Journal of Algebra and Its Applications
Vol. 11, No. 6 (2012) 1250107 (13 pages)

DOI: 10.1142/S0219498812501071

PURE PROJECTIVITY AND PURE INJECTIVITY OVER FORMAL TRIANGULAR MATRIX RINGS

A. HAGHANY*, M. MAZROOEI† and M. R. VEDADI‡

Department of Mathematical Sciences
Isfahan University of Technology
Isfahan 84156-83111, Iran
**aghagh@cc.iut.ac.ir*
†m.mazrooei@math.iut.ac.ir
‡mrvedadi@cc.iut.ac.ir

Received 20 January 2010
Accepted 31 December 2011
Published 9 November 2012

Communicated by S. K. Jain

Over a formal triangular matrix ring we study pure injective, pure projective and locally coherent modules. Some applications are then given, in particular the (J-)coherence of the ring $\left[\begin{smallmatrix} A & 0 \\ M & B \end{smallmatrix}\right]$ is characterized whenever ${}_BM$ is flat.

Keywords: Pure injective; pure projective; locally coherent.

Mathematics Subject Classification 2000: Primary: 16D10, 16D90; Secondary: 16P40

1. Introduction and Preliminaries

All the rings we consider will be associative rings with $1 \neq 0$ and all the modules will be unital. Unless otherwise mentioned we will be working with right modules. For any ring R, the category of right R-modules is denoted by Mod-R. Throughout the paper, A and B will denote two (arbitrary but fixed) rings, and M stands for a left B- right A-bimodule. We let T denote the set of formal triangular matrices of the form $\left[\begin{smallmatrix} a & 0 \\ m & b \end{smallmatrix}\right]$; $a \in A$, $b \in B$ and $m \in M$, which is a ring under componentwise addition and multiplication given by the rule:

$$\begin{bmatrix} a & 0 \\ m & b \end{bmatrix} \begin{bmatrix} a' & 0 \\ m' & b' \end{bmatrix} = \begin{bmatrix} aa' & 0 \\ ma' + bm' & bb' \end{bmatrix}.$$

This ring T is known as a formal triangular matrix ring. These rings were initially used as sources of examples in ring theory, showing unsymmetric properties, such as being right, but not left, hereditary. Goodearl in his book [5] constructed new examples of nonsingular rings by means of formal triangular matrix rings, and gave necessary and sufficient conditions for such rings to be right hereditary by first characterizing their projective one sided ideas; see [5, Chap. 4, Sec. A]. It should be pointed out that the formal triangular matrix rings have applications in the subject of representing rings in matrix forms [1]. The ordinary lower triangular matrix ring $T_n(R)$ over a ring R is a particular case of a formal triangular matrix ring (see the proof of Corollary 4.5), and even for the case $n = 2$, some interesting questions have been raised. For example, the Auslander–Reiten question asks when $T_2(R)$ is of finite representation type. Recall that a left and right Artinian ring with only a finite number of isomorphism classes of indecomposable finitely generated modules is said to be of finite representation type. The above question has been dealt with in [14]. An associative ring with identity is called right pure semisimple if it is left Artinian and every right module is a direct sum of modules of finite length. It is well known that a ring R is right and left pure semisimple if and only if R is of finite representation type. The pure semisimple conjecture asserts that a right pure semisimple ring must be of finite representation type. Possible counter examples to pure semisimple conjecture were investigated by Ivo Herzog [10], who proved the following deep result: If R is a counter example, then there are division rings F and G and a simple G–F-bimodule ${}_GB_F$ such that the ring $\left(\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right)$ is a counter example. This "minimal" counter example motivated the study of purity over triangular matrix rings, and because of its relation to the pure semisimple conjecture, the problem of full description of indecomposable modules over potential minimal counter examples was considered; see for example [11, 15, 16]. In this paper we aim to complement the knowledge of purity over lower triangular matrix rings by studying pure projective modules, pure injective modules and locally coherent modules over a formal triangular matrix ring. To some extent, the general module theory over T has been investigated in [2, 7–9, 12, 13].

In the rest of this introduction some notations that will remain fixed throughout the paper are mentioned. We shall adapt the well-known description of T-modules from [5] which is afforded by the equivalence of category Mod-T with a category Ω, described below.

Let Ω denote the category whose objects are triples $(X, Y)_f$ where $X \in$ Mod-A, $Y \in$ Mod-B and $f : Y \otimes_B M \to X$ is a map in Mod-A. If $(X, Y)_f$ and $(X', Y')_g$ are objects in Ω, the morphisms from $(X, Y)_f$ to $(X', Y')_g$ in Ω are pairs (φ_1, φ_2) where $\varphi_1 : X \to X'$ is a map in Mod-A, $\varphi_2 : Y \to Y'$ is a map in Mod-B satisfying the condition $\varphi_1 f = g(\varphi_2 \otimes 1_M)$ where 1_M denotes the identity map on M. It is well known [6] that the category Mod-T is equivalent to the category Ω. The right T-module corresponding to the triple $(X, Y)_f$ is the additive group $X \oplus Y$ with the right T-action given by

$$(x, y) \begin{bmatrix} a & 0 \\ m & b \end{bmatrix} = (xa + f(y \otimes m), yb).$$

Conversely if V_T is given then by using the idempotents $e_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $e_2 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and ring identifications $e_1 T \equiv A$ and $Te_2 \equiv B$, the triple corresponding to V is constructed. It is $(X, Y)_f$ where $X = Ve_1$, $Y = Ve_2$ and $f : Ve_2 \otimes_B M \to Ve_1$ is given by $f(ve_2 \otimes m) = ve_2 \left[\begin{smallmatrix} 0 & 0 \\ m & 0 \end{smallmatrix}\right] = v \left[\begin{smallmatrix} 0 & 0 \\ m & 0 \end{smallmatrix}\right] e_1$.

