Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2013) 37: 925 – 933

doi:10.3906/mat-1206-31

On the Krull dimension of endo-bounded modules

Ahmad HAGHANY, Majid MAZROOEI, Mohammad Reza VEDADI*
Department of Mathematical Sciences, Isfahan University of Technology, Isfahan, Iran

Received: 19.06.2012 • **Accepted:** 11.04.2013 • **Published Online:** 23.09.2013 • **Printed:** 21.10.2013

Abstract: Modules in which every essential submodule contains an essential fully invariant submodule are called endo-bounded. Let M be a nonzero module over an arbitrary ring R and $X = \mathrm{Spec}_2(M_R)$, the set of all fully invariant $\mathcal{L}_2$-prime submodules of M_R. If M_R is a quasi-projective $\mathcal{L}_2$-Noetherian such that $(M/P)_R$ is endo-bounded for any $P \in X$, then it is shown that the Krull dimension of M_R is at most the classical Krull dimension of the poset X. The equality of these dimensions and some applications are obtained for certain modules. This gives a generalization of a well-known result on right fully bounded Noetherian rings.

Key words: Classical Krull dimension, endo-bounded module, FBN ring, Krull dimension, $\mathcal{L}_2$-Noetherian module, $\mathcal{L}_2$-prime module

1. Introduction

Throughout this paper rings will have unit elements and modules will be right unitary. The concept of the *classical Krull dimension* of an arbitrary poset X was originally defined in [2], denoted by $\mathrm{Cl.K.dim}(X)$. For $X = \mathrm{Spec}(R)$, the set of all prime ideals of a ring R, $\mathrm{Cl.K.dim}(X)$ was already denoted by $\mathrm{Cl.K.dim}(R)$ and called the classical Krull dimension of R; see [7, Chapter 14]. The latter dimension is a crucial concept in commutative algebra. It is well known that a commutative Noetherian ring R is Artinian if and only if $\mathrm{Cl.K.dim}(R) = 0$. A suitable tool that measures how far a module M_R is from being Artinian is the *Krull dimension* of M_R, $\mathrm{K.dim}(M_R)$, in the sense of Gabriel and Rentchler; see [7, Chapter 15] for an excellent reference on the subject.

Generalizing commutative rings to *right bounded rings* R (i.e. every essential right ideal of R contains an ideal that is essential as a right ideal), it was proven that $\mathrm{Cl.K.dim}(R) = \mathrm{K.dim}(R_R)$ for every right fully bounded right Noetherian (r.FBN) ring R [7, Theorem 15.13]. A generalization of the latter equality to modules is the aim of the present work.

By a *prime module* M_R, we mean the "classical" notion of a prime module, that is, $\mathrm{ann}_R(M) = \mathrm{ann}_R(N)$ for any $0 \neq N \leq M_R$. The set of all fully invariant submodules of a module M_R is denoted by $\mathcal{L}_2(M)$. Some generalizations of the concept of prime ideal and the classical Krull dimension of a ring were given by earlier authors. In [6], the poset of all prime submodules of a module was considered, and a principal ideal theorem analogue for modules was obtained. In [1], $\mathrm{Cl.K.dim}(X)$ was called the dimension of M_R when X is the poset of all distinguished prime submodules of M_R, and it was proven for faithful R-modules that the dimension is at

*Correspondence: mrvedadi@cc.iut.ac.ir
2010 *AMS Mathematics Subject Classification:* Primary 16P60, 16P70; Secondary 16P40.

most equal to Cl.K.dim(R). Also in [3], the classical Krull dimension of a module was defined by considering the certain chains of prime submodules, where it is shown that this classical Krull dimension is equal to the Krull dimension for a multiplication module M_R. Although various generalizations of the classical Krull dimension of rings are already given, no comparison has been made between the Krull dimension and the classical Krull dimension of a module over an arbitrary ring.

In this paper, we consider the classical Krull dimension of the poset $\mathrm{Spec}_2(M_R)$, the set of all fully invariant proper submodules P of M_R with the property $\mathrm{Hom}_R(M, W_1)W_2 \subseteq P \Rightarrow W_1 \subseteq P$ or $W_2 \subseteq P$ where $W_i \in \mathcal{L}_2(M)(i = 1, 2)$. Proper submodules of M_R having the latter property were called $\mathcal{L}_2$-*prime* in [17]; see also [18] where the term "fully prime" was used for $\mathcal{L}_2$-prime. If (0) is an $\mathcal{L}_2$-prime submodule of M_R, then M is called an $\mathcal{L}_2$-*prime* R-*module*; see [4] as an original reference of such R-modules. Every fully invariant $\mathcal{L}_2$-prime submodule of a module is a prime submodule [17, Proposition 2.1(ii)]. We define fully endo-bounded modules that form a class of modules properly containing both the class of multiplication modules and the class of (fully) bounded modules in the sense of [8] and [11]. For a quasi-projective fully endo-bounded $\mathcal{L}_2$-Noetherian module M_R, it is shown that $\mathrm{K.dim}(M_R)$ is at most equal to the classical Krull dimension of $\mathrm{Spec}_2(M_R)$, and the equality is obtained for certain modules. This generalizes the similar result for r.FBN rings. Any unexplained terminology and all the basic results on rings and modules that are used in the sequel can be found in [7] and [13].

