

If vector integrals over a closed path are calculated precisely, based on a physical law and in accordance with mathematical principles, what useful information do they provide?

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This article explores the insights gained from evaluating vector integrals over closed paths, adhering to fundamental physical laws and mathematical principles, while highlighting the power of vector analysis in understanding physical phenomena. It demonstrates how the choice of the integration path affects the results and shows the conditions necessary for accurate calculations. Using examples from electromagnetic theory, including Faraday's law and Ampère's circuital law, the study illustrates how induced electric and magnetic fields behave along arbitrary paths and how their fluxes can be precisely calculated. Emphasis is placed on the right-hand rule and proper handling of surface element vectors, showcasing the effectiveness of vector analysis in teaching and applying physics concepts

I. INTRODUCTION

As is well understood, the proper language for describing physical phenomena is the precise and correct application of mathematics. One important mathematical tool frequently introduced and employed in calculations involving electromagnetic fields is the line integral of a vector over a closed path. For a vector field $\vec{A}$, this integral is expressed as follows¹

$$\oint_C \vec{A} \cdot d\vec{\ell}$$

In the equation above, $d\vec{\ell}$ represents the differential element of length along the path over which the integral is to be evaluated. This differential length is entirely arbitrary. The numerical value of the integral, and whether it is positive or negative, is a topic that is rarely addressed in standard physics and mathematics textbooks. In this article, we aim to explore this subject in greater detail.

II. ESTIMATING AND INTERPRETING THE PATH INTEGRAL OF A VECTOR FOR DIFFERENT SPATIAL DISTRIBUTIONS OF THE VECTOR FIELD

As illustrated in Figures 1(a) and 1(b), two path integrals have been considered. For each case, the same spatial distribution for the $\vec{A}$ has been assumed, and we seek to calculate the path integral for these two scenarios. As shown in Figure 1(a), the spatial distribution of the vector $\vec{A}$ is such that along most of the path, it forms an acute angle with the differential length element. Therefore, it is evident that the result of the integral $\oint_C \vec{A} \cdot d\vec{\ell}$ will be positive, since in a dot product, if the angle between the two vectors is acute, the result is positive,

whereas if the angle is obtuse, the dot product becomes negative. However, as seen in Figure 1(b), due to the specific path integral of $d\vec{\ell}$, the integral yields a negative result. Therefore in Figure 1(b) most of the angles between $\vec{A}$ and $d\vec{\ell}$ are obtuse. Well, what was observable in Figures 1(a) and 1(b) was that we assumed the spatial distribution of the vector $\vec{A}$ in space to be known, and we expected the results of the integrals in Figures 1(a) and 1(b) to become positive and negative, respectively. However, if we do not have a visual representation of the spatial distribution of vector $\vec{A}$ at every point, is there any computational method that can reliably estimate the numerical value of the line integral of $\vec{A}$ along the path, so that we can determine the dominant orientation of the vector throughout the path? The answer to this question is affirmative, provided that, first, the nature of the vector $\vec{A}$ is clearly defined, and second, a robust mathematical principle underlies the related calculations. To clarify the matter further, we will next examine several well-known examples in electromagnetic theory and demonstrate that the governing equations in electromagnetism are far more fundamental than they might appear, controlling the behaviors of electric and magnetic fields with great rigor.

III. FARADAY'S LAW OF INDUCTION AND THE FUNDAMENTAL MATHEMATICS GOVERNING IT

Consider Figure 2(a), in which the magnetic field $\vec{B}$ points out of the plane of the page and has a specific component $\vec{B} = +B(t)\hat{z}$ that varies with time where $\hat{z}$ is the unit vector along the z-axis, pointing out of the plane of the page. For the sake of simplicity in calculations, assume that the spatial variations of this magnetic field are negligible. Here, the rate of change of $B(t)$ with time can be decreasing, increasing, or even periodic. Undoubtedly, as we know, electric fields arise in space due to the variations of the magnetic field $\vec{B}$, and these fields have their