Thus, the regular module T_T corresponds to $(A \oplus M, B)_f$ where f is the map $B \otimes_B M \to A \oplus M$ given by $f(b \otimes m) = (0, bm)$. It is convenient to view such triples as T-modules, and the morphisms between them as T-homomorphisms. Often the map f occurring in a triple will be clear from the context. This is particularly evident for T_T and its submodules. If $(\varphi_1, \varphi_2) : (X, Y)_f \to (X', Y')_g$ is a map in Ω, then it is clear that (φ_1, φ_2) is injective (respectively, surjective) if and only if φ_1 and φ_2 are injective (respectively, surjective). Let $V = (X, Y)_f$ be a T-module. Any T-submodule of V corresponds to a triple $(X', Y')_{f'}$ where X' is a submodule of X_A, Y' is a submodule of Y_B such that $f' = f(\iota \otimes 1_M)$ maps $(Y' \otimes_B M)$ into the submodule X' where $\iota : Y' \to Y$ is the natural inclusion. Also it is easy to verify that $(X, Y)_f$ is a finitely generated T-module if and only if $(X/\operatorname{Im} f)_A$ and Y_B are finitely generated modules. A detailed description of the factor modules of V_T is given in [9]. We now introduce the following ideals of T:

$$I = \begin{bmatrix} 0 & 0 \\ M & B \end{bmatrix}; \quad P = \begin{bmatrix} A & 0 \\ M & 0 \end{bmatrix}.$$

Throughout, *these ideals and the following notation will be retained.* If V is a T-module corresponding to $(X, Y)_f$, we let: $\tilde{f} : Y \to \operatorname{Hom}_A(M, X)$; $\tilde{f}(y)(m) = f(y \otimes m)$ for $y \in Y$, $m \in M$. Note that $\tilde{f}$ is a B-homomorphism. Moreover, we often write VM for the T-submodule $V \left[\begin{smallmatrix} 0 & 0 \\ M & 0 \end{smallmatrix}\right]$ of V. It is easy to verify that $V/VI \simeq X/\operatorname{Im} f$ in Mod-A, and $V/VP \simeq Y$ in Mod-B.

Goodearl in [5, p. 114] has introduced some useful adjoint pairs of functors, one of which in our notation is the following:

$$(\mathcal{J}_{23}, \mathcal{P}_3) : \text{Mod-}B \to \text{Mod-}T,$$

$$\mathcal{J}_{23}(N) = (N \otimes_B M, N)_{1_{N \otimes_B M}}, \quad \mathcal{P}_3((X, Y)_f) = Y.$$

For future use we introduce two new adjoint pairs below. Define

$$\mathcal{K} : \text{Mod-}A \to \text{Mod-}T \quad \text{by } \mathcal{K}(Z) = (Z, \operatorname{Hom}_A(M, Z))_f,$$

$$f(\alpha \otimes m) = \alpha(m) \quad \forall \alpha \in \operatorname{Hom}_A(M, Z), \quad m \in M;$$

and $\mathcal{K}(g) = (g, g_*) \,\forall\, g : Z_A \to Z'_A$, where g_*: $\operatorname{Hom}_A(M, Z) \to \operatorname{Hom}_A(M, Z')$ is the left multiplication map by g. Also, define Γ: Mod-$T \to$ Mod-A by $\Gamma[(X, Y)_f] = X$ and $\Gamma[(\theta_1, \theta_2)] = \theta_1$. It is left to the reader to verify that $\mathcal{K}$ and Γ are functors and the map $\lambda_{(V,Z)} : \operatorname{Hom}_A(\Gamma(V), Z) \to \operatorname{Hom}_T(V, \mathcal{K}(Z))$ defined by $\lambda_{(V,Z)}(\theta) = (\theta, \theta_* \tilde{f})$ $\forall\, V = (X, Y)_f \in$ Mod-T, $Z \in$ Mod-A, is a bijection and natural in each variable.

Therefore, $(\Gamma, \mathcal{K})$ is an adjoint pair. The other adjoint pair is as follows: (F, G) : Mod-$T \to$ Mod-$(A \oplus B)$ where $F[(X,Y)_f] = (X/\operatorname{Im} f) \oplus Y$ and $G(C \oplus D) = (C, D)_0$. Here we have used the fact that any right module over the ring $A \oplus B$ is of the form $C \oplus D$ with suitable C_A and D_B.