2. Preliminaries

We begin by recalling some definitions from [17]. An R-module M is called $\mathcal{L}_2$-*Noetherian* if M finitely generates all of its fully invariant submodules and has ascending chain condition (acc) on them. Some examples of $\mathcal{L}_2$-Noetherian modules are Noetherian self-generator modules and modules without nontrivial fully invariant submodules. Note that the module R_R is $\mathcal{L}_2$-Noetherian if and only if every ideal of R is finitely generated as a right ideal. A *minimal* $\mathcal{L}_2$-prime submodule means a minimal member among all $\mathcal{L}_2$-prime submodules of M_R. We shall use the notation $N \trianglelefteq M_R$, $N \leq_e M_R$ to denote respectively that N is a fully invariant, essential submodule of M_R, and $W \star K$ for $\mathrm{Hom}_R(M, W)K$ where $W, K \leq M_R$. In the following, we present some facts on $\mathcal{L}_2$-Noetherian and $\mathcal{L}_2$-prime modules for later use.

Lemma 2.1 *Let M be an R-module.*
(i) *If $N \trianglelefteq M_R$ and $Q/N \trianglelefteq M/N$, then $Q \trianglelefteq M_R$.*
(ii) *If M is quasi-projective and $K \leq L \trianglelefteq M_R$, then $L/K \trianglelefteq M/K$.*
(iii) *If $M = M_1 \oplus M_2$, then $N \trianglelefteq M_R$ if and only if $N = N_1 \oplus N_2$ for some $N_i \trianglelefteq M_i$ with $\mathrm{Hom}_R(M_1, M_2)N_1 \subseteq N_2$ and $\mathrm{Hom}_R(M_2, M_1)N_2 \subseteq N_1$.*
(iv) *Let $n \geq 1$. Every fully invariant submodule of $M^{(n)}$ has the form $N^{(n)}$ for some $N \trianglelefteq M_R$.*

Proof These have routine arguments. □

Proposition 2.2 *Let R be a ring, $I \trianglelefteq R$ and M be an R-module.*
(i) *If $MI = 0$ then M_R is $\mathcal{L}_2$-Noetherian if and only if $M_{R/I}$ is $\mathcal{L}_2$-Noetherian.*
(ii) *M_R is $\mathcal{L}_2$-Noetherian if and only if M/N is $\mathcal{L}_2$-Noetherian for any $N \trianglelefteq M_R$.*

Proof We only prove (ii). M/N has acc on its fully invariant submodules by Lemma 2.1(i). On the other hand, if $L/N \trianglelefteq M/N$ then $L \trianglelefteq M$ and so M finitely generates L by our assumption. Hence, there exists an R-epimorphism $f : M^{(n)} \to L$ for some positive integer n. Let $\iota_i : M \to M^{(n)}$ be the natural injection for $i = 1, ..., n$. Since $N \trianglelefteq M$, $f\iota_i(N) \subseteq N$. This shows that the map $g : (M/N)^{(n)} \to L/N$ with $g(x_1 + N, ..., x_n + N) = f(x_1, ..., x_n) + N$ is well defined. Clearly g is also an R-epimorphism. Thus, M/N finitely generates L/N, proving that M/N is $\mathcal{L}_2$-Noetherian. The converse is clear using $N = 0$. □

Proposition 2.3 *Let M be an R-module and $P \lhd M$.*
(i) *Let M_R be quasi-projective. Then $P \in \mathrm{Spec}_2(M)$ if and only if for any $W_1, W_2 \trianglelefteq M$, $P \subsetneq W_i$ $(i = 1, 2)$ implies $W_1 \star W_2 \nsubseteq P$ if and only if M/P is an $\mathcal{L}_2$-prime R-module.*
(ii) *Let M_R be quasi-projective. If $N \lhd M$ and $N \leq P$ then $P/N \in \mathrm{Spec}_2(M/N)$ if and only if $P \in \mathrm{Spec}_2(M)$.*
(iii) *Let $n \geq 1$. Then $K \in \mathrm{Spec}_2(M^{(n)})$ if and only if $K = N^{(n)}$ for some $N \in \mathrm{Spec}_2(M)$.*

