Journal of Mathematical Extension
Vol. 15, No. 3, (2021) (18)1-12
URL: https://doi.org/10.30495/JME.2021.1550
ISSN: 1735-8299
Original Research Paper

Matricial Radius: A Relation of Numerical Radius with Matricial Range

M. Kian
University of Bojnord

M. Dehghani*
University of Kashan

M. Sattari
University of Zabol

Abstract. It has been shown that if T is a complex matrix, then

$$\begin{aligned}\omega(T) &= \frac{1}{n}\sup\{|\mathrm{Tr}\ X|;\ X \in W^n(T)\}\\ &= \frac{1}{n}\sup\{\|X\|_1;\ X \in W^n(T)\}\\ &= \sup\{\omega(X);\ X \in W^n(T)\}\end{aligned}$$

where n is a positive integer, $\omega(T)$ is the numerical radius and $W^n(T)$ is the n'th matricial range of T.

AMS Subject Classification: 15A60 ; 47A12.
Keywords and Phrases: Numerical range, matricial range, completely positive mapping, numerical radius.

Received: February 2020; Accepted: January 2021
*Corresponding Author

1 Introduction and Preliminaries

One of the most well-known concept in study of Hilbert space operators is the notion of numerical range. Assume that $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ is a Hilbert space and $\mathbb{B}(\mathscr{H})$ is the C^*-algebra of all bounded linear operators on $\mathscr{H}$ with the identity operator I. When $\mathscr{H}$ has finite dimension n, we identify $\mathbb{B}(\mathscr{H})$ with the algebra $\mathbb{M}_n := \mathbb{M}_n(\mathbb{C})$ of all $n \times n$ complex matrices and I_n denotes the $n \times n$ identity matrix. The numerical range of $T \in \mathbb{B}(\mathscr{H})$ is well-known:

$$W(T) = \{\langle Tx, x\rangle; \quad x \in \mathscr{H}, \ \|x\| = 1\}.$$

This set is an important tool which gives many information about T, particularly about its eigenvalues and eigenspaces. The numerical range has a unique nature in numerical analysis and differential equations. It has many desirable properties, which probably the most famous of them is the Toeplitz-Hausdorff result. It asserts that $W(T)$ is convex for every $T \in \mathbb{B}(\mathscr{H})$, see e.g. [7]. The basic properties of the numerical range of bounded linear operators on Hilbert spaces can be found in [7].

We summarize some basic properties of the numerical range in the following theorem.

Theorem A.[7] For $T \in \mathbb{B}(\mathscr{H})$;

(i) $W(\alpha I + \beta T) = \alpha + \beta W(T), \ \alpha, \beta \in \mathbb{C}$;

(ii) $W(U^*TU) = W(T),$ for every unitary $U \in \mathbb{B}(\mathscr{H})$;

(iii) $sp(T) \subseteq \overline{W(T)},$ where $sp(T)$ is the spectrum of T.

A related concept is the numerical radius. The numerical radius of $T \in \mathbb{B}(\mathscr{H})$ is defined by

$$\omega(T) = \sup\{|\lambda|, \ \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|; \ \|x\| = 1\}.$$

Some of basic properties of the numerical radius are listed below.

Theorem B. For every $T, S \in \mathbb{B}(\mathscr{H})$

(i) $\omega(T) = \omega(T^*)$ and $\omega(U^*TU) = \omega(T)$ for every unitary $U \in \mathbb{B}(\mathscr{H})$;

(ii) $\dfrac{1}{2}\|T\| \leq \omega(T) \leq \|T\|$ and $\omega(T) = \|T\|$ if T is normal;

(iii) $\omega(T \oplus S) = \max\{\omega(T), \omega(S)\}$;

Recently, some new generalization of the numerical range and and the numerical radius are introduced in [1, 16].