2. Pure Monomorphisms

Let R be any ring and W be an R-module. Suppose $U \leq W_R$. Then U is called a *pure submodule* of W if for any left R-module H, the additive homomorphism $\iota \otimes 1_H : U \otimes_R H \to W \otimes_R H$ is one to one where $\iota : U \to W$ is the inclusion map. It is well known that U is a pure submodule of W if and only if any system of equations $\sum_{i=1}^{n} x_i a_{ij} = u_j \in U$ $(j = 1, \ldots, m,\ a_{ij} \in R)$ which is solvable in W, is also solvable in U. A short exact sequence $0 \to U \xrightarrow{f} W \xrightarrow{g} K \to 0$ is called a *pure exact sequence* if $\operatorname{Im} f$ is a pure submodule of W. In this case f is called a *pure monomorphism*. An R-module N is called *pure projective* (*pure injective*) if it is projective (injective) relative to any pure exact sequence. For some useful information on these concepts see [3].

Lemma 2.1. (i) *If $R \xrightarrow{\epsilon} S$ is a ring homomorphism and N is a pure submodule of some S-module L, then N is a pure R-submodule of L with R-module structure induced by ϵ.*

(ii) *Let $X' \leq X_A$ and $Y' \leq Y_B$. Then X' and Y' are pure submodules of X_A and Y_B respectively, if and only if $(X' \oplus Y')$ is a pure $(A \oplus B)$-submodule of $(X \oplus Y)$.*

(iii) *Let $(X', Y')_{f'} \xrightarrow{(\varphi,\theta)} (X, Y)_f$ be a homomorphism in* Mod-T.
If $f' = 0$ then $\tilde{f}\theta = 0$. The converse is true provided that $\ker \varphi = 0$.

(iv) *Let R and S be rings, $N_R \xrightarrow{f} L_R$ be a pure monomorphism and ${}_RW_S$ be a bimodule. Then $N \otimes_R W \xrightarrow{f \otimes 1} L \otimes_R W$ is a pure monomorphism in* Mod-S.

Proof. (i)–(iii) are proved by routine arguments.

(iv) Since f is pure, $f \otimes 1$ is a monomorphism, and it is a pure monomorphism by the associativity of tensors. □

Proposition 2.2. *Let $(X', Y')_{f'} \xrightarrow{(\varphi,\theta)} (X, Y)_f$ be a T-module homomorphism.*

(i) *If (φ, θ) is a pure monomorphism, then φ and θ are pure monomorphisms in* Mod-A *and* Mod-B, *respectively.*

(ii) *Suppose $f' = 0$. If φ and $Y' \xrightarrow{\theta} \ker \tilde{f}$ are pure monomorphisms in* Mod-A *and* Mod-B, *respectively, then $(X', Y')_0 \xrightarrow{(\varphi,\theta)} (X, \ker \tilde{f})_0$ is a pure monomorphism in* Mod-T.

Proof. (i) Since (φ,θ) is one to one, φ and θ are so. By hypothesis the image of $((\varphi,\theta)$ is a pure T-submodule of $(X,Y)_f$. Using the ring homomorphism $(a,b)\to\left[\begin{smallmatrix}a & 0\\ 0 & b\end{smallmatrix}\right]$ from $A\oplus B$ to T and Lemma 2.1(i), we can deduce that $(\operatorname{Im}\varphi\oplus\operatorname{Im}\theta)$ is a pure $(A\oplus B)$-submodule of $(X\oplus Y)$. The proof is now completed by Lemma 2.1(ii).

(ii) By Lemma 2.1(iii), $\operatorname{Im}\theta\subseteq\ker\tilde{f}$. Since $f(y\otimes M)=0$ for all $y\in\ker\tilde{f}$, we have $(X\oplus\ker\tilde{f})\left[\begin{smallmatrix}0 & 0\\ M & 0\end{smallmatrix}\right]=0$. Hence, $(\operatorname{Im}\varphi\oplus\operatorname{Im}\theta)$ is a T/M-submodule as well as an $(A\oplus B)$-submodule of $(X\oplus\ker\tilde{f})$. Therefore, by hypothesis $(X',Y')_0\xrightarrow{(\varphi,\theta)}(X,\ker\tilde{f})_0$ is a pure monomorphism in Mod-T/M by Lemma 2.1(ii). The proof is completed by Lemma 2.1(i). □

Proposition 2.3. *Let $(X,Y)_f$ be a T-module, X' be a pure submodule of X_A and $\operatorname{Im} f\leq X'$. Then $(X',Y)_f$ is a pure submodule of $(X,Y)_f$.*

Proof. Suppose that $\sum_{j=1}^{n}(x_j,y_j)A_{ij}=(x'_i,v_i)\in(X'\oplus Y)$ where $A_{ij}=\left[\begin{smallmatrix}a_{ij} & 0\\ m_{ij} & b_{ij}\end{smallmatrix}\right]\in T$ and $(x_j,y_j)\in(X\oplus Y)$ $(i=1,\dots,m)$. Then we have $\sum_{j=1}^{n}x_ja_{ij}+f(y_j\otimes m_{ij})=x'_i$ and $\sum_{j=1}^{n}y_jb_{ij}=v_i$ $(i=1,\dots,m)$. Since X' is a pure submodule of X and $\operatorname{Im} f\leq X'$, there are $u_1,\dots,u_n\in X'$ such that $\sum_{j=1}^{n}u_ja_{ij}=x'_i-\sum_{j=1}^{n}f(y_j\otimes m_{ij})$. It follows that $\sum_{j=1}^{n}(u_j,y_j)A_{ij}=(x'_i,v_i)$, and so $(X',Y)_f$ is a pure submodule of $(X,Y)_f$. □