Proof (i) We only prove the first equivalence. One direction is clear. Assume that $N_1, N_2 \trianglelefteq M$ with $N_1 \star N_2 \subseteq P$. We shall prove that $N_1 \subseteq P$ or $N_2 \subseteq P$. If not, $P \subset N_i + P$ for $i = 1, 2$. Let $W_i = N_i + P$ (i=1,2). Since $P \lhd M$, $W_i \trianglelefteq M$ (i=1,2). We show that $W_1 \star W_2 \subseteq P$, which contradicts our assumption. Let $f \in \mathrm{Hom}_R(M, W_1)$ and $x + y \in W_2$ where $x \in N_2$ and $y \in P$. Since $f(y) \in P$, it is enough to show that $f(x) \in W_1$. Let $\pi : W_1 \to W_1/P$ and $\eta : N_1 \to N_1/(N_1 \cap P)$ be the natural projections and $\theta : W_1/P \to N_1/(N_1 \cap P)$ be the natural isomorphism. Since M is quasi-projective, there exists $g \in \mathrm{Hom}_R(M, N_1)$ such that $\eta g = \theta\pi f$. Thus, $g(N_2) \subseteq N_1$ and $g(x) - f(x) \in N_1 \cap P$. Hence, $f(x) = g(x) + (f(x) - g(x)) \in N_1 + P = W_1$. The proof is now completed.
(ii) This follows from (i) and the fact that M/N is quasi-projective when $N \trianglelefteq M$.
(iii) Let $K \in \mathrm{Spec}_2(M^{(n)})$. By Lemma 2.1(iv), $K = N^{(n)}$ for some $N \trianglelefteq M$. Now if $A \star B \subseteq N$ for some $A, B \trianglelefteq M$, then $A^{(n)} \star B^{(n)} \subseteq K$. Thus, $A^{(n)} \subseteq K$ or $B^{(n)} \subseteq K$, and hence $A \subseteq N$ or $B \subseteq N$. This shows that $N \in \mathrm{Spec}_2(M)$.
Similarly, the converse is proven by Lemma 2.1(iv). □

By [17, Proposition 2.5], $\mathrm{Spec}_2(M) \neq \varnothing$ when M_R is $\mathcal{L}_2$-Noetherian. In the following, we see an analogous result for certain quasi-projective modules.

Corollary 2.4 *Let M_R be quasi-projective and $P \lhd M$ such that P is maximal among all proper fully invariant submodules of M. Then $P \in \mathrm{Spec}_2(M)$. In particular, $\mathrm{Spec}_2(M) \neq \varnothing$ if M_R is a nonzero quasi-projective with acc on fully invariant submodules.*

Proof Apply Proposition 2.3(i). □

Let M be an R-module. If M is $M^{(\Lambda)}$-projective for every index set Λ, then we say that M_R is $\sum$-*projective*. Finitely generated quasi-projective modules are known to be $\sum$-projective. It is easy to verify that a module M_R is $\sum$-projective if and only if $(M/N)_R$ is so for any $N \in \mathcal{L}_2(M_R)$. In Proposition 2.6, for a $\sum$-projective $\mathcal{L}_2$-prime module, we obtain a generalization of the fact that "nonzero ideals in a prime ring are essential as right ideals". First we prove the following Lemma.

Lemma 2.5 *If M_R is $\sum$-projective and $B, A \leq M_R$ then $(B \star A) \star B \subseteq B \star (A \star B)$.*

Proof Let $D = A^{(\Lambda)}$ where $\Lambda = \mathrm{Hom}_R(M, B)$. For any $g \in \Lambda$ let $\pi_g : D \to A$ be the natural projection map. Define $h : D \to B \star A$ by $h(x) = \sum_{g\in\Lambda} g(\pi_g(x))$. Then h is an R-epimorphism. Suppose now that $f : M \to B \star A$ is an R-homomorphism. Since M_R is $\sum$-projective, M is a D-projective R-module. Thus, there exists $\alpha : M \to D$ such that $h\alpha = f$. Now for any $b \in B$, $f(b) = h(\alpha(b)) = \sum_{g\in\Lambda} g(\pi_g\alpha(b)) \in B \star (A \star B)$ because $\pi_g\alpha \in \mathrm{Hom}_R(M, A)$ for any $g \in \Lambda$. □

Proposition 2.6 *Let M_R be a $\sum$-projective and $P \in Spec_2(M_R)$. Then in the R-module M/P, every nonzero fully invariant submodule is essential.*

Proof Without loss of generality, we may suppose that M_R is an $\mathcal{L}_2$-prime module. Let $0 \neq A \trianglelefteq M_R$ and $A \cap N = 0$ for some $N \leq M_R$. Then $N \star A \subseteq N \cap A = 0$. Thus, $A \star (N \star A) = 0$, and hence $(A \star N) \star A = 0$ by Lemma 2.5 . Since now $(A \star N) \trianglelefteq M$ and M_R is $\mathcal{L}_2$-prime, $A \star N = 0$. It follows that $IN = 0$ where $I = \mathrm{Hom}_R(M, A)$. Since $I \trianglelefteq S = \mathrm{End}_R(M)$, we see that $0 = I(SN) = A \star SN$. Thus, SN and hence N must be zero, proving that $A \leq_e M_R$.