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own specific spatial distribution. This distribution is illustrated in Figure 2(b) for a fixed moment in time and a given rate of change of B. In Figures 2(c) and 2(d), two observers are seen who, completely arbitrarily (we emphasize that it is entirely optional), selects a path with an arbitrary loop and wishes to examine whether the induced electric fields along this path form mostly acute or obtuse angles. In precise mathematical terms, the observer aims to determine the magnitude and sign of the following expression without having any knowledge of the spatial distribution of the induced electric fields.

$$\oint_C \vec{E} \cdot d\vec{\ell} = ?$$

The mathematics underlying Faraday's law of induction which arises from an empirical law combined with the principle of energy conservation provides a precise method to calculate the exact value of the above integral without knowledge of the spatial distribution of the electric field. This is expressed in the well-known relation shown below²⁻⁷.

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\frac{d}{dt} \Phi_B = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{s}$$

In the above relation, Φ_B refers to the magnetic flux, B through the open surface S, with units of Weber in the SI system. There are specific points to consider when calculating the magnetic flux in the above relation. We know that for closed surfaces, vector analysis always takes the normal surface vector as pointing outward from the surface¹. For open surfaces, the direction of the surface normal vector is determined systematically by the loop along which the contour integral is taken. In other words, on the left hand side of Faraday's law, the observer has complete freedom in choosing the direction of $d\vec{\ell}$, but once a direction for traversing the path is selected, there is no longer any freedom in choosing the surface normal vector on the right hand side; it must be determined precisely according to fundamental mathematical rules¹. This rule, known as the right-hand rule, states that if the four fingers of the right hand follow the path of $d\vec{\ell}$, the thumb points in the direction of the standard, correct normal vector¹. The magnetic flux must then be calculated precisely using this standard surface normal. Based on the above explanations, it becomes clear why in Fig. 2(c) the surface-element vector must be taken into the page, while in Fig. 2(d) the surface-element vector must be taken out of the page. This very important point causes the magnetic flux through the arbitrary contour shown in Fig. 2(c) to be inherently negative, whereas the magnetic flux through the arbitrary contour shown in Fig. 2(d) becomes inherently positive.

Thus, in evaluating the right-hand side of Faraday's law, three points must be taken into careful consideration:

1:) Determining the exact direction of the surface-element vector used for calculating the flux.

2:) Accurately computing the time derivative of the flux, which may be positive or negative depending strongly on the rate of change of the magnetic field.

3:) Properly accounting for the negative sign in front of the time derivative of the flux, as it appears in the law.

By observing these three points precisely, one can fully rely on the resulting numerical value and accurately describe whether the induced electric fields along the chosen arbitrary path form mostly acute angles or obtuse angles. As we know, the line integral of the induced electric fields along a mathematical path leads to a physical concept associated with that path, known as the electromotive force, or the "equivalent battery present in the path," which is expressed by the following relation²⁻⁷:

$$\varepsilon_{emf} = \oint_C \vec{E} \cdot d\vec{\ell} = -\frac{d}{dt} \Phi_B$$

The physical application of this equivalent battery in electromagnetic theory has been fully established, and we do not intend to restate it here²⁻⁷. We only point out that if, instead of the mathematical paths shown in Figs. 2(c) and 2(d), we place a thin metallic wire with an ohmic resistance R , the resulting current can be calculated from the following relation. Without any doubt, the direction of the current or more precisely, the direction of the current density generated in that thin metallic wire is exactly the direction in which the induced electric fields form acute angles. The most effective way to teach and apply the set of relations presented is to introduce a numerical example. In this example using a specified time rate of change for the magnetic field (as shown in Figures 3 and 4) we determine both the magnitude and direction of the current induced in the two conducting loops shown in the mentioned figures.

IV. AMPÈRE'S CIRCUITAL LAW, THE DENSITY OF NON-MATERIAL CURRENTS, AND THE GOVERNING EQUATIONS

Without delving into the theoretical details regarding how Ampere's circuital law was formulated for magnetic fields arising from changing electric fields, we turn to an example of its application in environments where no material current densities exist, and only a time dependent electric field is present in the system, as illustrated in Figure 5.