Corollary 2.4. *Let $(X,Y)_f$ be a T-module. If either A is a regular ring or X_A is semisimple, then $(\operatorname{Im} f,Y)_f$ is a pure T-submodule of $(X,Y)_f$.*

Proof. In any case, $\operatorname{Im} f$ is a pure submodule of X_A. Hence the result is true by Proposition 2.3. □

3. Pure Projectivity

Following [17, 5.25] a module M_R is called *finitely presented* if M_R is finitely generated and for every exact sequence $0\to K\to L\to M\to 0$ of R-modules with L finitely generated, K_R is finitely generated. By [3, Theorem 18-2.10], a module is pure projective if and only if it is isomorphic to a direct summand of a direct sum of finitely presented modules. The following result contains some facts that we shall use in the sequel.

Proposition 3.1. (i) *A module M_R is finitely presented if and only if it is finitely generated pure projective, if and only if there exists an exact sequence $0\to K\to R^{(n)}\to M\to 0$ for some $n\in\mathbb{N}$ and K finitely generated.*

(ii) *Let X be an R-module and H be an ideal of R such that H is a finitely generated right ideal and $XH=0$. Then X is a finitely presented R/H-module if and only if it is finitely presented as an R-module.*

Proof. (i) This follows by [17, 25.4 and 25.1(iii); 3, Theorem 18-2.10].

(ii) Let X be a finitely presented R/H-module. Then there exists an exact sequence $0 \to K \to (R/H)^{(n)} \to X \to 0$ in Mod-R/H such that K is a finitely generated R/H-module. Clearly K is finitely generated as an R-module. Since now H_R is finitely generated, R/H and so $(R/H)^{(n)}$ are finitely presented R-modules. It follows that X_R is finitely presented [17, 25.1(i)]. The converse is true because any finitely generated R/H-module is also a finitely generated R-module. □

Let R be a ring and H be a proper ideal of R. If $K \xrightarrow{f} L$ is an R-homomorphism, we use $\bar{f} : K/KH \to L/LH$ for the natural R/H-homomorphism defined by $\bar{f}(k + KH) = f(k) + LH$.

Proposition 3.2. *Let H be a proper ideal in the ring R.*

(i) *If W_R is pure projective then W/WH is a pure projective R/H-module.*

(ii) *If H is a finitely generated right ideal and W is a pure projective R/H-module then W_R is pure projective.*

Proof. (i) Let $0 \to K \xrightarrow{\varphi} L \xrightarrow{\theta} N \to 0$ be a pure exact sequence in Mod-R/H and $f : W/WH \to N$ be an R/H-homomorphism. By Lemma 2.1(i), the above sequence is also a pure exact sequence in Mod-R. Consider the homomorphism $f\pi : W_R \to N_R$ where $\pi : W \to W/WH$ is the natural projection. By the pure projective condition on W_R, there exists an R-homomorphism $g : W \to L$ such that $f\pi = \theta g$. Since now $LH = 0$, we have $\theta\bar{g}(\bar{w}) = f(\bar{w})$ for all $\bar{w} \in W/WH$, proving that W/WH is a pure projective R/H-module.

(ii) This is clear by [3, Theorem 18-2.10] and Proposition 3.1. □

Corollary 3.3. *Suppose that H is a proper ideal in the ring R such that H_R is finitely generated and W is an R-module with $WH = 0$. Then W is a pure projective R/H-module if and only if it is a pure projective R-module.*

Corollary 3.4. *Let $V = (X, Y)_f$ be a T-module. Then $(X/\mathrm{Im}\, f)_A$ is pure projective (respectively, finitely presented) if and only if $(V/VI)_T$ is pure projective (respectively, finitely presented).*

Proof. Note that $I = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} T$ is a finitely generated right ideal in T and $A \simeq T/I$. Thus, the result is obtained by Corollary 3.3, Proposition 3.1(ii) and the fact that $X/\mathrm{Im}\, f \simeq V/VI$ in Mod-A. □

Lemma 3.5. *Let $V = (X, Y)_f$ be a T-module. If f is a monomorphism in* Mod-A, *then $VI \simeq (Y \otimes_B M, Y)_{1_{(Y \otimes_B M)}}$ in* Mod-T.

Proof. Note that $VI = (X, Y)\begin{bmatrix} 0 & 0 \\ M & B \end{bmatrix} = (\mathrm{Im}\, f, Y)$. Now it is easy to verify that $(\mathrm{Im}\, f, Y)_f \simeq (Y \otimes_B M, Y)_{1_{(Y \otimes_B M)}}$ in Mod-T. □

Theorem 3.6. *If $(X, Y)_f$ is a pure projective module in* Mod-T *then Y_B and $(X/\operatorname{Im} f)_A$ are pure projective modules. The converse is true provided that f is a pure monomorphism.*

Proof. Assume $V = (X, Y)_f$ is pure projective. Applying Proposition 3.2(i) to $H = I$ (respectively, $H = P$), we deduce that $(V/VI)_{T/I}$ (respectively, $(V/VP)_{T/P}$) is pure projective. It follows that $(X/\operatorname{Im} f)_A$ and Y_B are pure projective modules.