A module M_R is called *endo-bounded* if every essential submodule of M_R contains a fully invariant essential submodule of M_R. The module M_R is called *fully endo-bounded* if M/P is endo-bounded as a module over $R/\mathrm{ann}_R(M/P)$ for any $P \in \mathrm{Spec}_2(M_R)$. □

Proposition 2.7 *Let M_R be a module with $MI = 0$ for some $I \lhd R$.*

(i) *$M_{R/I}$ is (fully) endo-bounded if and only if M_R is (fully) endo-bounded.*

(ii) *If M_R is quasi-projective then M_R is fully endo-bounded if and only if M/N is fully endo-bounded for any $N \trianglelefteq M_R$.*

Proof (i). This has a routine proof using the facts that " $N \leq_e M_R$ if and only if $N \leq_e M_{R/I}$ " and " $\mathrm{Spec}_2(M_R) = \mathrm{Spec}_2(M_{R/I})$ ".

(ii) This follows by Proposition 2.3(ii) and part (i). □

Following [8], a ring R is called *pre semi-Artinian* if for every prime ideal P of R, the (right) socle of the ring R/P is nonzero. In the following, we give instances where (fully) endo-bounded modules appear.

Lemma 2.8 *Let R be a pre semi-Artinian ring.*

(i) *If M_R is prime, then M_R is endo-bounded.*

(ii) *If M_R is $\sum$-projective $\mathcal{L}_2$-prime, then it is endo-bounded or singular.*

Proof (i) Let $I = \mathrm{ann}_R(M)$ and $T = R/I$. In view of Proposition 2.7(i), it is enough to show that M_T is endo-bounded. Assume that $N \leq_e M_T$ and $J = \mathrm{ann}_T(M/N)$. For any $m \in M$, let $J_m = \mathrm{ann}_T(m + N)$. Since M/N is a singular T-module, $J_m \leq_e T_T$ for any $m \in M$. Hence, $J = \cap_{m\in M} J_m \supseteq \mathrm{Soc}(T_T)$ is a nonzero ideal by hypothesis. Thus, it is enough to show that $MJ \leq_e M_T$. Let $MJ \cap K = 0$ for some $K \leq M_T$. Then $KJ = 0$. Since now M_T is prime and faithful, we must have $K = 0$, as desired.

(ii) Suppose that M_R is not singular. As we see in (i), if $N \leq_e M_R$, then $MI \subseteq N$ for some $I \trianglelefteq_e R_R$. By our assumption MI is nonzero. Hence, the result is obtained by Proposition 2.6. □

Corollary 2.9 *Over a pre semi-Artinian ring R, all R-modules are fully endo-bonded.*

Proof Apply Lemma 2.8(i) and the fact that for all $P \in \text{Spec}_2(M)$, M/P is a prime module [17, Proposition 2.1]. □

Remark 2.10 *In [8], a bounded module M_R was defined by the condition $ann_R(M/N) \leq_e R_R$ for any $N \leq_e M_R$, and similarly M_R was called fully bounded if for all $P \in Spec_2(M_R)$, M/P is bounded. The proof of Lemma 2.8(i) shows that if M_T is a bounded faithful prime module, then it is an endo-bounded module. Thus, every fully bounded module is fully endo-bounded. However, it is easy to verify that $\mathbb{Q}_{\mathbb{Z}}$ is fully endo-bounded because $Spec_2(\mathbb{Q}_{\mathbb{Z}}) = \{0\}$, but not fully bounded because $ann_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}) = 0$.*

3. Classical Krull dimension for modules

In this section we introduce the classical Krull dimension for an arbitrary module M as the classical Krull dimension of the poset $\text{Spec}_2(M)$ and investigate it for quasi-projective modules.

Definition 3.1 *Let M be an R-module and $X_{-1} = \varnothing$. Suppose that γ is an ordinal number and X_β is defined for all $\beta < \gamma$. Let X_γ be the set of all $P \in \text{Spec}_2(M)$ with the property that for any $Q \in \text{Spec}_2(M)$ with $P \subset Q$, there exists $\beta < \gamma$ such that $Q \in X_\beta$. Note that $X_i \subseteq X_j$ for all ordinal numbers i, j with $i \leq j$. The chain $\{X_i\}_{i \geq -1}$ will be called the* $\mathcal{L}_2$-classical Krull chain *of M. We say that M_R has* $\mathcal{L}_2$-classical Krull dimension *if there exists an ordinal number α such that $X_\alpha = \text{Spec}_2(M)$. The smallest α with this property will be called the $\mathcal{L}_2$-classical Krull dimension of M and will be denoted by $\mathcal{L}_2$-dim(M_R). Note that if $0 \in X_\beta$ for some ordinal number β, then $\mathcal{L}_2$-dim(M_R) $\leq \beta$.*