The numerical radius is also defined for elements of a C^*-algebra. Recall that a linear functional τ on a C^*-algebra $\mathscr{A}$ is positive if $\tau(a) \geq 0$ for every positive element $a \in \mathscr{A}$. Also, a state is a positive linear functional whose norm is equal to one. If $\mathscr{A}$ is a unital C^*-algebra, the numerical radius of $A \in \mathscr{A}$ is defined by

$$\nu(A) = \sup\{|\tau(A)|;\ \tau \text{ is a state on } \mathscr{A}\}.$$

The reader is referred to [5, 7, 10, 11, 15] and references therein for more result concerning the numerical radius and the numerical range.

2 Matricial Range

Let $\mathscr{A}, \mathscr{B}$ be unital C^*-algebras and let $\mathscr{A}_+$ denotes the cone of positive elements of $\mathscr{A}$. Recall that a mapping $\Phi : \mathscr{A} \to \mathscr{B}$ is called positive, whenever $\Phi(\mathscr{A}_+) \subseteq \mathscr{B}_+$. Moreover, for $n \in \mathbb{N}$, Φ is called n-positive if the mapping $\Phi_n : \mathbb{M}_n(\mathscr{A}) \to \mathbb{M}_n(\mathscr{B})$ defined by $\Phi_n([A_{ij}]) = [\Phi(A_{ij})]$ is positive. If $\Phi : \mathscr{A} \to \mathscr{B}$ is n-positive for every $n \in \mathbb{N}$, then Φ is called completely positive.

For $T \in \mathbb{B}(\mathscr{H})$, assume that $CP_n(T)$ is the set of all unital completely positive linear mappings from $C^*(T)$ to $\mathbb{M}_n$:

$$CP_n(T) = \{\Phi|\ \Phi : C^*(T) \to \mathbb{M}_n \text{ is unital and completely positive}\},$$

in which $C^*(T)$ is the unital C^*-algebra generated by T. Arveson [2] defined the n'th matricial range of an operator $T \in \mathbb{B}(\mathscr{H})$ by

$$W^n(T) = \{\Phi(T)|\ \ \Phi \in CP_n(T)\}.$$

This is a matrix valued extension of the numerical range, say

$$W^1(T) = \overline{W(T)}.$$

It follows from the definition of $W^n(T)$ that

Theorem C. If $T \in \mathbb{B}(\mathscr{H})$ and $n \in \mathbb{N}$, then

(i) $W^n(U^*TU) = W^n(T)$ for each unitay $U \in \mathbb{B}(\mathscr{H})$;

(ii) $W^n(\alpha I) = \{\alpha I_n\}$ and $W^n(\alpha T + \beta I) = \alpha W^n(T) + \beta I_n$ for all $\alpha, \beta \in \mathbb{C}$.

Moreover, as a non-commutative Toeplitz-Hausdorff result, it is known that $W^n(T)$ is C^*-convex[13]. A set $\mathcal{K} \subseteq \mathbb{B}(\mathscr{H})$ is called C^*-convex, if $X_1, \ldots, X_m \in \mathcal{K}$ and $A_1, \ldots, A_m \in \mathbb{B}(\mathscr{H})$ with $\sum_{j=1}^m A_j^* A_j = I$ imply that $\sum_{j=1}^m A_j^* X_j A_j \in \mathcal{K}$. Indeed, this is a noncommutative generalization of linear convexity. It is evident that the C^*-convexity of a set implies its convexity in the usual sense. But the converse is not true in general. For more information about C^*-convexity see [9, 12] and the references therein.

Matricial ranges are closely connected with C^*-convex sets. In fact, the matrix ranges turn out to be the compact C^*-convex sets. However, except in some special cases, it is not routine to obtain the matricial ranges of an operator. The reader is referred to [2, 4, 6, 14] and the references therein for more information about matricial ranges.

The main purpose of this note is to define an analogues of the numerical radius related to the matricial range. However, we will find relations between the numerical radius and matricial range of an operator. The tone of the paper is mostly expository.