Suppose now $(X/\operatorname{Im} f)_A$ and Y_B are pure projective and $f : Y \otimes_B M \to X$ is a pure monomorphism in Mod-A. Since $\operatorname{Im} f$ is a pure A-submodule of X, $VI = (\operatorname{Im} f, Y)_f$ is a pure T-submodule of V by Proposition 2.3. By Corollary 3.4, $(V/VI)_T$ is pure projective. Thus the pure exact sequence $0 \to VI \to V \to V/VI \to 0$ splits. Hence, in view of [3, Theorem 18-2.10], it is enough to show that VI is a pure projective T-module. By Lemma 3.5, we shall show that $(Y \otimes_B M, Y)_{1_{(Y \otimes_B M)}}$ is a pure projective T-module. Let $0 \to (X', Y')_g \xrightarrow{(\varphi_1, \varphi_2)} (X'', Y'')_h \xrightarrow{(\theta_1, \theta_2)} (X''', Y''')_k \to 0$ be a pure exact sequence and $(Y \otimes_B M, Y)_{1_{(Y \otimes_B M)}} \xrightarrow{(\alpha_1, \alpha_2)} (X''', Y''')_k$ be a homomorphism in Mod-T. By Proposition 2.2, $0 \to Y' \overset{\varphi_2}{\to} Y'' \overset{\theta_2}{\to} Y''' \to 0$ is a pure exact sequence in Mod-B. Since $Y \overset{\alpha_2}{\to} Y'''$ is a B-homomorphism and Y_B is pure projective by hypothesis, there exists $Y \overset{\gamma_2}{\to} Y''$ such that $\theta_2 \gamma_2 = \alpha_2$. Define $\gamma_1 : Y \otimes_B M \to X''$ by $\gamma_1 = h(\gamma_2 \otimes 1_M)$. The map $(\gamma_1, \gamma_2) : (Y \otimes_B M, Y)_{1_{(Y \otimes_B M)}} \to (X'', Y'')_h$ is a homomorphism in Mod-T. Also we have the following commutative diagrams

$$\begin{array}{ccc} Y'' \otimes_B M & \xrightarrow{h} & X'' \\ {\scriptstyle \theta_2 \otimes 1_M} \downarrow & & \downarrow {\scriptstyle \theta_1} \\ Y''' \otimes_B M & \xrightarrow{k} & X''' \end{array} \qquad \begin{array}{ccc} Y \otimes_B M & \xrightarrow{1_{Y \otimes_B M}} & Y \otimes_B M \\ {\scriptstyle \alpha_2 \otimes 1_M} \downarrow & & \downarrow {\scriptstyle \alpha_1} \\ Y''' \otimes_B M & \xrightarrow{k} & X''' \end{array}$$

Thus, $\theta_1 \gamma_1 = \theta_1 h(\gamma_2 \otimes 1_M) = k(\theta_2 \otimes 1_M)(\gamma_2 \otimes 1_M) = k(\theta_2 \gamma_2 \otimes 1_M) = k(\alpha_2 \otimes 1_M) = \alpha_1$. Hence $(\theta_1, \theta_2)(\gamma_1, \gamma_2) = (\alpha_1, \alpha_2)$, proving that VI is a pure projective T-module. Therefore, V_T is pure projective. □

Corollary 3.7. *Let $(X, Y)_f$ be a T-module and f be an A-isomorphism. Then $(X, Y)_f$ is pure projective in* Mod-T *if and only if Y_B is pure projective.*

Corollary 3.8. *If $(X, Y)_f$ is a finitely presented T-module then $(X/\operatorname{Im} f)_A$ and Y_B are finitely presented. The converse is true provided that f is monic.*

Proof. If $V = (X, Y)_f$ is finitely presented, then by Proposition 3.1, it is finitely generated pure projective. Thus $(X/\operatorname{Im} f)_A$ and Y_B are finitely presented by Theorem 3.6.

Conversely assume $(X/\mathrm{Im}\, f)_A$ and Y_B are finitely presented and f is monic. Since f is monic, the T-module VI is isomorphic to $(Y \otimes_B M, Y)_{1_{Y\otimes_B M}}$ by Lemma 3.5. Since now Y is a finitely generated pure projective B-module, $(Y \otimes_B M, Y)_{1_{Y\otimes_B M}}$ is also a finitely generated pure projective T-module by Lemma 3.7. Thus VI is a finitely presented T-module by Proposition 3.1. On the other hand, by hypothesis and Corollary 3.4, $(V/VI)_T$ is finitely presented. Therefore, V is a finitely presented T-module by [17, 25.1(ii)]. □

Corollary 3.9. *The functors $\mathcal{J}_{23}$, $\mathcal{P}_3$ and F preserve pure projective modules.*

Proof. This is obtained by Theorem 3.6 and the fact that all modules over $(A \oplus B)$ have the form $X \oplus Y$ for some $X \in$ Mod-A, $Y \in$ Mod-B with pointwise module structure so that $X \oplus Y$ is pure projective if and only if X_A and Y_B are so. □

Remark 3.10. Suppose B is a right Noetherian ring and M_A is not finitely generated. Then the T-module T/P that corresponds to $(X, Y)_f$ where $X = 0$, $Y = B$ and $f = 0$, is not finitely presented (pure projective) because P_T is not finitely generated. But clearly $(X/\mathrm{Im}\, f)_A$ and Y_B are finitely presented (pure projective). This shows that the converse of Theorem 3.6 is not true in general, and the functor G does not necessarily preserve pure projective modules.