By [2, Proposition 1.4], $\mathcal{L}_2$-dim(M_R) exists if and only if $\text{Spec}_2(M)$ is a Noetherian poset, i.e. it satisfies the acc. If M_R is semisimple, then every element in $\mathcal{L}_2(M_R)$ has the form $\oplus_i M_i$ where each M_i is a homogeneous component of M_R. Hence, it is easy to verify that $\mathcal{L}_2$-dim(M_R) $= 0$. Therefore, Proposition 2.3(ii) shows that $\mathcal{L}_2$-dim(M_R) $= 0$ when M_R is quasi-projective and R is a ring with $(R/P)_R$ Artinian for all $P \in \text{Spec}(R)$.

Proposition 3.2 *Let M be an R-module, $n \geq 1$ and $\mathcal{L}_2$-dim(M_R) exists. Then $\mathcal{L}_2$-dim($M^{(n)}$) exists and $\mathcal{L}_2$-dim(M) $= \mathcal{L}_2$-dim($M^{(n)}$).*

Proof This is obtained by Proposition 2.3(iii).

If R is a r.FBN ring, then by Proposition 3.2 and the fact that Cl.K.dim(R) $=$ K.dim(R), we have $\mathcal{L}_2$-dim(M_R) $=$ K.dim(M_R) for every finitely generated free R-module M. In Theorems 4.1 and 4.3, we will see a generalization of the latter equality. □

Proposition 3.3 *Let M be a quasi-projective R-module and $N \trianglelefteq M_R$.*
(i) *Assume that $\{X_i\}_{i \geq -1}$ and $\{Y_j\}_{j \geq -1}$ are the $\mathcal{L}_2$-classical Krull chains of M and M/N, respectively. Then $P \in X_\alpha$ if and only if $P/N \in Y_\alpha$ for any $\alpha \geq 0$ and $P \in \text{Spec}_2(M)$ with $N \subseteq P$.*
(ii) *If $\mathcal{L}_2$-dim(M_R) $= \alpha$ exists then $\mathcal{L}_2$-dim($(M/N)_R$) is at most equal to α.*

Proof (i). The equivalence follows by definition and Proposition 2.3(ii).
(ii). This is obtained by (i). □

Proposition 3.4 *Suppose that M_R is quasi-projective and $\mathcal{L}_2$-dim(M_R) exists. Then $\mathcal{L}_2$-dim$(M/(W_1 \star W_2)) =$ Max $\{\mathcal{L}_2$-dim$(M/W_1), \mathcal{L}_2$-dim$(M/W_2)\}$ for every $W_1, W_2 \trianglelefteq M_R$.*

Proof Let $M_i = M/W_i$, $L_i = W_i/(W_1 \star W_2)$ (i=1,2), $L = M/(W_1 \star W_2)$, $\alpha = \mathcal{L}_2$-dim(M_1), and $\beta = \mathcal{L}_2$-dim(M_2). Since $L_i \trianglelefteq L_R$ and $M_i \simeq L/L_i$ (i=1,2), $\mathcal{L}_2$-dim$(M_i) = \mathcal{L}_2$-dim$(L/L_i) \leq \mathcal{L}_2$-dim$(L)$ by Proposition 3.3(ii). This shows that $\mathcal{L}_2$-dim$(L) \geq$ Max$\{\alpha, \beta\}$. For the converse, suppose that $\{X_i\}_{i \geq -1}$, $\{Y_i\}_{i \geq -1}$, and $\{Z_i\}_{i \geq -1}$ are the $\mathcal{L}_2$-classical Krull chains of M_1, M_2, and M, respectively. Let $P/(W_1 \star W_2) \in \mathrm{Spec}_2(L)$. Since $W_1 \star W_2 \trianglelefteq M_R$, $P \in \mathrm{Spec}_2(M)$ by Proposition 2.3(ii). Thus, $W_1 \leq P$ or $W_2 \leq P$. This in turn implies $P/W_1 \in X_\alpha$ or $P/W_2 \in Y_\beta$. Therefore, $P \in Z_\alpha$ or $P \in Z_\beta$ and so $P \in Z_{\mathrm{Max}\{\alpha,\beta\}}$. It follows that $\mathcal{L}_2$-dim$(L) \leq$ Max$\{\alpha, \beta\}$. The proof is now complete. □

Corollary 3.5 *If M_R is quasi-projective with $\mathcal{L}_2$-classical Krull dimension and $P, P_0 \in \mathrm{Spec}_2(M)$ with $P_0 \subset P$, then $\mathcal{L}_2$-dim$(M/P) < \mathcal{L}_2$-dim(M/P_0).*

