}^{\infty} (A'_n \cos(n\phi) + B'_n \sin(n\phi)) \rho^{-n} + A_0 + A'_0 \ln\rho$$

3 On the boundary of dielectric cylinder :

because of boundary condition, $\Phi_1$, *and* $\Phi_2$ should not include terms of cos n$\varphi$ where $n > 2$ and terms of sin n$\varphi$ where $n \geq 1$.
This result is directly related to the fact that sin n$\varphi$ and cos n$\varphi$ are linearly independent functions.

$$\Phi_1 (\rho = \rho_0) = \Phi_2 (\rho = \rho_0)$$

4

Outside dielectric cylinder :

Because the cylinder has no free charges, then $\Phi_2$, should not include a Inρ ***term, since*** such a term is proportional to the total charge on the cylinder.

$$\Phi_1(\rho, \phi) = A_0 + A_1 \rho \cos\phi \qquad \rho < \rho_0$$

$$\Phi_2(\rho, \phi) = B_0 + \frac{B_1}{\rho} \cos\phi + C_1 \rho \cos\phi \qquad \rho > \rho_0$$

The constants $A_1$, $A_2$, $B_1$, $B_2$, and $C_1$ can now be. evaluated using some more conditions.

5 Outside dielectric cylinder :

$$\rho \rightarrow \text{large},\ \Phi_2(\rho, \phi) = -E_0 \rho \cos\phi + V_0$$

$$B_0 + C_1 \rho \cos\phi = -E_0 \rho \cos\phi + V_0$$

$$\Rightarrow \quad B_0 = V_0 \text{ and } C_1 = -E_0$$

6 On the boundary of dielectric cylinder :

The continuity of the potential at $\rho = \rho_0$ gives

$$\Phi_1 (\rho = \rho_0) = \Phi_2 (\rho = \rho_0)$$

$$A_0 + A_1 \rho_0 \cos\phi = V_0 + \frac{B_1}{\rho_0}\cos\phi - E_0 \rho_0 \cos\phi$$

$$A_0 = V_0, \text{ and } A_1 = (B_1/\rho_0^2) - E_0$$

7 On the boundary of dielectric cylinder :

Another equation between A, and B, can now be found from the continuity condition of the normal D components:

$$D_{\text{٢}n} - D_{\text{١}n} = \sigma \quad \xrightarrow{\sigma = 0} \quad \mathbf{D}_2 \cdot \hat{\mathbf{n}} = \mathbf{D}_1 \cdot \hat{\mathbf{n}}$$

$$-\varepsilon \frac{\partial \Phi_1}{\partial \rho}\bigg|_{\rho=\rho_0} = -\varepsilon_0 \frac{\partial \Phi_2}{\partial \rho}\bigg|_{\rho=\rho_0}$$

$$-KA_1 = (B_1/\rho_0^2) + E_0$$

$$\begin{cases} A_0 = V_0 \\ A_1 = -2E_0/(K+1) \\ B_0 = V_0 \\ B_1 = \rho_0^2 E_0 (K-1)/(K+1) \\ C_1 = -E_0 \end{cases}$$

$$\Phi_1(\rho, \phi) = V_0 - \frac{2E}{K+1}\rho \cos\phi$$

$$\Phi_2(\rho, \phi) = V_0 + \frac{\rho_0^2 E_0}{\rho}\frac{K-1}{K+1}\cos\phi - E_0 \rho \cos\phi$$

If the material of which the cylinder is made has a very high dielectric constant (K >> ***1***)

$$\Phi_1(\rho, \phi) = V_0$$

$$\Phi_2(\rho, \phi) = V_0 + \frac{\rho_0^2 E_0}{\rho} \cos \phi - E_0 \rho \cos \phi$$

**Example**: A Dipole at the Center of a Dielectric Sphere

The angular dependence is produced by inserting a dipole of moment **p** at the center of a dielectric sphere of radius $\boldsymbol{R}$ and permittivity $\boldsymbol{\varepsilon_1}$. The permittivity of the material external to the sphere is $\boldsymbol{\varepsilon_2}$. potential is a function of an angle as well as a distance.