4. Locally Coherent Modules and the Coherence of T

We now investigate the coherent T-modules. Following [17, 26.1], a module N_R is called *locally coherent* if every finitely generated submodule of N is finitely presented. A module is called *coherent* if it is finitely generated and locally coherent. A ring R is called *right coherent* if R_R is coherent. It is well known that if a ring R is either right Noetherian or regular, then R is a right coherent ring. If $(X, Y)_f$ is a T-module, $Y_1 \leq Y_B$ and $\iota : Y_1 \to Y$ is the inclusion map, then we have the natural A-module homomorphism $\iota \otimes 1_M : Y_1 \otimes_B M \to Y \otimes_B M$, and in this case we set $f_1 = f(\iota \otimes 1_M)$. It is easy to verify that for any A-submodule K of X containing $\mathrm{Im}\, f_1$, the natural map $(K, Y_1)_{f_1} \xrightarrow{(1,1)} (X, Y)_f$ is a monomorphism in Mod-T.

Theorem 4.1. *If $(X, Y)_f$ is a locally coherent module in* Mod-T *then Y_B is locally coherent and for every finitely generated submodule $Y_1 \leq Y_B$, the module $(X/\mathrm{Im}\, f_1)_A$ is locally coherent. The converse is true provided that ${}_BM$ is flat and f is monic.*

Proof. Let $(X, Y)_f$ be a locally coherent T-module, Y_1 be a finitely generated submodule of Y_B and K be a submodule of X_A containing $\mathrm{Im}\, f_1$ such that $(K/\mathrm{Im}\, f_1)_A$ is finitely generated. Then $(K, Y_1)_{f_1}$ is a finitely generated T-submodule of $(X, Y)_f$, hence $(K, Y_1)_{f_1}$ is finitely presented by our assumption. This shows that $(K/\mathrm{Im}\, f_1)_A$ and $(Y_1)_B$ are finitely presented modules by Corollary 3.8, proving that $(X/\mathrm{Im}\, f_1)_A$ and $(Y_1)_B$ locally coherent.

Conversely, suppose that ${}_BM$ is flat and f is monic and $(X_1, Y_1)_{f_1}$ is a finitely generated T-submodule of $(X, Y)_f$. Then $(X_1/\mathrm{Im}\, f_1)$ and Y_1 are finitely generated submodules of $(X/\mathrm{Im}\, f_1)_A$ and Y_B respectively, and so they are finitely presented by our assumption. Since now ${}_BM$ is flat and f is monic, f_1 is monic. The proof is completed by Corollary 3.8. □

Theorem 4.2. *If T is a right coherent ring, then A and B are right coherent rings and $(M/LM)_A$ is locally coherent for any finitely generated right ideal L of B. The converse is true provided that ${}_BM$ is flat.*

Proof. Suppose that T is a right coherent ring. Consider the T-module $T = (X, Y)_\theta$ where $X = A \oplus M$, $Y = B$ and $\theta : B \otimes_B M \to A \oplus M$ defined by $\theta(b \otimes m) = (0, bm)$ is an A-homomorphism. Applying Theorem 4.1, we deduce that B is a right coherent ring and for every finitely generated right ideal L_1 of B, $(X/\mathrm{Im}\, f_1) \simeq A \oplus (M/L_1M)$ is a locally coherent A-module. For $L_1 = B$, we deduce that A is a right coherent ring. Hence, for every finitely generated right ideal L_1 of B, $(M/L_1M)_A$ is locally coherent by [17, 26.1(1)]. The converse is also obtained by Theorem 4.1 because θ is monic. □

Corollary 4.3. *If A is a right Noetherian ring and B is a regular ring then T is a right coherent ring.*

Proposition 4.4. *Let ${}_BM$ be flat and M_A be finitely generated. Then T is a right coherent ring if and only if A_A, B_B and M_A are coherent.*

Proof. Since M_A is finitely generated, $(LM)_A$ is finitely generated for any finitely generated right ideal L of B. Thus M_A is locally coherent if and only if $(M/LM)_A$ is locally coherent for any finitely generated right ideal L of B [17, 26.1]. The result is then obtained by Theorem 4.2. □

It is well known that being finitely presented, and hence being (locally) coherent is a Morita invariant property. Thus, in view of [17, 26.6(c)], a ring R is right coherent if and only if $\mathrm{Mat}_n(R)$ is so. In the following we observe that a similar result holds for the lower triangular matrix ring $T_n(R)$.