As shown in Figure 5(a), it is assumed that in a vacuum with permittivity ε_0 and permeability μ_0 , the electric field points out of the plane of the paper and varies with a certain temporal rate. For the sake of simplicity, it is assumed that no material current density exists in the medium. We know that in the same environment, the induced magnetic fields arise from the variations of the electric field and have different magnitudes and directions in space, as illustrated in Figure 5(b).

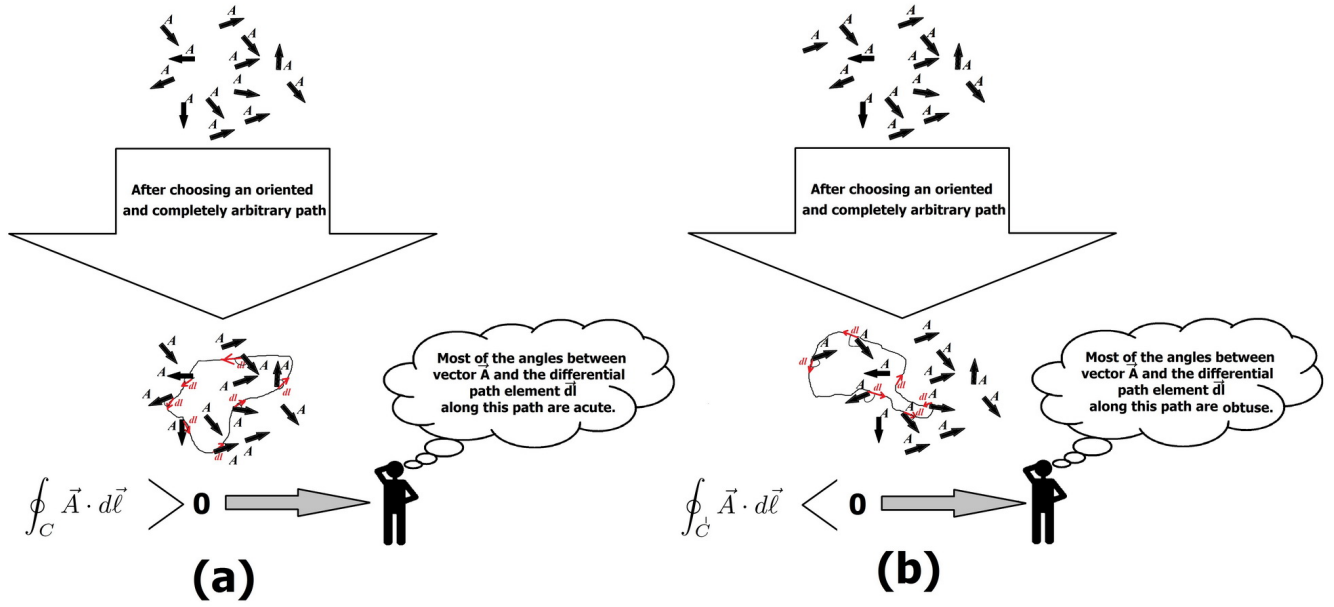


FIG. 1. Choosing two different paths to examine the spatial behavior of the vector $\vec{A}$ while traversing the path. In these two figures, the observer knows the spatial distribution of the vector $\vec{A}$, and therefore knows that along the first path (Fig. 1a) the result of the vector line integral is positive, while along the second path (Fig. 1b) the result of the vector line integral is negative..