3 Matricial Radius

Similar to the connection of numerical radius and numerical range, it is natural to define the matricial radius of an operator to be the maximum norm of the elements of its matricial range. However, as pointed out in [6], unlike the numerical radius, the matricial radius is not interesting. For every $T \in \mathbb{B}(\mathscr{H})$ and $n \geq 2$, it holds

$$\max\{\|X\|;\ X \in W^n(T)\} = \|T\|.$$

As another candidate for the matricial radius, we consider the next definition.

Definition 3.1. For every operator $T \in \mathbb{B}(\mathscr{H})$ and every positive integer n, set

$$\nu_n(T) = \sup\{|\mathrm{Tr}X|;\ X \in W^n(T)\} = \sup\{|\mathrm{Tr}\ \Phi(T)|;\ \Phi \in CP_n(T)\},$$

where $\mathrm{Tr}(\cdot)$ denotes the canonical trace. It is easy to see that

(i) $\nu_1(T) = \nu(T)$;
(ii) $\nu_n(T^*) = \nu_n(T)$;
(iii) $\nu_n(U^*TU) = \nu_n(T)$ for every unitary U.

Moreover, it can be shown that

$$\nu_n(T) \leq n\|T\|$$

and the equality holds if T is normal. Although, ν_n has some favorite properties, it is not interesting too.

Example 3.2. Consider

$$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathbb{M}_2,$$

so that $\omega(T) = \frac{1}{2}$ and $\|T\| = 1$. Moreover, it is known that [2]

$$W^n(T) = \left\{ B \in \mathbb{M}_n \; ; \; \omega(B) \leq \frac{1}{2} \right\}.$$

Therefore

$$\nu_2(T) = 1 = 2\ \omega(T).$$

We will show that the equality $\nu_n(\cdot) = n\ \omega(\cdot)$ holds in general. We need some lemmas to continue our work.

Lemma 3.3. *[2] Let S and T be Hilbert space operators (perhaps acting on different spaces) and S is normal. Then the followings are equivalent:*
1. $W^n(S) \subseteq W^n(T)$
2. $sp(S)$ is contained in the closed numerical range of T.

The next theorem reveals that ν_n can not be a proper extension of the numerical radius.

Theorem 3.4. *For every $T \in \mathbb{M}_k$*

$$\omega(T) = \frac{1}{n}\nu_n(T) \qquad (n \in \mathbb{N}).$$

Proof. Assume that $\Phi : C^*(T) \to \mathbb{M}_n$ is a unital completely positive linear mapping. The Arveson's extension theorem (see for example [3, Theorem 3.1.5]) guarantees the existence of a unital completely positive linear mapping $\widetilde{\Phi} : \mathbb{M}_k \to \mathbb{M}_n$, which is an extension of Φ. Moreover, the Stinespring theorem (See [3, Theorem 3.1.2]) yields that $\widetilde{\Phi}(A) = V^*\pi(A)V$ in which $V : \mathbb{C}^n \to \mathbb{C}^{k^2n}$ and $V^*V = I$ and $\pi : \mathbb{M}_k \to \mathbb{M}_{k^2n}$ is an $*$-homomorphism so that $\pi(A) = A \oplus \cdots \oplus A$. Now, assume that $\{u_1, \cdots, u_n\}$ is an orthonormal system of eigenvectors for $\widetilde{\Phi}(T)$. Then $Vu_j \ \ (j = 1, \cdots, n)$ are unit vectors in $\mathbb{C}^{k^2n}$. Therefore