Corollary 4.5. *Let R be a ring and $n \geq 1$. Then R is a right coherent ring if and only if $T_n(R)$ is so.*

Proof. We proceed by induction on n and apply Proposition 4.4 for $M = R^{(n-1)}$, $A = T_{n-1}(R)$ and $B = R$. We have ${}_BM$ is flat and $T_n(R) \simeq \left[\begin{smallmatrix} A & 0 \\ M & B \end{smallmatrix}\right]$. By induction assumption A is a right coherent ring. Now if $q = n - 1$, then $M = mA$ where

$m = (1, 1, \ldots, 1)_{1\times q}$ and $r \cdot \mathrm{ann}_A(m) = aA$ where

$$a = \begin{bmatrix} I_{(q-1)\times(q-1)} & 0 \\ & \vdots \\ -1 \ \cdots \ -1 & 0 \end{bmatrix}_{q\times q}.$$

Hence $M \simeq A/aA$ is a locally coherent A-module [17, 26.1]. Thus $T_n(R)$ is right coherent by Proposition 4.4. □

In [4] *right J-coherent* rings are introduced and studied (a ring R is called right J-coherent if the Jacobson radical of R is coherent as a right R-module). Next we use Theorem 4.1 to investigate the J-coherence of T.

Proposition 4.6. *If T is a right J-coherent ring then A and B are right J-coherent rings and $(M/LM)_A$ is coherent for any finitely generated right ideal L of B that is contained in $J(B)$. The converse is true provided that ${}_BM$ is flat.*

Proof. Note that by [8, Corollary 2.2], $J(T) = (X, Y)_{\theta_1}$ where $X = J(A) \oplus M$ and $Y = J(B)$. Also a finite direct sum of modules is coherent if and only if each direct summand is so. The rest of the proof is similar to that of Theorem 4.2. □

Corollary 4.7. *If $T_2(R)$ is a right J-coherent ring then R is right coherent.*

Remark 4.8. (i) The converse of Corollary 4.7 is not true in general. For example $R = \left[\begin{smallmatrix} \mathbb{Z} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{smallmatrix}\right]$ is a coherent ring by Theorem 4.2, but $J(R) = \left[\begin{smallmatrix} 0 & 0 \\ \mathbb{Q} & 0 \end{smallmatrix}\right]$ is not finitely generated. This also shows that the coherence of T does not imply that M_A is finitely generated.

(ii) The flatness of ${}_BM$ cannot be relaxed from the hypothesis of Theorem 4.2. Set $A = B =$ the coherent ring $\mathbb{Q}[x_1, x_2, \ldots]$; see [17, 26.7] and let I be the ideal of A generated by $\{x_1x_j \,|\, j = 2, 3, \ldots\}$ and $M = A/I$. Then ${}_BM$ is not flat and $U := r \cdot \mathrm{ann}_T(\left[\begin{smallmatrix} 1 & 0 \\ 0 & x_1 \end{smallmatrix}\right]) = \left[\begin{smallmatrix} 0 & 0 \\ C/I & 0 \end{smallmatrix}\right]$ where C is the ideal of A generated by $\{x_j \,|\, j = 2, 3, \ldots\}$. It is easy to see that $(C/I)_A$ and hence U_T is not finitely generated. This shows that T is not a right coherent ring.

Proposition 4.9. *If A is a right Noetherian ring, M_A finitely generated and B a right coherent ring then T is a right coherent ring.*

Proof. We use [17, 26.6(e)]. First note that if $(X, Y)_f$ is a right ideal of T, then $Y \leq B_B$ and X being a submodule of $(A \oplus M)_A$ is Noetherian. Hence $(X, Y)_f$ is a finitely generated right ideal if and only if Y is a finitely generated right ideal of B. Second, for every $t = \left[\begin{smallmatrix} a & 0 \\ m & b \end{smallmatrix}\right] \in T$, we have $r \cdot \mathrm{ann}_T(t) = (X', Y)_\theta$ where $Y = r \cdot \mathrm{ann}_B(b)$ and $X' \leq (A \oplus M)_A$. Therefore, the right coherence of B implies that of T. □

5. Pure Injectivity

In this section we investigate pure injective T-modules. The following Lemma is needed and may be found in the literature, but we give a proof for completeness.

Lemma 5.1. *Let R be a ring and $0 \to K \xrightarrow{f} L \xrightarrow{g} N \to 0$ be a pure exact sequence in* Mod-R. *Then $0 \to K/KH \xrightarrow{\bar{f}} L/LH \xrightarrow{\bar{g}} N/NH \to 0$ is a pure exact sequence in* Mod-R/H, *for every proper ideal H of R.*

Proof. It is known that the R/H-module $K \otimes_R R/H$ is naturally isomorphic to K/KH. Now consider the following commutative diagram

$$\begin{array}{ccc} K \otimes_R R/H & \xrightarrow{f\otimes 1} & L \otimes_R R/H \\ \| & & \| \\ K/KH & \xrightarrow{\bar{f}} & L/LH. \end{array}$$

By Lemma 2.1(iv), $f \otimes 1$, and hence $\bar{f}$ are pure monomorphisms in Mod-R/H. The rest of the proof is routine. □

Proposition 5.2. *Let H be a proper ideal of a ring R and W be an R-module with $WH = 0$. Then W is a pure injective R/H-module if and only if it is a pure injective R-module.*