Proof We may assume that $P_0 = 0$. Let $\{X_i\}_{i \geq -1}, \{Y_i\}_{i \geq -1}$ be the $\mathcal{L}_2$-classical Krull chains of M and M/P, respectively, and suppose $\mathcal{L}_2$-dim$(M_R) = \alpha$. By our assumption $0 \in \mathrm{Spec}_2(M)$, and since now $0 \subset P$, there exists $\beta < \alpha$ such that $P \in X_\beta$. It is enough to show that $0 \in Y_\beta$. Note that $0 \in \mathrm{Spec}_2(M/P)$ and suppose that $0 \neq Q/P \in \mathrm{Spec}_2(M/P)$. Then $P \subset Q$ and $Q \in \mathrm{Spec}_2(M)$ by Proposition 3.3. Thus, there exists $\gamma < \beta$ such that $Q \in X_\gamma$. Again by Proposition 3.3, $Q/P \in Y_\gamma \subseteq Y_\beta$. It follows that $0 \in Y_\beta$, as desired. □

Proposition 3.6 *Suppose that M_R is quasi-projective, $\mathcal{L}_2$-prime with acc on fully invariant submodules. Then $\mathcal{L}_2$-dim$(M/N) < \mathcal{L}_2$-dim(M_R) for any nonzero $N \in \mathcal{L}_2(M)$.*

Proof If not, we shall have $\mathcal{L}_2$-dim$(M/N) = \mathcal{L}_2$-dim(M_R) for some $0 \neq N \trianglelefteq M$. Let $\mathcal{A} = \{0 \neq K \trianglelefteq M \mid \mathcal{L}_2$-dim$(M/K) = \mathcal{L}_2$-dim$(M_R)\}$. Then $\mathcal{A} \neq \varnothing$. Since M has acc on fully invariant submodules, $\mathcal{A}$ has a maximal member P. Applying Proposition 2.3(i), we first show that $P \in \mathrm{Spec}_2(M)$. Thus, suppose that there exist nonzero fully invariant submodules W_1, W_2 such that $P \subseteq W_i$ (i=1,2) and $W_1 \star W_2 \subseteq P$. Since M is $\mathcal{L}_2$-prime, $0 \neq W_1 \star W_2$, and $\mathcal{L}_2$-dim$(M_R) = \mathcal{L}_2$-dim$(M/P) \leq \mathcal{L}_2$-dim$(M/(W_1 \star W_2))$. Thus, $W_1 \star W_2 \in \mathcal{A}$. Hence, by Proposition 3.4, $\mathcal{L}_2$-dim$(M/W_1) = \mathcal{L}_2$-dim(M_R) or $\mathcal{L}_2$-dim$(M/W_2) = \mathcal{L}_2$-dim(M_R), which in turn implies that $W_1 \in \mathcal{A}$ or $W_2 \in \mathcal{A}$. It follows that $W_1 = P$ or $W_2 = P$, proving that $P \in \mathrm{Spec}_2(M)$. Now an application of Corollary 3.5 for $P_0 = 0$ shows that $\mathcal{L}_2$-dim$(M/P) < \mathcal{L}_2$-dim(M_R), a contradiction. □

Proposition 3.7 *Let M_R be quasi-projective with acc on fully invariant submodules. Then $\mathcal{L}_2$-dim$(M_R) = \mathcal{L}_2$-dim(M/P) for some minimal $\mathcal{L}_2$-prime submodule P of M_R.*

Proof First note that $\mathcal{L}_2(M)$ contains $\{P_1, ..., P_n\}$, the set of all minimal $\mathcal{L}_2$-prime submodules of M_R by [17, Proposition 2.2]. Let $\{X_i\}_{i \geq -1}$ be the $\mathcal{L}_2$-classical Krull chain of M, $\mathcal{L}_2$-dim$(M_R) = \alpha$ and $\beta =$ Max$\{\mathcal{L}_2$-dim$(M/P_1), ..., \mathcal{L}_2$-dim$(M/P_n)\}$. Then $\alpha \geq \beta$ by Proposition 3.3(ii). If $\alpha > \beta$, then there exists

$P \in \mathrm{Spec}_2(M)$ such that $P \notin X_\beta$. By [17, Proposition 2.1(i)], there exists $1 \leq k \leq n$ with $P_k \subseteq P$. Let $\{Y_i\}_{i \geq -1}$ be the $\mathcal{L}_2$-classical Krull chain of M/P_k. By hypothesis, $P_k \in \mathcal{L}_2(M)$. Since now $P \notin X_\beta$, $P/P_k \notin Y_\beta$ by Proposition 3.3(i). Also, $P/P_k \in \mathrm{Spec}_2(M/P_k)$ by Proposition 2.3(ii). This shows that $Y_\beta \neq \mathrm{Spec}_2(M/P_k)$ while $\mathcal{L}_2$-dim$(M/P_k) \leq \beta$, a contradiction. Therefore, $\alpha = \beta$. □