$$\begin{aligned}
|\mathrm{Tr}\ \Phi(T)| = |\mathrm{Tr}\ \widetilde{\Phi}(T)| &= \left|\sum_{j=1}^{n} \langle \widetilde{\Phi}(T)u_j, u_j\rangle\right| \\
&= \left|\sum_{j=1}^{n} \langle V^*\pi(T)Vu_j, u_j\rangle\right| \\
&\le \sum_{j=1}^{n} |\langle \pi(T)Vu_j, Vu_j\rangle| \\
&\le \sum_{j=1}^{n} \omega(\pi(T)) \\
&= n\ \omega(T \oplus \cdots \oplus T) \\
&= n\ \omega(T),
\end{aligned}$$

where the last inequality follows from (iii) of Theorem B. Taking supremum over all Φ, we conclude that

$$\nu_n(T) \le n\ \omega(T). \tag{1}$$

Furthermore, let $T \in \mathbb{M}_k$. Put $S = \omega(T)I$ so that S is normal and $W^n(S) = \{\omega(T)I_n\}$ by (iii) of Theorem C. Moreover, $sp(S) = \{\omega(T)\} \subseteq \overline{W(T)}$. Lemma 3.3 then implies that $W^n(S) \subseteq W^n(T)$ and so $\nu_n(S) \le \nu_n(T)$. Therefore

$$n\ \omega(T) = \nu_n(S) \le \nu_n(T). \tag{2}$$

The result now follows from (1) and (2). $\square$

The next definition provide another choice for the matricial radius.

Definition 3.5. For every $T \in \mathbb{B}(\mathscr{H})$

$$\omega_n(T) = \sup\{\mathrm{Tr}\ |\Phi(T)|;\ \Phi \in CP_n(T)\} = \sup\{\|X\|_1;\ X \in W^n(T)\}.$$

It is easy to see that

(i) $\omega_1(T) = \nu(T)$;
(ii) $\omega_n(T^*) = \omega_n(T)$;
(iii) $\omega_n(U^*TU) = \omega_n(T)$ for every unitary U.

Moreover, the following desirable property holds for ω_n.

Proposition 3.6. *For every $T \in \mathbb{B}(\mathscr{H})$*

$$\omega_n(T) \leq n\|T\| \qquad (n \in \mathbb{N}).$$

If T is normal, then equality holds.

Proof. It is not hard to see that if Φ is completely positive, then

$$\Phi(T)^*\Phi(T) \leq \|\Phi\|\Phi(T^*T). \tag{3}$$

Noting that $\|\Phi\| = \|\Phi(I)\| = 1$ and using the Löwner–Heinz inequality, (3) implies that

$$|\Phi(T)| \leq \Phi\left(|T|^2\right)^{1/2} \tag{4}$$

for every unital completely positive linear mapping Φ. Moreover,

$$|T|^2 \leq \|T\|^2 I.$$

Now assume that $\Phi : C^*(T) \to \mathbb{M}_n$ is a unital completely positive linear mapping. It follows from the last inequality that

$$\Phi\left(|T|^2\right)^{1/2} \leq \|T\|\ I_n. \tag{5}$$

From (4) and (5) we get

$$|\Phi(T)| \leq \|T\|\ I_n$$

and so

$$\mathrm{Tr}\ |\Phi(T)| \leq n\ \|T\|.$$

This concludes the inequality $\omega_n(T) \leq n\|T\|$ for every $T \in \mathbb{B}(\mathscr{H})$.

Now assume that T is normal. Then the Gelfand mapping $\Gamma : C^*(T) \to C(sp(T))$ is an isometric $*$-isomorphism, where $C(sp(T))$ is the C^*-algebra of all continuous functions on $sp(T)$. Consider two facts:
1. Every positive linear mapping $\Phi : C(\Omega) \to \mathscr{A}$ is completely positive for each arbitrary C^*-algebra $\mathscr{A}$ [3];
2. The composition of every two completely positive linear mapping is completely positive too.

Every positive linear mapping $\Phi : C^*(T) \to \mathbb{M}_n$ can be written as $\Phi = \Psi o \Gamma$, where $\Psi = \Phi o \Gamma^{-1} : C(sp(T)) \to \mathbb{M}_n$. Therefore, every positive linear mapping $\Phi : C^*(T) \to \mathbb{M}_n$ is completely positive.