Proof. ($\Rightarrow$) Let $K \xrightarrow{f} L$ be a pure monomorphism in Mod-R and $\alpha : K \to W$ be an R-homomorphism. By Lemma 5.1, we have the pure monomorphism $K/KH \xrightarrow{\bar{f}} L/LH$ in Mod-R/H. Since $WH = 0$, we have $\bar{\alpha} : K/KH \to W$. Now W is a pure injective R/H-module, hence there exists an R/H-homomorphism $\gamma : L/LE \to W$ such that $\gamma\bar{f} = \bar{\alpha}$. Let $\beta = \gamma\pi$ where $L \xrightarrow{\pi} L/LH$ is the natural projection. Then for any $k \in K$, $\beta f(k) = \gamma\pi(f(k)) = \gamma\bar{f}(\bar{k}) = \bar{\alpha}(\bar{k}) = \alpha(k)$ that is $\beta f = \alpha$, proving that W_R is pure injective.

($\Leftarrow$) By definition and Lemma 2.1(i). □

Recall that a ring R is right pure semisimple if any right R-module is pure injective. The class of right pure semisimple rings is closed under finite direct sums and homomorphic images; see [17, 53.7(b)].

Corollary 5.3. *Suppose that A and B are right pure semisimple rings and V is a T-module. Then M, V/VM, V/VP and V/VI are pure injective T-modules.*

Proof. Use Proposition 5.2 and the fact that $T/M \simeq A \oplus B$ is a pure semisimple ring. □

Theorem 5.4. *Let $V = (X, Y)_f$ be a T-module. If V_T is pure injective then so is X_A. The converse is true provided that $\tilde{f}$ is an isomorphism.*

Proof. Assume that V_T is pure injective. Let $X_1 \xrightarrow{g} X_2$ be a pure monomorphism in Mod-A and $\alpha : X_1 \to X$ be an A-homomorphism. It is easy to see that $(X_1, 0)_0 \xrightarrow{(\alpha,0)} (X, Y)_f$ and $(X_1, 0)_0 \xrightarrow{(g,0)} (X_2, 0)_0$ are homomorphisms in Mod-T. By Proposition 2.2(ii), $(g, 0)$ is a pure monomorphism. Thus by our assumption, there exists a T-homomorphism $(\beta_1, \beta_2) : (X_2, 0)_0 \to (X, Y)_f$ such that $(\beta_1, \beta_2)(g, 0) = (\alpha, 0)$. It follows that $\beta_1 g = \alpha$, as desired.

Conversely, assume that X_A is pure injective and $\tilde{f}$ is an isomorphism. Let $(X_1, Y_1)_{f_1} \xrightarrow{(g_1,g_2)} (X_2, Y_2)_{f_2}$ be a pure monomorphism in Mod-T and $(\alpha_1, \alpha_2) : (X_1, Y_1)_{f_1} \to (X, Y)_f$ be a T-homomorphism. By Proposition 2.2(i), $X_1 \xrightarrow{g_1} X_2$ is a pure monomorphism in Mod-A. Since X_A is pure injective, there exists A-homomorphism $\beta_1 : X_2 \to X$ such that $\beta_1 g_1 = \alpha_1$. For every $z \in Y_2$, define $\theta_z : M \to X$ by $\theta_z(m) = \beta_1 f_2(z \otimes m)$ for all $m \in M$. Then $\theta_z \in \mathrm{Hom}_A(M, X)$. Since now $\tilde{f}$ is an isomorphism there exists unique $y_z \in Y$ such that $\tilde{f}(y_z) = \theta_z$. Consequently, y_z is the unique element of Y that satisfies $f(y_z \otimes m) = \beta_1 f_2(z \otimes m)$ for all $m \in M$. Define $\beta_2 : Y_2 \to Y$ by $\beta_2(z) = y_z$ for all $z \in Y_2$. Then we can deduce that β_2 is a B-homomorphism such that $(\beta_1, \beta_2) : (X_2, Y_2)_{f_2} \to (X, Y)_f$ is a T-homomorphism. Since now $(\beta_1 g_1, \beta_2 g_2) : (X_1, Y_1)_{f_1} \to (X, Y)_f$ is a T-homomorphism, we have $f(\beta_2 g_2 \otimes 1_M) = (\beta_1 g_1) f_1$. Therefore, $f(\beta_2 g_2 \otimes 1_M) = \alpha_1 f_1 = f(\alpha_2 \otimes 1_M)$. This shows that $(\alpha_2 - \beta_2 g_2)(y) \in \mathrm{Ker}\, \tilde{f} = 0\ \forall\, y \in Y_1$. Thus $\beta_2 g_2 = \alpha_2$, proving that V is a pure injective T-module.

We have now the following result related to the adjoint pair of functors $(\Gamma, \mathcal{K})$: Mod-$T \to$ Mod-A introduced in Sec. 1. □

Corollary 5.5. *The functors $\mathcal{K}$ and Γ map pure injective modules to pure injective modules.*

Proof. For every X_A, $\mathcal{K}(X) = (X, \mathrm{Hom}_A(M, X))_f$ such that $\tilde{f} = 1_{\mathrm{Hom}_A(M,X)}$. Therefore, the result is obtained by Theorem 5.4. □

Acknowledgment

The authors are indebted to the referee for comments and suggestions that helped improve the presentation.

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