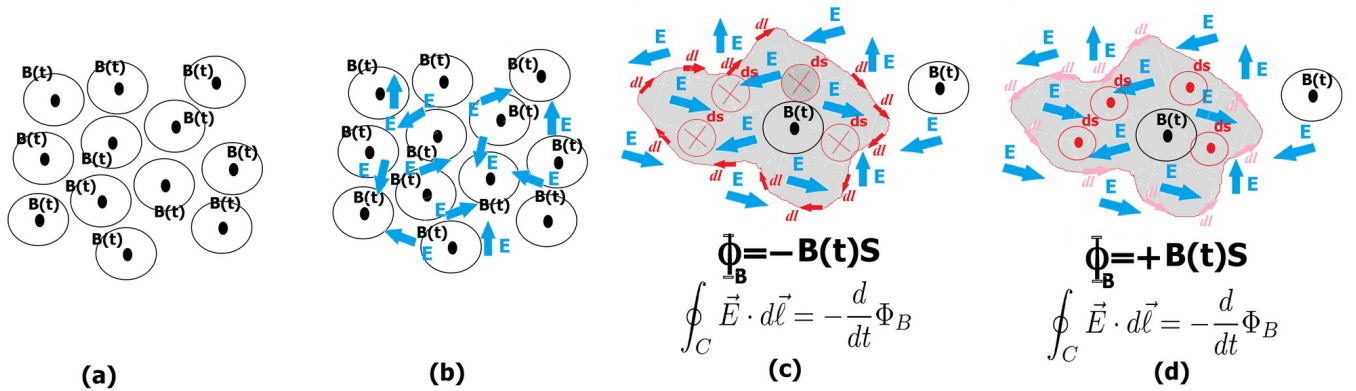


FIG. 2. The time variation of the magnetic field and the creation of electric fields according to Faraday's law of induction. Figure 2(a) shows the presence of a time-varying magnetic field. Figure 2(b) shows the presence of induced electric fields in space, with a spatial distribution that is unknown to the observer. Figure 2(c) represents the observer's choice of a path in order to examine and accurately evaluate the line integral of the electric field vector along that path, together with its corresponding standard surface-element vector. Finally, Figure 2(d) illustrates the change in the direction of traversal used to compute the line integral of the electric field vector, and the evaluation of this integral using another standard surface-element vector..

Just as in Faraday's induction for the electric field shown in Figure 2, in Figure 5(c) we choose an entirely

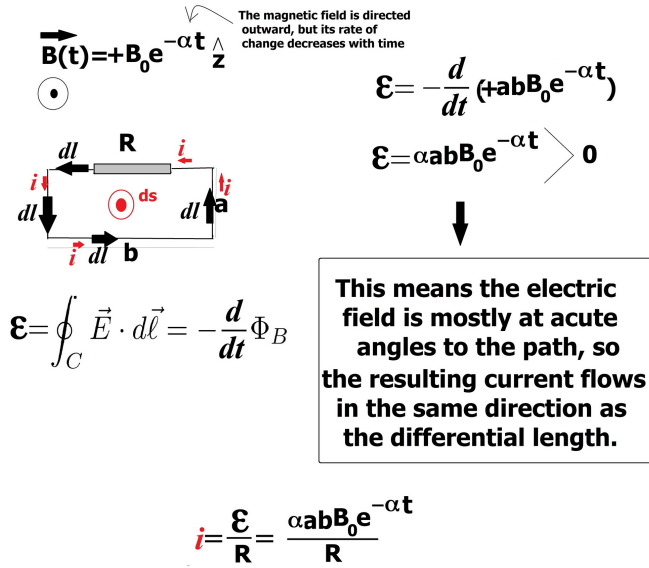


FIG. 3. A thin rectangular conducting loop of area $a \times b$ and resistance R has induced electric fields acting like a battery, driving current along the direction where the fields form mostly acute angles with the path

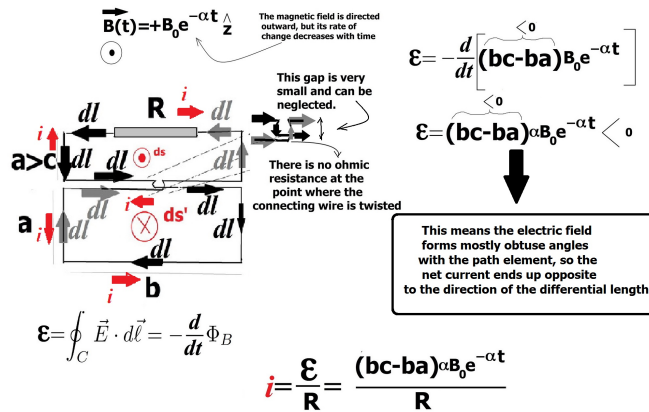


FIG. 4. A twisted rectangular loop of area $(a \times b < c \times b)$ and resistance R with a negligible gap and no contact at the twist still obeys Faraday's law reliably.