Now let $x \in \mathscr{H}$ be a unit vector. The linear mapping $\Phi_x : C^*(T) \to \mathbb{M}_n$ defined by $\Phi(Z) = \langle Zx, x\rangle I_n$ is positive and so is completely positive. Therefore,

$$\omega_n(T) \geq \mathrm{Tr}\ |\Phi_x(T)| = \mathrm{Tr}\ |\langle Tx, x\rangle I_n| = n|\langle Tx, x\rangle|,$$

whence

$$\omega_n(T) \geq n\ \omega(T) = n\|T\|.$$

□

Proposition 3.6 gives an extension of (ii) of Theorem B. Note that there exists other norms on $\mathbb{M}_n$ which can be used in Definition 3.5 rather than $\|\cdot\|_1$. Typical norms on $\mathbb{M}_n$ are

$$\|A\|_p = \mathrm{Tr}\,(|A|^p)^{1/p} \quad \text{and} \quad \|A\| = \lim_{p\to\infty} \|A\|_p \qquad (A \in \mathbb{M}_n)$$

in which $\|A\|$ is the operator norm. Except when $p = 1$, Proposition 3.6 does not hold in general. To see this, consider the unilateral shift operator defined on a separable Hilbert space by $Te_j = e_{j+1}$ $(j \geq 1)$. It is known that [6]

$$W^n(T) = \{B \in \mathbb{M}_n\ ;\ B^*B \leq I_n\}.$$

Therefore,

$$\omega_n(T) = n = n\|T\|.$$

Considering the p-norm ($p \neq 1$) in Definition 3.5 concludes

$$\sup\{\|X\|_p;\ X \in W^n(T)\} = \sqrt[p]{n} \neq n\ \|T\|.$$

Unfortunately, Definition 3.5 can not be a proper extension of the numerical radius too.

Theorem 3.7. *For every* $T \in \mathbb{M}_k$

$$\omega(T) = \frac{1}{n}\omega_n(T) \qquad (n \in \mathbb{N}).$$

Proof. It is known that (see [8, Theorem 3.7])

$$\|A\|_1 \leq n\ \omega(A) \qquad (A \in \mathbb{M}_n).$$

Moreover, for $T \in \mathbb{M}_k$, it is known that $W^m(W^n(T)) \subseteq W^m(T)$ for all $m, n \in \mathbb{N}$ [6], i.e., if $A \in W^n(T)$, then $W^m(A) \subseteq W^m(T)$. Therefore, $\omega(A) \leq \omega(T)$. It follows that

$$\|X\|_1 \leq n\ \omega(X) \leq n\ \omega(T) \qquad (X \in W^n(T)),$$

whence

$$\omega_n(T) \leq n\ \omega(T).$$

Furthermore, applying an argument as in the last part of the proof of Theorem 3.4 shows that

$$n\ \omega(T) \leq \omega_n(T).$$

This completes the proof. $\square$

Example 3.8. Assume that

$$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathbb{M}_2,$$

so that $\omega(T) = \frac{1}{2}$ and $\|T\| = 1$ and

$$W^n(T) = \left\{ B \in \mathbb{M}_n\ ;\ \omega(B) \leq \frac{1}{2} \right\}.$$

We have

$$\begin{aligned}\omega_n(T) = \sup\{\|X\|_1;\ X \in W^n(T)\} &\leq n \sup\{\omega(X);\ X \in W^n(T)\}\\ &\leq \frac{n}{2} = n\omega(T).\end{aligned} \tag{6}$$