4. Main results

Theorem 4.1 *If M_R is quasi-projective, $\mathcal{L}_2$-Noetherian, and fully endo-bounded, then K.dim(M_R), if it exists, is at most equal to $\mathcal{L}_2$-dim(M_R).*

Proof By induction on $\mathcal{L}_2$-dim(M_R). Suppose K.dim(M_R) exists. Since M_R is $\mathcal{L}_2$-Noetherian, by [17, Theorem 3.1] there exists $P \in \mathrm{Spec}_2(M)$ such that K.dim$(M_R) =$ K.dim$((M/P)_R)$. If $\mathcal{L}_2$-dim$(M_R) = 0$, we will show that $(M/P)_R$ is semisimple. The existence of the Krull dimension then implies that $(M/P)_R$ is Artinian; see, for example, [7, Ex. 15C]. Let N/P be a proper essential R-submodule of M/P. By hypothesis $(M/P)_R$ is endo-bounded and so there exists $K/P \trianglelefteq_e M/P$ such that $K/P \subseteq N/P$. By Proposition 2.2(ii), the quasi-projective R-module $(M/P)/(K/P) \simeq M/K$ is $\mathcal{L}_2$-Noetherian. Apply now Corollary 2.4 for the R-module M/K to deduce that $\mathrm{Spec}_2(M/K) \neq \varnothing$. It follows that M/P has a nonzero fully invariant $\mathcal{L}_2$-prime submodule and hence $\mathcal{L}_2$-dim$((M/P)_R) \neq 0$. Thus, by Proposition 3.3(ii), $\mathcal{L}_2$-dim$(M_R) \neq 0$, a contradiction. Therefore, $(M/P)_R$ has no proper essential submodules, proving that M/P is a semisimple R-module.
Now assume that $\mathcal{L}_2$-dim$(M_R) = \alpha$ and the result holds for any fully endo-bounded quasi-projective $\mathcal{L}_2$-Noetherian R-module with $\mathcal{L}_2$-classical Krull dimension less than α. By [13, Lemma 2.8], it is enough to show that K.dim$(M/N) < \alpha$ for any $N/P \leq_e M/P$. Suppose that $N/P \leq_e M/P$. Note that M/P is also a fully endo-bounded quasi-projective $\mathcal{L}_2$-Noetherian R-module. Hence, there exists $0 \neq K/P \trianglelefteq_e M/P$ such that $K/P \subseteq N/P$. Apply Proposition 3.6 for the $\mathcal{L}_2$-prime R-module M/P to deduce that $\mathcal{L}_2$-dim$(M/K) < \mathcal{L}_2$-dim$(M/P) \leq \alpha$. Therefore, by the induction assumption, we have K.dim$(M/K) \leq \mathcal{L}_2$-dim$(M/K) < \alpha$. Because $K \subseteq N$, we must have K.dim$(M/N) < \alpha$ [7, Lemma 15.1], as desired.

In the following we give some applications of our results for modules over pre semi-Artinian rings. Clearly every commutative ring with zero classical Krull dimension is pre semi-Artinian. There are also noncommutative pre semi-Artinian rings R that do not have Krull dimensions; for example, say $R = \begin{bmatrix} F & 0 \\ M & F \end{bmatrix}$, F is a field, and M_F is nonfinitely generated free. □

Corollary 4.2 *Let R be a pre semi-Artinian ring and M_R be a quasi-projective Noetherian self-generator module. Then* K.dim$(M_R) \leq \mathcal{L}_2$*-dim*(M_R).

Proof By Corollary 2.9 and Theorem 4.1.

We are now going to investigate the inequality $\mathcal{L}_2$-dim$(M_R) \leq$ K.dim(M_R). In [15] an R-module M was called *essentially compressible* if for every $N \leq_e M$, there exists an R-monomorphism $M \to N$. If R is a semiprime right Goldie ring, then nonsingular essentially compressible R-modules are precisely submodules of free R-modules [15, Theorem 2.3]. In particular, if R is a right Noetherian ring then $(R/P)_R$ is essentially compressible for any prime ideal P of R. Hence, the well-known result Cl.K.dim$(R) \leq$ K.dim(R_R) on right Noetherian rings may be obtained by the following result. □

Theorem 4.3 *Assume that M_R is a $\sum$-projective $\mathcal{L}_2$-Noetherian module with Krull dimension such that $\mathcal{L}_2$-prime factors of M are essentially compressible. Then $\mathcal{L}_2$-dim(M_R) $\leq$* K.dim(M_R).