arbitrary mathematical path to examine whether the induced magnetic fields in space form more acute or obtuse angles with the differential length vector. Undoubtedly, the only relation that can reliably answer this question

is Ampere's circuital law, which again leads to the calculation of the flux of the field this time, the electric field over a surface. In other words, to apply Ampere's law to induced magnetic fields, careful attention must be paid to the normal vector of the chosen surface. For example, in Figures 5(c) and 5(d), the standard surface element vectors are shown for two arbitrary paths. In short, the principles previously applied to electric fields induced by changing magnetic fields now apply to magnetic fields induced by changing electric fields, using Ampère's law in the absence of material currents²⁻⁷.

$$\oint_C \vec{B} \cdot d\vec{\ell} = \epsilon_0 \mu_0 \frac{d}{dt} \Phi_E = \epsilon_0 \mu_0 \frac{d}{dt} \int_S \vec{E} \cdot d\vec{s}$$

In other words, it is emphasized again that while the direction of the differential length vector $d\vec{\ell}$ on the left-hand side of the above relation is entirely arbitrary, the direction of the surface element vector on the right-hand side, used to calculate the electric field flux, is determined systematically and must be identified according to the right-hand rule¹.

In the set of Figures 5 and the formalism presented therein, it is again assumed that the electric field has no spatial dependence. For this reason, in calculating the electric field flux, it can be easily taken out of the integral exactly as assumed in Figure 2. Undoubtedly, students can easily provide examples for Ampère's circuital law similar to those given for Faraday's law of induction. They should only keep in mind that the contour considered for Ampère's law is purely a mathematical construct and, unlike in Faraday's law, cannot be applied to thin conductive circuits with resistance R^{2-7} . This is because, fundamentally, the line integral of the magnetic field along a path does not carry any conceptual meaning analogous to a battery in the context of electrical circuit theory.

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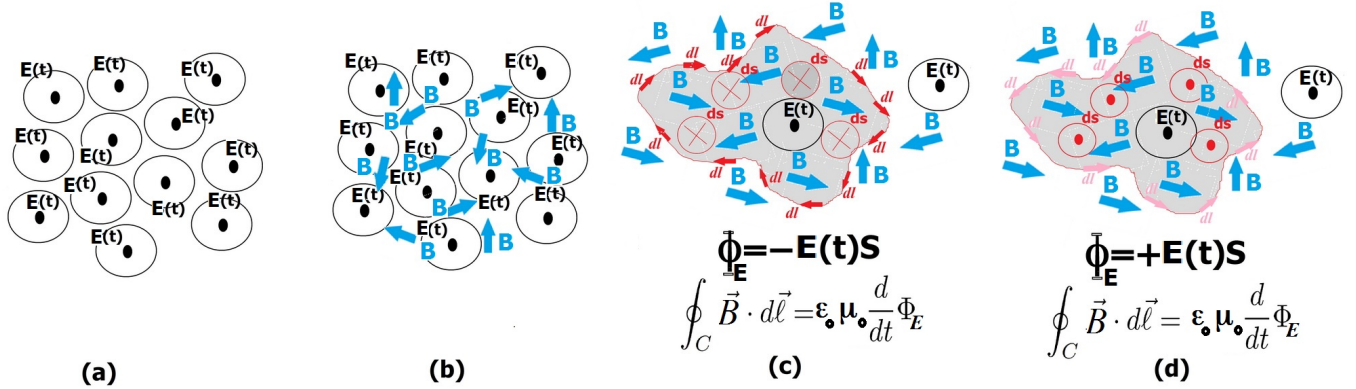


FIG. 5. The time variation of the electric field and the creation of magnetic fields according to Ampère's circuital law. Figure 5(a) shows the presence of a time-varying electric field. Figure 5(b) shows the presence of induced magnetic fields in space, with a spatial distribution that is unknown to the observer. Figure 5(c) represents the observer's choice of a path in order to examine and accurately evaluate the line integral of the magnetic field vector along that path, together with its corresponding standard surface-element vector. Finally, Figure 5(d) illustrates the change in the direction of traversal used to compute the line integral of the magnetic field vector, and the evaluation of this integral using another standard surface-element vector.