Moreover, put $Y = \frac{1}{2} I_n \in W^n(T)$ and then

$$\omega_n(T) = \sup\{\|X\|_1;\ X \in W^n(T)\} \geq \|Y\|_1 = \frac{n}{2} = n\omega(T),$$

whence,

$$\omega_n(T) = n\omega(T).$$

Remark 3.9. First, we can not find a suitable extension of the numerical radius based on the matricial range. So, we would like to pose this question that is there such an extension. Second, we obtain some relations of the numerical radius of an operator with its matricial range. In particular,

$$\begin{aligned}\omega(T) = &= \frac{1}{n} \sup\{|\mathrm{Tr}\ X|;\ X \in W^n(T)\}\\ &= \frac{1}{n} \sup\{\|X\|_1;\ X \in W^n(T)\}\\ &= \sup\{\omega(X);\ X \in W^n(T)\}.\end{aligned}$$

The last equality follows from (6).

Acknowledgements
The authors would like to express their sincere gratitude to the anonymous referees for their helpful comments.

References

[1] A. Abu-Omar, F. Kittaneh, A generalization of the numerical radius, *Linear Algebra Appl.*, **569** (2019), 323–334.

[2] W.B. Arveson, Subalgebras of C^*-algebras, II, *Acta Math.*, **128** (1972), 271–308.

[3] R. Bhatia, *Positive definite matrices*, Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2007.

[4] M. Dehghani, M. Kian, On Matricial Ranges of Some Matrices, *Journal of Mathematical Extension*, **13** no. 1, (2019), 83–102.

[5] S.S. Dragomir, *Inequalities for the Numerical Radius of Linear Operators in Hilbert Spaces*, SpringerBriefs in Mathematics. Springer, Cham, 2013.

[6] D.R. Farenick, Matricial extension of the numerical range: A brief survey, *Linear and Multilinear Algebra*, **34** (1993), no. 3-4, 197–211.

[7] E.K. Gustafson, K.M. Rao, *Numerical Range: The Field of Values of Linear Operators and Matrices*, Universitext, Springer-Verlag, New York, 1997.

[8] M. Kian, C^*-convexity of norm unit balls, *J. Math. Anal. Appl.*, **445** (2017), 1417–1427.

[9] M. Kian, Epigraph of operator functions, *Quaest. Math.*, **39** (2016), 587–594.

[10] F. Kittaneh, Numerical radius inequalities for Hilbert space operator, *Studia Math.*, **168** (2005), 73–80.

[11] F. Kittaneh, M.S. Moslehian and T. Yamazaki, Cartesian decomposition and numerical radius inequalities, *Linear Algebra Appl.*, **471** (2015), 46–53.

[12] R.I. Leoebl, V.I. Paulsen, Some remarks on C^*-convexity, *Linear Algebra Appl.*, **35** (1981), 63–78.

[13] E.J. Narcowich, J.D. Ward, A Toeplitz-Hausdorff theorem for matrix ranges, *J. Operator Theory.* **6** (1982), 87–101.

[14] S.-H. Tso, P.Y. Wu, Matricial ranges of quadratic operators, Rocky *Mountain J. Math.*, **29** (1999), 1139–1152.

[15] A. Zamani, Some lower bounds for the numerical radius of Hilbert space operators, *Adv. Oper. Theory.*, **2** no. 2 (2017), 98–107.

[16] A. Zamani, P. Wójcik, Another generalization of the numerical radius for Hilbert space operators, *Linear Algebra Appl.*, **609** (2021), 114–128.

Mohsen Kian
Associate Professor of Mathematics
Department of Mathematics
University of Bojnord
Bojnord, Iran
E-mail: kian@ub.ac.ir

Mahdi Dehghani
Assistant Professor of Mathematics
Department of Mathematics
Faculty of Mathematical Sciences, University of Kashan
Kashan, Iran
E-mail: m.dehghani@kashanu.ac.ir

Mostafa Sattari
Assistant Professor of Mathematics
Faculty of Basic Sciences
Department of Mathematics, University of Zabol
Zabol, Iran.
E-mail: Sattari@uoz.ac.ir