Proof In view of Proposition 3.7 and [17, Theorem 3.1], without loss of generality, we may suppose that M_R is $\mathcal{L}_2$-prime. Now we give a proof by induction on K.dim(M_R). If K.dim(M_R) $= 0$, then M_R is Artinian, and hence M_R is a homogeneous semisimple R-module by [17, Theorem 2.4]. This shows that $\mathcal{L}_2$-dim(M_R) $= 0$. Now assume that K.dim(M_R) $= \alpha$ and let $\{X_i\}_{i\geq -1}$ be the $\mathcal{L}_2$-classical Krull chain of M_R. We shall show that $0 \in X_\alpha$. Thus, suppose that $0 \neq P \in \text{Spec}_2(M)$. By hypothesis, M_R is $\sum$-projective and so $P \leq_e M$ by Proposition 2.6. Also by our assumption, M is essentially compressible. Hence, there exists an R-monomorphism $f: M \to P$. It follows that K.dim($(M/P)_R$) $<$ K.dim(M_R) by [7, Lemma 15.6]. Now by induction assumption, we have $\mathcal{L}_2$-dim(M/P) $\leq$ K.dim($(M/P)_R$) $< \alpha$. Thus, if $\beta = \mathcal{L}_2$-dim(M/P), then $P \in X_\beta$ by Proposition 3.3(i), and the proof is complete. □

Corollary 4.4 *Let R be a pre semi-Artinian ring and M_R be $\sum$-projective $\mathcal{L}_2$-Noetherian such that $\mathcal{L}_2$-prime factors of M are essentially compressible. Then* K.dim(M_R) $=$ *$\mathcal{L}_2$-dim(M_R) provided that* K.dim(M_R) *exists.*

Proof By Corollary 2.9 and Theorem 4.1.

The following result is a generalization of a well-known fact stating that the classical Krull dimension of a right Noetherian ring R is at most equal to the Krull dimension of R_R [13, 6.4.5]. □

Corollary 4.5 *Let R be a ring with Krull dimension in which every ideal is finitely generated as a right ideal. Then Cl.K.dim(R) $\leq$ K.dim(R_R).*

Proof This is obtained by Theorem 4.3 [15, Theorem 2.3]. □

References

[1] Abu-Saymeh, S.: On dimension of finitely generated modules. Comm. Alg. 23, 1131–1144 (1995).

[2] Albu, T.: Sur la dimension de Gabriel des modules. In: Seminar F. Kasch- B. Pareigis. Algebra Berchte (21). Munich. Verlag Uni-Druck 1974.

[3] Behboodi, M., Molakarimi, M.: Multiplication modules with Krull dimension. Turk. J. Math. 36, 550–559 (2012).

[4] Bican, L., Jambor, P., Kepka, T., Nemec, P.: Prime and coprime modules. Fund. Math. 57, 33–45 (1980).

[5] Cho, Y.H.: On distinguished prime submodules. Comm. Korean Math. Soc. 15. 493–498 (2000).

[6] George, S.M., McCasland, R.L., Smith, P.F.: A principal ideal theorem analogue for modules over commutative rings. Comm. Alg. 22, 2083–2099 (1994).

[7] Goodearl, K.R., Warfield, R.B.: An Introduction to Noncommutative Noetherian Rings. New York. Cambridge University Press 2004.

[8] Haghany, A., Mazrooei, M., Vedadi, M.R.: Bounded and fully bounded modules. Bull. Aust. Math. Soc. 84, 433–440 (2011).

[9] Heakyung, L.: Strongly right FBN rings. Bull. Aust. Math. Soc. 38, 457–464 (1988).

[10] Heakyung, L.: On relatively FBN rings. Comm. Alg. 23, 2991–3001 (1995).

[11] Heakyung, L.: Right fbn rings and bounded modules. Comm. Alg. 16, 977–987 (1988).

[12] Kok-Ming, T.: Homological properties of fully bounded Noetherian rings. J. London Math. Soc. 55, 37–54 (1997).

[13] McConnell, J.C., Robson, J.C.: Noncommutative Noetherian Rings. New York. Wiley- Interscience 1987.

[14] Smith, P.F.: Modules with many homomorphism. J. Pure Appl. Algebra 197, 305–321 (2005).

[15] Smith, P.F., Vedadi, M.R.: Essentially compressible modules and rings. J. Algebra 304, 812–831 (2006).

[16] Smith, P.F., Vedadi, M.R.: Submodules of direct sums of compressible modules. Comm. Alg. 36, 3042–3049 (2008).

[17] Vedadi, M.R.: $\mathcal{L}_2$-prime and dimensional modules. Int. Electr. J. Alg. 7, 47–58 (2010).

[18] Wijayanti, I.E., Wisbauer, R.: On coprime modules and comodules. Comm. Alg. 37, 1308–1333 (2009).